# Optimal Bounds for Noisy Computing

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Joint work with Ziao Wang, Banghua Zhu, and Lele Wang

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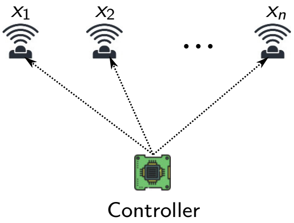


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- Goal: Compute a function $f(x_1, \ldots, x_n)$ from noisy measurements

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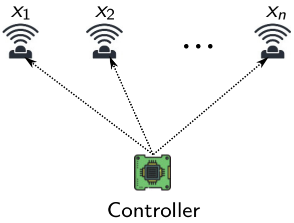


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- Applications
  - Fault tolerance
  - Active ranking
  - Recommendation systems
  - $\cdots$

# Problem Statement (OR Function)

- Let $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$.
- **OR function:**

$$\mathrm{OR}(\mathbf{x}) = \begin{cases} 1, & \text{if } \exists\, i \in [n] : x_i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

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  - At $k$th time step, submit query $U_k = x_i$ for some $i \in [n]$.
  - Receive noisy response

    $$Y_k = U_k \oplus Z_k,$$

    where $Z_k \sim \mathrm{Bern}(p)$, for some fixed and known $p < 1/2$.
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- **Question:** How many queries are needed to find $\widehat{\mathrm{OR}}$ s.t.

$$\sup_{\mathbf{x}} \mathrm{P}(\widehat{\mathrm{OR}} \neq \mathrm{OR}(\mathbf{x})) \leq \delta?$$

# Related Work (OR Function)

- Noisy boolean decision trees
  - Computation of boolean functions in the presence of noise
  - $\Omega(n \log n)$ queries are necessary when querying strategy is non-adaptive[1][2][3]
  - $\mathcal{O}(n)$ queries are sufficient when querying strategy is adaptive using a tournament algorithm[4]

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- Multi-armed bandits
  - Evaluating OR function of $n$ bits is the same as evaluating their maximum.
  - Best arm identification problem
  - Reward is $\text{Bern}(p)$ when bit is 0, and $\text{Bern}(1-p)$ when bit is 1
  - $\mathcal{O}\left(\frac{n \log(1/\delta)}{(1-2p)^2}\right)$ queries are sufficient[5]

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**Dependence on $p$ is not tight in prior work.**

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## Main Result

**Theorem 1 (OR function)**

*It is both sufficient and necessary to use*

$$(1 \pm o(1)) \frac{n \log \frac{1}{\delta}}{D_{\mathrm{KL}}(p \| 1-p)}$$

*queries in expectation to compute* OR *function with vanishing error probability* $\delta = o(1)$.

- $D_{\mathrm{KL}}(p \| 1-p)$: Kullback-Leibler (KL) divergence between Bern($p$) and Bern($1-p$)

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- $D_{\mathrm{KL}}(p \| 1-p)$: Kullback-Leibler (KL) divergence between Bern($p$) and Bern($1-p$)
- Lower bound: Based on Le Cam's two point method
- Upper bound: Devise an adaptive querying strategy to compute the OR function

**Lemma (Le Cam's Two Point Lemma)**

*Let $(P_x)_{x \in \mathcal{X}}$ be a family of distributions, and let $\ell : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}_+$ be any loss function. Let $x_1, x_2 \in \mathcal{X}$ satisfy that*

$$\ell(x_1, \hat{x}) + \ell(x_2, \hat{x}) \geq \Delta, \qquad \forall\, \hat{x} \in \hat{\mathcal{X}}.$$

*Then,*

$$\inf_{\hat{x}} \sup_{x \in \mathcal{X}} E_x[\ell(x, \hat{x})] \geq \frac{\Delta}{2} \left(1 - \| P_{x_1} - P_{x_2} \|_{TV}\right)$$

# Lower Bound: Le Cam's Two Point Method (1/3)

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- For computing the OR function, use $\mathcal{X} = \{0,1\}^n$, $\hat{\mathcal{X}} = \{0,1\}$ and

$$\ell(\mathbf{x}, \hat{x}) = \mathbb{1}\{\mathrm{OR}(\mathbf{x}) \neq \hat{x}\},$$

and $P_{\mathbf{x}}$ is the distribution of observations when the underlying sequence is $\mathbf{x}$.

# Lower Bound: Le Cam's Two Point Method (2/3)

- Consider length-$n$ sequences:
  - $\mathbf{x}_0 \triangleq$ all-zero sequence,
  - $\mathbf{x}_j \triangleq 1$ in $j$th position, and zeros everywhere else.
- For any $\hat{x} \in \{0, 1\}$,

$$\mathbb{1}\{\mathrm{OR}(\mathbf{x}_0) \neq \hat{x}\} + \mathbb{1}\{\mathrm{OR}(\mathbf{x}_j) \neq \hat{x}\} \geq 1$$

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where:

- $(a)$: Bretagnolle-Huber inequality

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where:
  - $(a)$: Bretagnolle-Huber inequality
  - $(b)$: Divergence decomposition ($T_j$ is the number of times bit $j$ is queried)

# Lower Bound: Le Cam's Two Point Method (3/3)

- Recall

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- Bound holds for each $j$.

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- Since $\sum_{j=1}^{n} \mathrm{E}_{\mathbf{x}_0}[T_j] \leq T$, there must exist $j^*$ s.t. $\mathrm{E}_{\mathbf{x}_0}[T_{j^*}] \leq T/n$. Thus,

$$\inf_{\hat{x}} \sup_{\mathbf{x}\in\{0,1\}^n} \mathrm{P}(\hat{x} \neq \mathrm{OR}(\mathbf{x})) \geq \frac{1}{4} \exp\left(-\frac{T \cdot D_{\mathrm{KL}}(p \| 1-p)}{n}\right),$$

which gives the lower bound.

# Upper Bound: Proposed NoisyOR Algorithm (1/3)

- Proposed **NoisyOR** algorithm uses two subroutines:
  - EstimateSingleBit: estimates the value of a single bit using noisy queries
  - TournamentOR: existing algorithm that computes the $\mathrm{OR}$ function

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**Algorithm 1** EstimateSingleBit

**Input**: Single bit $x$, error probability $\delta$.
**Output**: Estimate of $x$.

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Set t ← 1.
while true do
    Make noisy observation y_t of bit x.
    Set α ← P(X = 1 | Y^t = y^t).
    Set t ← t + 1.
    if α ≥ 1 − δ then return 1.
    else if α ≤ δ then return 0.
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- EstimateSingleBit has error probability at most $\delta$ and uses at most

$$(1 + o(1)) \frac{\log(1/\delta)}{D_{\mathrm{KL}}(p \| 1 - p)}$$

queries in expectation.

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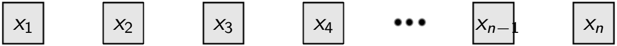
- TOURNAMENTOR: existing algorithm that computes the OR function[6][7]

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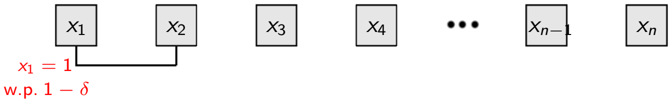


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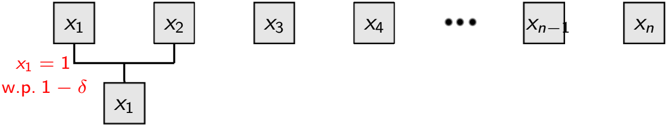


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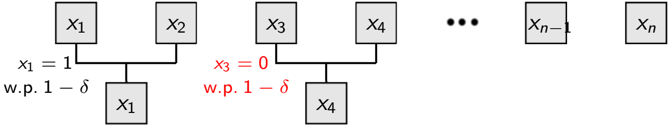


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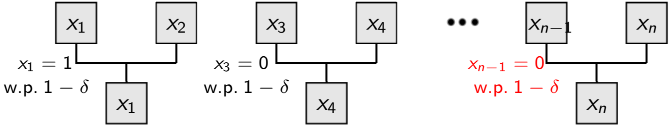


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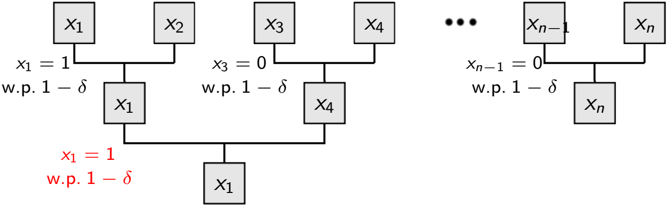


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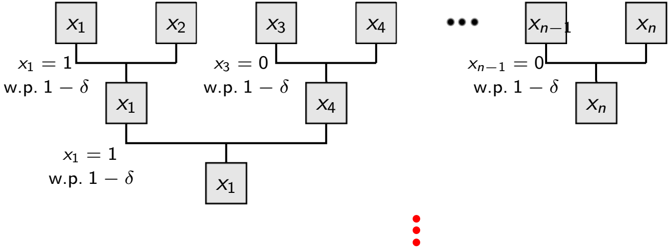


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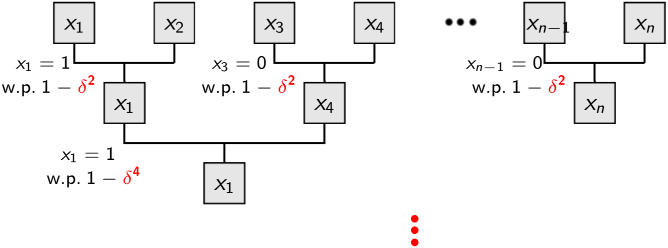


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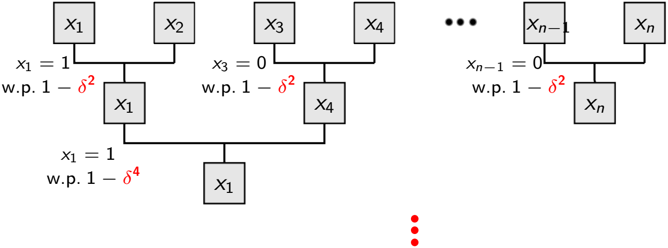


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# Upper Bound: Proposed NoisyOR Algorithm (3/3)

- Proposed **NoisyOR** algorithm

**Algorithm 2** NoisyOR

**Input**: Bit sequence $\mathbf{x} = (x_1, \ldots, x_n)$, error probability $\delta$.
**Output**: Estimate of OR($\mathbf{x}$).

```
Set y ← ∅.
for i ∈ [n] do
    if EstimateSingleBit(x_i, δ/2) = 1 then
        Append x_i to y.
if length(y) = 0 then
    return 0.
else if length(y) ≥ max(log n, nδ log 1/δ) then
    return 1.
else
    return TournamentOR(y, δ/2)
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else if length(y) ≥ max(log n, nδ log(1/δ)) then
    return 1.
else
    return TournamentOR(y, δ/2)
```

# Upper Bound: Proposed NoisyOR Algorithm (3/3)

- Proposed **NoisyOR** algorithm

**Algorithm 2** NoisyOR

**Input**: Bit sequence $\mathbf{x} = (x_1, \ldots, x_n)$, error probability $\delta$.
**Output**: Estimate of OR($\mathbf{x}$).

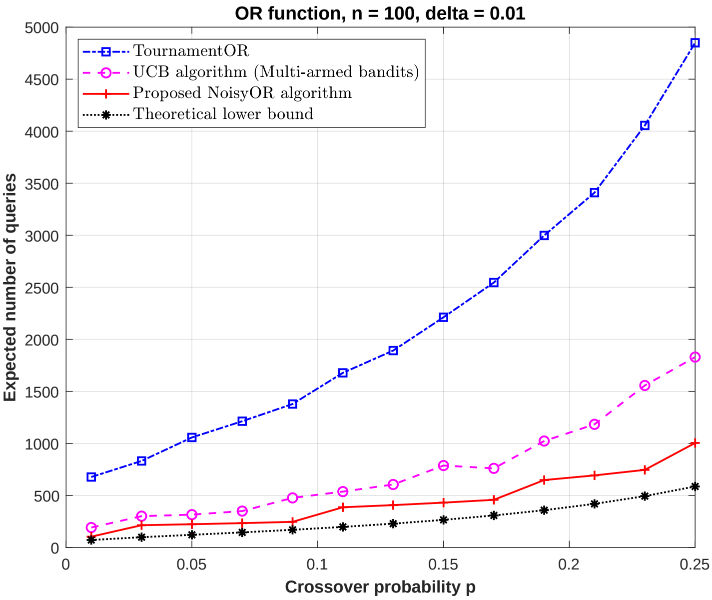
```
Set y ← ∅.
for i ∈ [n] do
    if EstimateSingleBit(x_i, δ/2) = 1 then
        Append x_i to y.
if length(y) = 0 then
    return 0.
else if length(y) ≥ max(log n, nδ log 1/δ) then
    return 1.
else
    return TournamentOR(y, δ/2)
```

- NoisyOR has error probability at most $\delta$ and uses at most

$$(1 + o(1)) \frac{n \log(1/\delta)}{D_{\mathrm{KL}}(p \| 1 - p)}$$

queries in expectation.

# Numerical Experiments



# Beyond the OR Function (1/2)

- Threshold function: For $\mathbf{x} \in \{0,1\}^n$,

$$\mathrm{TH}_k(\mathbf{x}) \triangleq \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\mathrm{OR}(\mathbf{x}) = \mathrm{TH}_1(\mathbf{x})$.

**Theorem 2 ($\mathrm{TH}_k$ function)**

*For $k = o(n)$, it is both sufficient and necessary to use*

$$(1 \pm o(1)) \frac{n \log \frac{\mathbf{k}}{\delta}}{D_{\mathrm{KL}}(p \| 1-p)}$$

*queries in expectation to compute $\mathrm{TH}_k$ with a vanishing error probability $\delta = o(1)$.*

# Beyond the OR Function (2/2)

- **Noisy Comparison Model:** When $\mathbf{x} \in \mathbb{R}^n$,
  - At $k$th time step, query $(U_k, V_k) \triangleq (x_i, x_j)$ for $i \neq j$.
  - Receive noisy response $Y_k = \mathbb{1}_{\{U_k < V_k\}} \oplus Z_k$, where $Z_k \sim \mathrm{Bern}(p)$.

[8] Z. Wang, N. Ghaddar, B. Zhu, and L. Wang. *Noisy Sorting Capacity*. 2023. arXiv: 2202.01446.

[9] Y. Gu and Y. Xu. "Optimal Bounds for Noisy Sorting". In: *STOC 2023*, 1502–1515.

# Beyond the OR Function (2/2)

- **Noisy Comparison Model:** When $\mathbf{x} \in \mathbb{R}^n$,
  - At $k$th time step, query $(U_k, V_k) \triangleq (x_i, x_j)$ for $i \neq j$.
  - Receive noisy response $Y_k = \mathbb{1}_{\{U_k < V_k\}} \oplus Z_k$, where $Z_k \sim \mathrm{Bern}(p)$.

| Function | Description | Optimal Query complexity ($\delta = o(1)$) |
|---|---|---|
| MAX | Returns index of maximum element | $\frac{n \log \frac{1}{\delta}}{D_{\mathrm{KL}}(p \| 1-p)}$ |
| SEARCH | Takes $w$ as input and returns $i$ s.t. $x_i < w < x_{i+1}$ ($\mathbf{x}$ is sorted) | $\frac{\log n}{1 - H(p)}$ |
| SORT[8][9] | Sorts $\mathbf{x}$ | $\left[\frac{1}{1-H(p)} + \frac{1}{D_{\mathrm{KL}}(p \| 1-p)}\right] n \log n$ |

[8] Z. Wang, N. Ghaddar, B. Zhu, and L. Wang. *Noisy Sorting Capacity*. 2023. arXiv: 2202.01446.

[9] Y. Gu and Y. Xu. "Optimal Bounds for Noisy Sorting". In: *STOC 2023*, 1502–1515.

# Final Remarks

- Optimal bounds for noisy computing: OR, $\text{TH}_k$, MAX, SEARCH, SORT functions
- Extensions:
  - General channel models
  - Different performance metric
  - Unknown $p$ and/or query-dependent $p$
- Arxiv version: https://arxiv.org/abs/2309.03986
- Any questions?