Permutation Action on Chow Rings of Matroids

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(Joint work with Robert Angarone and Vic Reiner)

Algebraic Aspects of Matroid Theory at BIRS

Background

1. *Chow rings*
2. *Group action*
3. *Unimodality and symmetry*

Chow ring of a matroid

$$A^\bullet(M) := \frac{\mathbb{R}[x_F : \varnothing \neq F \in \mathscr{L}(M)]}{I + J}$$

$$I = \left(x_F x_G : F, G \text{ incomparable in } \mathscr{L}\right)$$

$$J = \left(\sum_{G \supseteq F} x_G : F \text{ an atom}\right)$$

Theorem (Feichtner-Yuzvinsky '03):

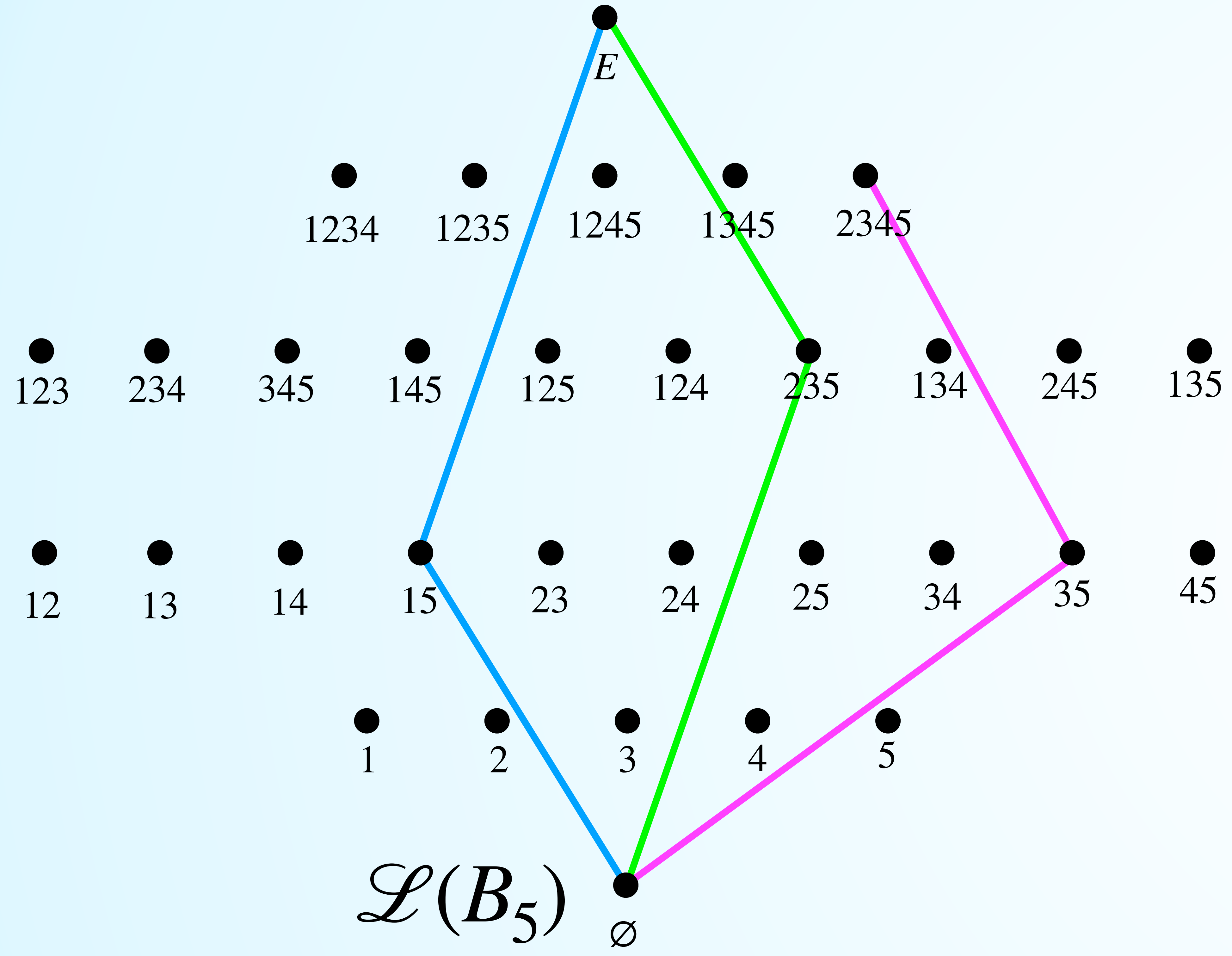
A basis for the Chow ring is given by monomials of the following form ("FY monomials"):

$$\prod_{i=1}^{k} x_{F_i}^{m_i} : \varnothing = F_0 \subset F_1 \subset \ldots \subset F_k$$

$$m_i \leq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1$$

Degree-k FY monomials, $\mathrm{FY}^k(M)$, form a basis for $A^k(M)$.

Example



$A^4(B_5)$	x_E^4 1	1
$A^3(B_5)$	$x_{ij}x_E^2$, 10 $x_{ijk}^2 x_E$, 10 x_{ijkl}^3, 5 x_E^3 1	26
$A^2(B_5)$	x_{ijk}^2, 10 x_{ijkl}^2, 5 x_E^2 1 $x_{ij}x_E$, 10 $x_{ijk}x_E$, 10 $x_{ij}x_{ijkl}$ 30	66
$A^1(B_5)$	x_{ij}, 10 x_{ijk}, 10 x_{ijkl}, 5 x_E 1	26
$A^0(B_5)$	1 1	1

$$A^k(M) \hookrightarrow A^{k+1}(M)$$
$$\Downarrow$$
$$\dim_{\mathbb{C}} A^k(M) \leq \dim_{\mathbb{C}} A^{k+1}(M)$$

$$A^k(M) \cong A^{r-1-k}(M)$$
$$\Downarrow$$
$$\dim_{\mathbb{C}} A^k(M) = \dim_{\mathbb{C}} A^{r-1-k}(M)$$

Group Action on Matroids

- Let $\mathrm{Aut}(M) := \{\text{maps } E \to E \text{ preserving } M\} \cong \{\text{poset automorphisms of } \mathscr{L}(M)\}$

  For $G \in \mathrm{Aut}(M)$, the Chow ring is a representation of G:

  - G acts on $\mathbb{R}[x_F : F \in \mathscr{L}]$ by swapping variables via $G \curvearrowright \mathscr{L}$
  - Since G preserves $I + J$, the quotient $A^\bullet(M)$ is a representation of G.

- Even further, $A^\bullet(M)$ is a **permutation representation** of G.

  G must preserve inclusion of flats and degrees of monomials, so it must permute the **Feichtner-Yuzvinsky basis.**

$$M = B_5 \qquad G = S_5$$

$$(12)(345) \cdot \left(x_{23}x_{1235}\right) = x_{14}x_{1234}$$

Unimodality and Symmetry

Theorem (Angarone–N–Reiner)

$$\mathrm{FY}^k(M) \hookrightarrow \mathrm{FY}^{k+1}(M)$$

(as G-sets)

$$\Downarrow$$

$$A^k(M) \hookrightarrow A^{k+1}(M)$$

(as G representations)

$$\Downarrow$$

$$A^k(M) \hookrightarrow A^{k+1}(M)$$

(as vector spaces)

$$\Downarrow$$

$$\dim_{\mathbb{C}} A^k(M) \leq \dim_{\mathbb{C}} A^{k+1}(M)$$

$$\mathrm{FY}^k(M) \cong \mathrm{FY}^{r-1-k}(M)$$

(as G-sets)

$$\Downarrow$$

$$A^k(M) \cong A^{r-1-k}(M)$$

(as G representations)

$$\Downarrow$$

$$A^k(M) \cong A^{r-1-k}(M)$$

(as vector spaces)

$$\Downarrow$$

$$\dim_{\mathbb{C}} A^k(M) = \dim_{\mathbb{C}} A^{r-1-k}(M)$$

Our Work

1. *Injective map*
2. *Example*
3. *Future directions*

Injective Map Idea

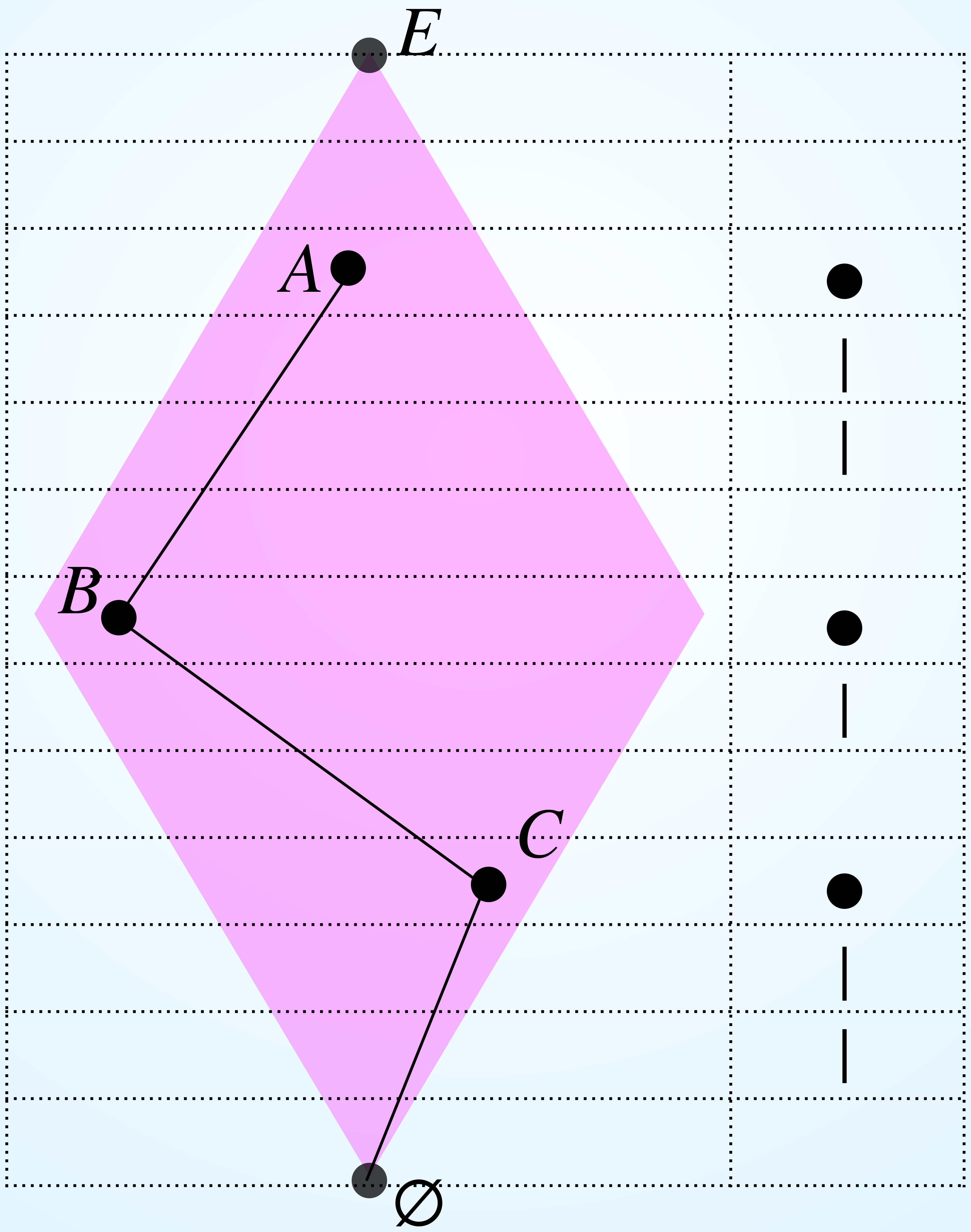
$$A^5(M) \hookrightarrow A^6(M)$$

$$\varnothing \subset C \subset B \subset A \subset E$$

$$x_A^2 x_B x_C^2 \in A^5(M)$$

Goal

Injectively raise the degree of this monomial by 1 while preserving the action of G.



Dots: ranks

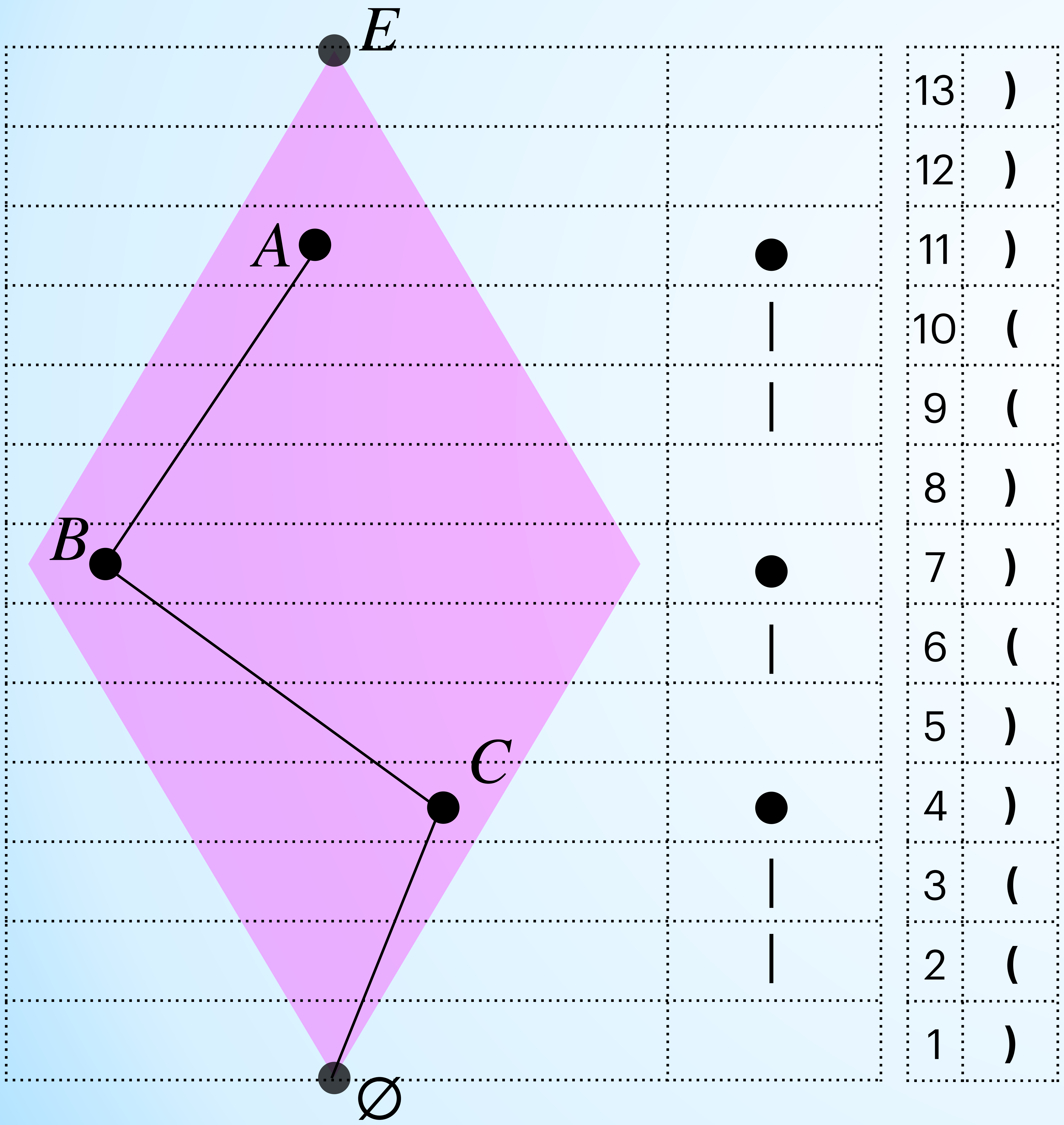
Dashes: degree

Flag: lost (!) (?)

New Goal

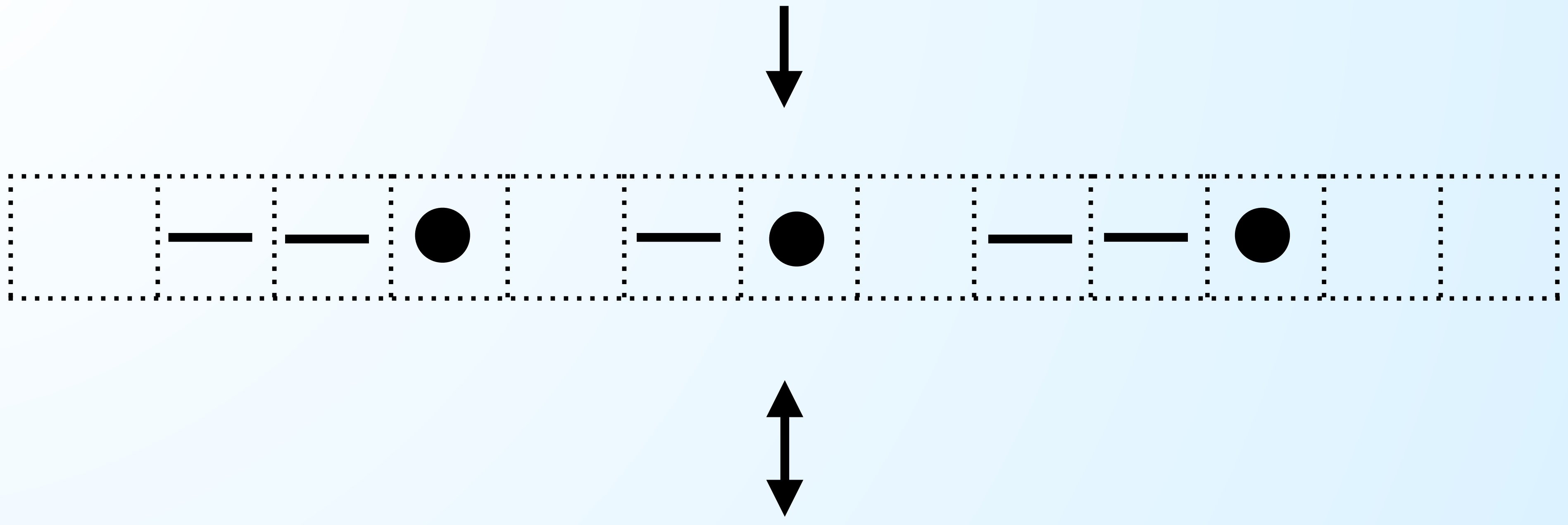
Add a dash, don't touch the dots, except possibly the top.

Making it Precise



Unlabelled dots-and-dashes diagrams for a rank r lattice are in bijection with subsets of $r-1$.

$$x_A^2 x_B x_C^2 \in A^5(M)$$



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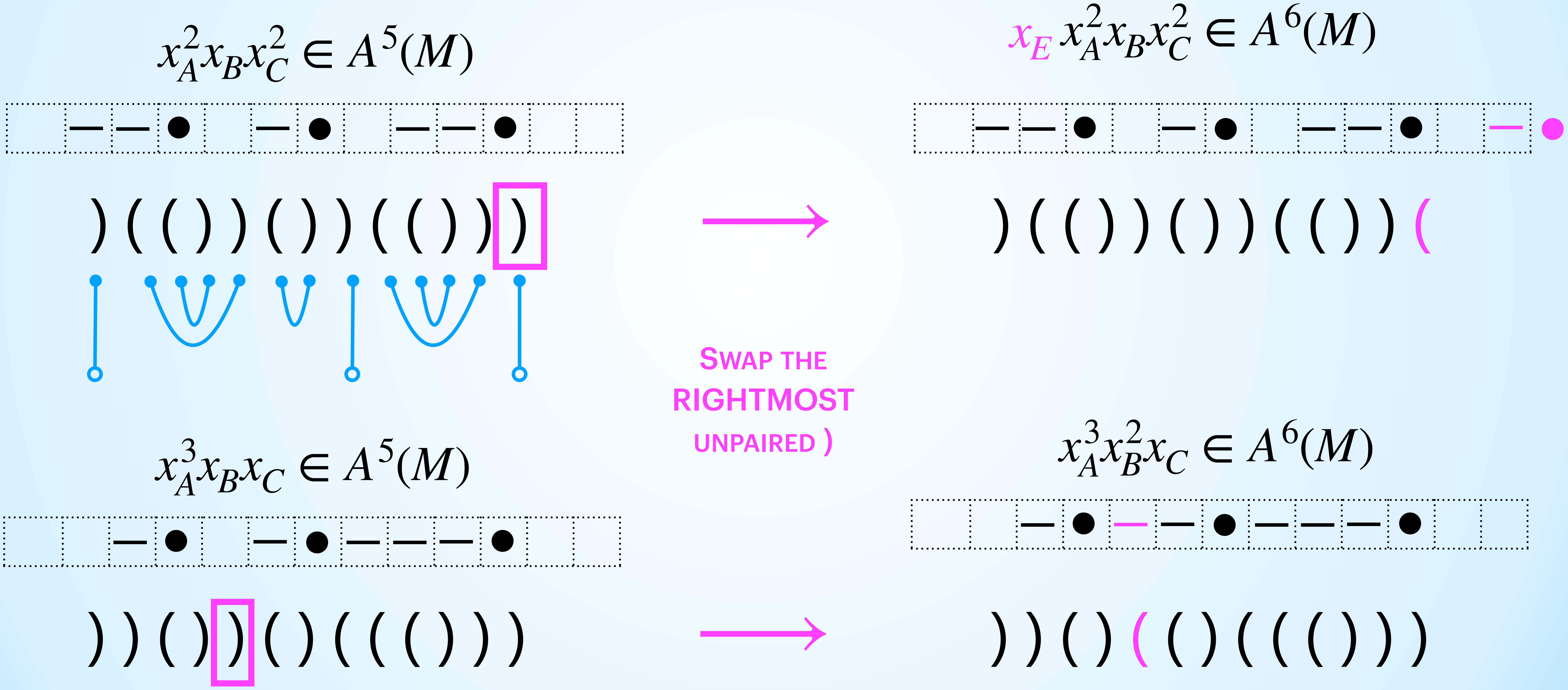
Our Map in Action



Punchline

$$\mathrm{FY}^k(M) \hookrightarrow \mathrm{FY}^{k+1}(M) \qquad\qquad \mathrm{FY}^k(M) \cong \mathrm{FY}^{r-1-k}(M)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$A^k(M) \hookrightarrow A^{k+1}(M) \qquad\qquad A^k(M) \cong A^{r-1-k}(M)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dim_{\mathbb{C}} A^k(M) \leq \dim_{\mathbb{C}} A^{k+1}(M) \qquad\qquad \dim_{\mathbb{C}} A^k(M) = \dim_{\mathbb{C}} A^{r-1-k}(M)$$

REAL ROOTEDNESS

$\Downarrow$

LOG-CONCAVITY

$\Downarrow$

UNIMODALITY

Future Directions

We hope this approach can be used to work toward a proof of the log-concavity of the Chow ring

Towards Log Concavity

Equivariant log concavity of $A^\bullet(M)$

$$A^i \otimes A^i - A^{i-1} \otimes A^{i+1}$$

is a genuine representation

Theorem (McCullough-Maestroni '22):
Chow rings are Koszul.

$$\Downarrow$$

Determinants of Toeplitz matricies with A^1 on the diagonal give genuine representations

Note that

$$A^i \otimes A^i - A^{i-1} \otimes A^{i+1} = \det \begin{vmatrix} A^i & A^{i+1} \\ A^{i-1} & A^i \end{vmatrix}$$

$$\begin{pmatrix} A^1 & A^2 & A^3 & A^4 & \dots \\ A^0 & A^1 & A^2 & A^3 & \dots \\ 0 & A^0 & A^1 & A^2 & \dots \\ 0 & 0 & A^0 & A^1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Conjecture (ANR)
These determinants are **permutation representations.**

THANK YOU

References

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