Open Problems Session #1, Wednesday

Scribe: Matt Baker

March 14, 2023

(Alex Fink) One might naturally ask, if $M \to N$ is a weak map of connected matroids of the same rank on the same ground set (i.e., every basis of N is a basis of M), if there exist a regular matroid polytope subdivision of M of which N is a face. The answer, as shown in a paper of Brandt-Speyer, is in general **no**.

Question: Can we salvage this by merely asking for a chain of subdivisions

$$M = M_0 \to M_1 \to \cdots \to M_k = N?$$

Rudi Pendavingh said that the Betsy Ross matroid should be a counterexample.

Question (June Huh): Can one prove that certain matroid polynomials (e.g. the Kazhdan–Lusztig polynomial and the Z-polynomial) are (coefficientwise) monotone with respect to weak maps?

Question (Alex Fink): Can we model weak maps using matroids over bands, in the sense of Baker–Bowler and Baker–Lorscheid?
Alex proposed a specific band which should do the job. What can one do with this?

(Oliver Lorscheid) If $f : M \to N$ is a strong map of matroids on the same ground set (i.e., N is a quotient of M), there is a factorization theorem which says that f factors as a restriction followed by a contraction. One might wonder if this also holds for B-matroids, where B is an *idyll*. However, the answer is **no**: for example, when B is the sign hyperfield, we are talking about oriented matroids and Richter-Gebert has given a counterexample to the factorization theorem.
Question: For which B is it true?

(Matt Larson) Suppose L is a linear subspace of K^n, where K is a field, giving rise to a loopless matroid M of rank r. Let P be a full-dimensional generalized permutohedron in $\mathbb{R}^n$, and define $R^{\cdot}(P, L)$ to be the image of

$$\oplus_{k \geq 0} H^0(X_{A_{n-1}}, \mathcal{O}(kP))$$

in $\mathbb{P}L \cap T$.

Conjecture 1:

$$\dim R^k(P, L) = \chi(W_L, \mathcal{O}(kP)) = \chi(M, \mathcal{O}(kP)).$$

This would follow if $H^i(W_L, \mathcal{O}(kP)) = 0$ for $i > 0$ and we have surjectivity on H^0.

Conjecture 2: $R^{\cdot}(P, L)$ is a Cohen-Macaulay ring.

Conjecture 3:

$$\sum_{k \geq 0} \chi(M, \mathcal{O}(kP)) t^k = \frac{Q(t)}{(1-t)^r},$$

where the coefficients of Q are nonnegative.

Remarks:

1. Conjecture 1 + Conjecture 2 implies Conjecture 3 if M is realizable.
2. These are true if $L = K^n$, P is the standard simplex, or P is the negative of the standard simplex.
3. If true, these conjectures would show that $[t^r]g_M(t) \geq 0$.

(Federico Ardila posed a problem but I had to step out to take a phone call.)

(Chris Eur):

Given a matroid quotient $M \twoheadrightarrow N$, we have a canonical **Higgs factorization**

$$M \twoheadrightarrow M_1 \twoheadrightarrow M_2 \twoheadrightarrow \cdots \twoheadrightarrow N.$$

Each successive quotient in the factorization comes from a matroid $\hat{M}_i$, where M_{i-1} is a single-element deletion of $\hat{M}_i$ and M_i is a single-element contraction of $\hat{M}_i$. Consider the beta-invariants of the matroids $\hat{M}_i$.

Question: Is the sequence $\beta(\hat{M}_1), \ldots, \beta(\hat{M}_{r-1})$ log-concave?