Exploring the p-adic Mandelbrot Set

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Women in Numbers 6

Arithmetic Dynamics

Let K be a number field and $f \in K(z)$.

$$f^n = \underbrace{f \circ f \circ f \ldots \circ f}_{n \text{ times}}$$

- α is *periodic* for f if $f^n(\alpha) = \alpha$ for some $n \in \mathbb{Z}_{\geq 0}$
- α is *preperiodic* for f if $f^k(\alpha) = f^{m+k}(\alpha)$ for some $m, k \in \mathbb{Z}_{\geq 0}$
- α is a *wandering point* if the orbit is not finite.

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \in \mathbb{Z}\}$$

Dynamics Notation

We say that $f \in K$ is $\bar{K}$-conjugate to g if $g = f^{\varphi}$ for some $\varphi \in \mathrm{PGL}_2(\bar{K})$, where

$$f^{\varphi} = \varphi \circ f \circ \varphi^{-1}.$$

If α is preperiodic of period (k, m) for f the $\phi(\alpha)$ is preperiodic of period (k, m) for f^{ϕ}.

$$[f] = \{f^{\phi} : \phi \in \mathrm{PGL}_2(\bar{K})\}$$

Example

Let f be a polynomial with a single finite critical point. Then f is conjugate to $z^d + c$ for some constant c.

Definitions

Let $\text{Crit}(f) = \{\text{critical points of } f\}$.

Definition

f is post-critically finite (PCF) if every element of $\text{Crit}(f)$ has finite forward orbit.

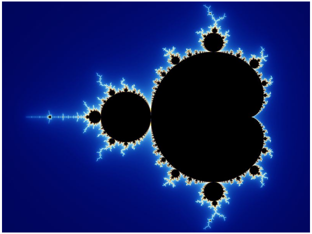
Definition

f is post-critically bounded (PCB) if every element of $\text{Crit}(f)$ has bounded forward orbit.

$$PCF \subseteq PCB$$

Mandelbrot Set

$$\mathcal{M} = \{c \in \mathbb{C} : z^2 + c \text{ is PCB}\}$$



Misiurewicz points are points in $\mathcal{M}$ for which 0 is preperiodic for $z^2 + c$. These points form a countable dense subset of the boundary of $\mathcal{M}$.

Misiurewicz points and Julia sets

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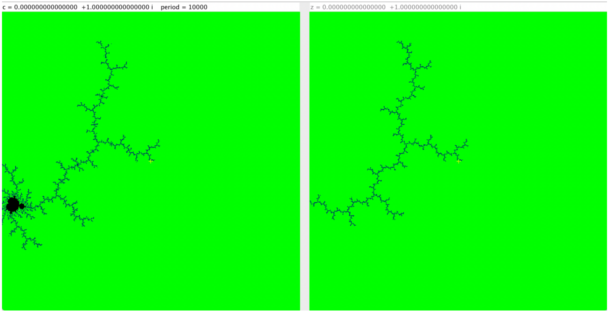
- Moreover, in 1989 Tan Lei proved that if you zoom in on the point c in the Julia set for f_c, the picture you see looks very similar to the Mandelbrot set zoomed in on c with the same level of magnification.

An example: $c = i$



On the left: Mandelbrot set zoomed in on $c = i$.
On the right: Julia set for $z^2 + i$ zoomed in on $z = i$.

Higher degrees

More generally, we can normalize a degree d polynomial and write it as

$$z^d + a_{d-1}z^{d-1} + \dots a_1 z + a_0.$$

Then we can consider the parameters that admit a post-critically bounded degree d polynomial.

In the complex setting, the unicritical case ($z^d + c$) has been studied and the corresponding PCB locus is sometimes called the multibrot set.

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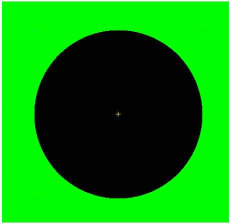
Answer:



p-adic analogue of $\mathcal{M}$

- For $z^2 + c$, 0 is PCB if and only if $|c|_p \leq 1$, so the p-adic analogue of the classical Mandelbrot set is just the p-adic unit disk (for all p).
- That isn't nearly as exciting as what we see in the complex setting, but what if we look at higher-degree polynomials?

Defining a p-adic Mandelbrot set of degree d

- Consider the space $\mathcal{P}_{d,p}$ of normalized degree d polynomials $z^d + a_{d-1}z^{d-1} + \dots a_1 z$ defined over $\mathbb{C}_p$.
- Define $\mathcal{M}_{d,p}$ to be the subset of $\mathcal{P}_{d,p}$ consisting of post-critically bounded polynomials.

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- Define $\mathcal{M}_{d,p}$ to be the subset of $\mathcal{P}_{d,p}$ consisting of post-critically bounded polynomials.
- If $d \leq p$ then $\mathcal{M}_{d,p}$ is the unit (poly)disk: a polynomial in $\mathcal{P}_{d,p}$ is PCB if and only if all of its coefficients lie in the p-adic unit disk. If $p < d$ then the situation can be more interesting and $\mathcal{M}_{d,p}$ can contain points outside the unit polydisk.

Example: A Cubic Family

$$f_t(z) = z^3 - \frac{3}{2}tz^2$$

- $\mathrm{Crit}(f) = \{0, t\}$ and 0 is fixed.
- t corresponds to a *Misiurewicz point* in $\mathcal{M}_{3,2}$ if t is strictly preperiodic under iteration of f_t, such that it eventually falls into a repelling periodic cycle.
- In my thesis, I showed that f_1 is a Misiurewicz point on the boundary of $\mathcal{M}_{3,2}$, meaning there are parameters t arbitarily close to 1 such that f_t is PCB and parameters t arbitrarily close to 1 such that t has an unbounded orbit. (When $p \geq d$, these Mandelbrot set analogues have no boundary.)

Self-Similar Behavior

We can visualize a disk in $\mathbb{Q}_2$ as a binary tree because each disk in $\mathbb{Q}_2$ of radius 2^k is the disjoint union of two disk of radius 2^{k-1}.

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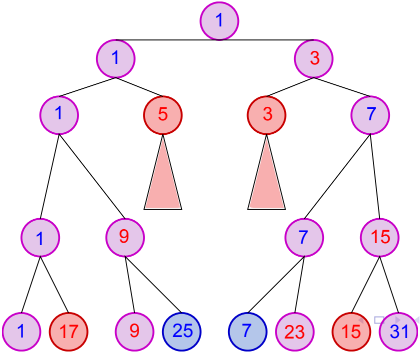
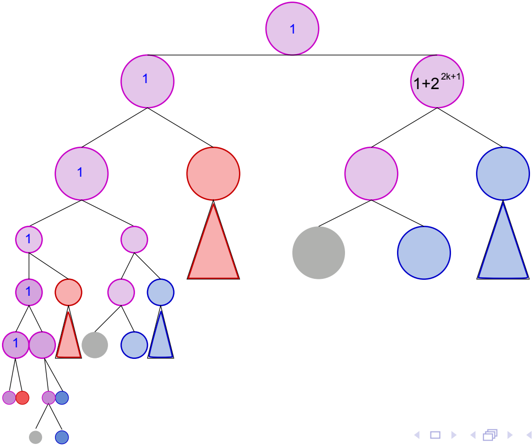
Color a disk blue if all parameters t in that disk correspond to PCB maps in our family $(z^3 - \frac{3}{2}tz^2)$, red if all parameters have a critical point with an unbounded orbit, and purple if the disk contains some of each. Here is what our parameter space looks like near $t = 1$:



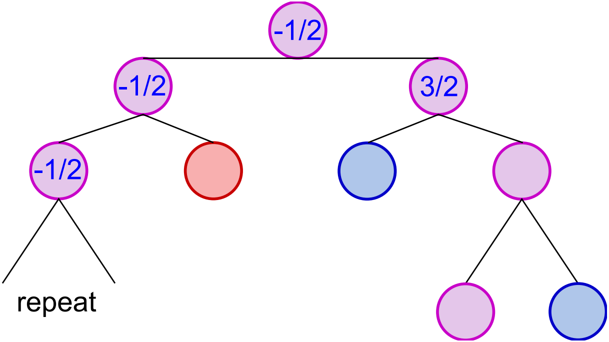
Self-Similar Behavior

When you zoom in on $t = 1$ (i.e., move down the left side of the tree) an arbitrary number of levels, you see the same pattern over and over:



Similarity between Mandelbrot and Julia sets

If we look at the map corresponding to $t = 1$ ($z^3 - \frac{3}{2}z^2$) and color disks in a similar way in a neighborhood of the repelling fixed point $z = -\frac{1}{2}$, we see a similar self-similar pattern (just like we do for Misiurewicz points in the classical Mandelbrot set!)



Our questions

- How widespread is this behavior?
  - Does a similar phenomenon occur for other Misiurewicz points in this cubic family?
  - ... or for analogous points in other families of fixed degree and prime?
  - Can we generalize this cubic family over $\mathbb{Q}_2$ to a degree $p+1$ family in degree p?
- Can we form an analogue of Tan Lei's theorem in the p-adic setting?
- Can we get a better sense of the size/structure of the p-adic Mandelbrot set when $p < d$?

Initial Conjectures:

Consider the family $f_t(z) = -\frac{3}{2}t(-2z^3 + 3z^2) + 1$.

The critical points 0 and 1 are preperiodic when $t = 1$.

$$0 \longrightarrow 1 \longrightarrow -\tfrac{1}{2} \circlearrowleft$$

Conjecture

If $t \equiv 13 \mod 16$ the forward orbit of 1 is bounded.

Conjecture

If $t \equiv 2n + 1 \mod 2^{n+1}$ the forward orbit of 1 escapes.