

Quadratic Pairs over Schemes and the Canonical Quadratic Pair on Clifford Algebras

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Usual Clifford Algebras

Let $q: V \rightarrow \mathbb{F}$ be a non-singular quadratic form, $\dim(V) = 2n$, $\text{char}(\mathbb{F}) \neq 2$.

$$\text{Cl}(V, q) = T(V) / \langle v \otimes v - q(v) \cdot 1 \mid v \in V \rangle.$$

Canonical involution $\underline{\sigma}: v_1 \otimes \dots \otimes v_k \mapsto v_k \otimes \dots \otimes v_1$.

$$\text{Cl}_0(V, q) = \text{Span}_{\mathbb{F}}(\{v_1 \otimes \dots \otimes v_k \mid k \text{ even}\}) \subset \text{Cl}(V, q)$$

$\underline{\sigma}_0 = \underline{\sigma}|_{\text{Cl}_0(V, q)}$ also called canonical involution.

$$\underline{\sigma}_0 \text{ is } \begin{cases} \text{unitary} & n \text{ odd} \\ \text{symplectic} & n \equiv 2 \pmod{4} \\ \text{orthogonal} & n \equiv 0 \pmod{4}. \end{cases}$$

$$\mathbf{O}(V, q) \rightarrow \mathbf{PGO}(\text{Cl}_0(V, q), \underline{\sigma}_0)$$

$$\varphi \mapsto (v_1 \otimes v_2 \mapsto \varphi(v_1) \otimes \varphi(v_2))$$

Usual Clifford Algebras

Let (A, σ) be a central simple \mathbb{F} -algebra, $\deg(A) = 2n$, σ orthogonal.

$$\text{Cl}(A, \sigma) = \frac{T(A)}{J_1 + J_2}$$

	(A, σ)	$(\text{End}_{\mathbb{F}}(V), \sigma_q) \cong (V \otimes_{\mathbb{F}} V, \text{sw})$
J_1	$\langle a - \frac{1}{2} \text{Trd}_A(a) \mid \sigma(a) = a \rangle$	$\langle v \otimes v - q(v) \cdot 1 \rangle$
J_2	complicated	$\langle (w_1 \otimes v) \otimes (v \otimes w_2) - q(v) w_1 \otimes w_2 \rangle$

Canonical involution $\underline{\sigma}(a_1 \otimes \dots \otimes a_k) = \sigma(a_k) \otimes \dots \otimes \sigma(a_1)$.

$$\underline{\sigma} \text{ is } \begin{cases} \text{unitary} & n \text{ odd} \\ \text{symplectic} & n \equiv 2 \pmod{4} \\ \text{orthogonal} & n \equiv 0 \pmod{4}. \end{cases}$$

$$\mathbf{PGO}(A, \sigma) \rightarrow \mathbf{PGO}(\text{Cl}(A, \sigma), \underline{\sigma})$$

$$\varphi \mapsto (a \mapsto \varphi(a)).$$

Conventions

Over arbitrary \mathbb{F} (later over a scheme S). Let

$$\sigma: A \rightarrow A$$

be an involution of the first kind, split A

$$\rightarrow \sigma': \text{End}_{\mathbb{F}'}(V) \rightarrow \text{End}_{\mathbb{F}'}(V)$$

adjoint to a regular bilinear form $b_\sigma: V \times V \rightarrow \mathbb{F}'$.

$$\text{call } \sigma \begin{cases} \text{orthogonal} & \text{if } b_\sigma \text{ symmetric} \\ \text{weakly-symplectic} & \text{if } b_\sigma \text{ skew-symmetric} \\ \text{symplectic} & \text{if } b_\sigma \text{ alternating.} \end{cases}$$

Quadratic Pairs

\mathbb{F} an arbitrary field, A central simple \mathbb{F} -algebra.

Definition [KMRT]

A *quadratic pair* on A is (σ, f) where

- σ is an orthogonal involution,
- $f: \text{Sym}(A, \sigma) \rightarrow \mathbb{F}$ is linear such that

$$f(a + \sigma(a)) = \text{Trd}_A(a) \quad \forall a \in A.$$

If $\text{char}(\mathbb{F}) \neq 2$, $s \in \text{Sym}(A, \sigma)$

$$f(s) = f\left(\frac{1}{2}(s + \sigma(s))\right) = \frac{1}{2}f(s + \sigma(s)) = \frac{1}{2}\text{Trd}_A(s).$$

Connection to Quadratic Forms

Theorem [KMRT]

Consider $(\text{End}_{\mathbb{F}}(V), \sigma_b)$ for a regular symmetric bilinear $b: V \times V \rightarrow \mathbb{F}$. We have $\varphi_b: (V \otimes_{\mathbb{F}} V, \text{sw}) \rightarrow (\text{End}_{\mathbb{F}}(V), \sigma_b)$. There is a bijection

$$\left\{ \begin{array}{l} f \text{ such that } (\sigma_b, f) \\ \text{is a quadratic pair} \end{array} \right\} \leftrightarrow \{q: V \rightarrow \mathbb{F} \text{ with polar } b\}$$

$$f \mapsto q_f(v) = f(\varphi_b(v \otimes v))$$

$$f_q(\varphi_b(v \otimes v)) = q(v) \leftarrow q$$

$$f_q(\varphi_b(v \otimes w + w \otimes v)) = b_q(v, w)$$

Classification [KMRT]

For all (A, σ, f) there exists $\ell \in A$ such that

- $\ell + \sigma(\ell) = 1$
- $f(_) = \text{Trd}_A(\ell \cdot _)$
- ℓ is unique up to adding an element of $\text{Alt}(A, \sigma) = \{a - \sigma(a) \mid a \in A\}$.

Conversely, given (A, σ) , any $\ell \in A$ with $\ell + \sigma(\ell) = 1$ defines

$$f_\ell = \text{Trd}_A(\ell _)$$

and (σ, f_ℓ) is a quadratic pair. $f_\ell = f_{\ell'} \Leftrightarrow \ell - \ell' \in \text{Alt}(A, \sigma)$.

More Clifford Algebras

Let (A, σ, f) be an \mathbb{F} -c.s.a. with quadratic pair, $\deg(A) = 2n$.

$$\text{Cl}(A, \sigma, f) = \frac{T(A)}{J_1 + J_2}$$

	(A, σ, f)	$(\text{End}_{\mathbb{F}}(V), \sigma, f), \varphi: V \otimes_{\mathbb{F}} V \xrightarrow{\sim} \text{End}_{\mathbb{F}}(V)$
J_1	$\langle a - f(a) \mid \sigma(a) = a \rangle$	$\langle v \otimes v - f(\varphi(v \otimes v)) \cdot 1 \rangle$
J_2	still complicated	$\langle (w_1 \otimes v) \otimes (v \otimes w_2) - f(\varphi(v \otimes v))w_1 \otimes w_2 \rangle$

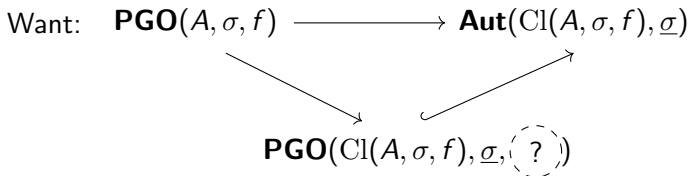
Canonical involution $\underline{\sigma}(a_1 \otimes \dots \otimes a_k) = \sigma(a_k) \otimes \dots \otimes \sigma(a_1)$.

$$\underline{\sigma} \text{ is orthogonal} \Leftrightarrow \begin{cases} n \equiv 0 \pmod{4}, \text{ or} \\ n \equiv 2 \pmod{4} \text{ and } \text{char}(\mathbb{F}) = 2. \end{cases}$$

For clarity, $c: A \rightarrow \text{Cl}(A, \sigma, f)$ is the map $a \mapsto c(a) = a$.

Canonical Quadratic Pair

When $\underline{\sigma}$ is orthogonal, $(\text{Cl}(A, \sigma, f), \underline{\sigma}, \text{?})$.



Theorem (Dolphin, Quéguiner-Mathieu (2021))

Let $\deg(A) \geq 8$ such that $\underline{\sigma}$ is orthogonal. Take any $a \in A$ with $\text{Trd}_A(a) = 1$ and use $\ell = c(a) \in \text{Cl}(A, \sigma, f)$. Then,

$$\underline{f} = \text{Trd}_{\text{Cl}(A, \sigma, f)}(c(a) \cdot \underline{\quad})$$

- does not depend on the choice of $a \in A$, and
- \underline{f} fills (?) in the diagram above.

Canonical Quadratic Pair

$$\begin{aligned}\ell + \underline{\sigma}(\ell) &= c(a) + \underline{\sigma}(c(a)) \\ &= c(a) + c(\sigma(a)) \\ &= c(a + \sigma(a)) \\ &= f(a + \sigma(a)) \cdot 1 \\ &= \text{Trd}_A(a) \cdot 1 \\ &= 1\end{aligned}$$

due to J_1

by definition of f

by assumption.

Over a Scheme

Let S be an arbitrary base scheme. $(\mathcal{Gch}_S, \mathcal{O})$ the ringed fppf-site over S . (\mathcal{A}, σ) is an Azumaya algebra of constant degree $2n$ with orthogonal involution.

- $\mathcal{A}: \mathcal{Gch}_S \rightarrow \mathcal{Ab}$ is a sheaf, in particular an \mathcal{O} -module, locally isomorphic to $M_{2n}(\mathcal{O})$.
- $\sigma: \mathcal{A} \rightarrow \mathcal{A}$ is a natural transformation, locally adjoint to symmetric bilinear forms.

$$\mathit{Sym}_{\mathcal{A}, \sigma} = \ker(\mathrm{Id} - \sigma) \subset \mathcal{A}$$

$$\mathit{Alt}_{\mathcal{A}, \sigma} = \mathrm{Im}(\mathrm{Id} - \sigma) \subset \mathcal{A}$$

$$\mathit{Symd}_{\mathcal{A}, \sigma} = \mathrm{Im}(\mathrm{Id} + \sigma) \subset \mathcal{A}.$$

Over a Scheme

Definition [Calmès, Fasel]

A *quadratic pair* on \mathcal{A} is (σ, f) where

- σ is an orthogonal involution, and
- $f: \text{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$ is \mathcal{O} -linear such that

$$f(a + \sigma(a)) = \text{Trd}_{\mathcal{A}}(a) \quad \forall T \in \text{Gch}_S, \forall a \in \mathcal{A}(T).$$

For (\mathcal{A}, σ, f) :

$$\mathcal{Cl}(\mathcal{A}, \sigma, f) = \frac{\mathcal{T}(\mathcal{A})}{\mathcal{I}_1 + \mathcal{I}_2}$$

$c: \mathcal{A} \rightarrow \mathcal{Cl}(\mathcal{A}, \sigma, f)$, canonical involution $\underline{\sigma}: c(a) \mapsto c(\sigma(a))$, and

$$\underline{\sigma} \text{ is orthogonal} \Leftrightarrow \begin{array}{l} n \equiv 0 \pmod{4}, \text{ or} \\ n \equiv 2 \pmod{4} \text{ and } 2 = 0 \in \mathcal{O}(S). \end{array}$$

Problem 1

Given (\mathcal{A}, σ, f) , there may not exist $\ell \in \mathcal{A}(S)$ such that $f(_) = \text{Trd}_{\mathcal{A}}(\ell \cdot _)$.

Example [Gille, Neher, R.]

$S = E$ an ordinary elliptic curve over \mathbb{F} , $\text{char}(\mathbb{F}) = 2$. Consider the $(\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z})$ -torsor $E' = E \xrightarrow{\cdot 2} E$. Then,

$$(Q, \sigma) = E' \wedge^{\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z}} (M_2(\mathcal{O}), \psi)$$

is a quaternion algebra over E .

- There exists f such that (Q, σ, f) is a quadratic pair.
- $Q(E) \cong \mathbb{F}$ and so $Q(E) = \text{Sym}_{Q, \sigma}(E)$.
- Then $\ell + \sigma(\ell) = 2\ell = 0$ for all $\ell \in Q(E)$.

Solution 1

Lemma [GNR]

If (\mathcal{A}, σ, f) is an Azumaya algebra with quadratic pair, then $1_{\mathcal{A}} \in \text{Symd}_{\mathcal{A}, \sigma}(S)$.

Given a *locally quadratic* (\mathcal{A}, σ) (i.e., with $1 \in \text{Symd}_{\mathcal{A}, \sigma}(S)$),

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{Id} + \sigma} & \text{Symd}_{\mathcal{A}, \sigma} \\ \downarrow & & \parallel \\ \mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma} & \xrightarrow{\xi} & \text{Symd}_{\mathcal{A}, \sigma} \end{array}$$

Theorem [GNR]

There is a bijection of sets

$$\left\{ \begin{array}{l} f \text{ such that } (\mathcal{A}, \sigma, f) \\ \text{is a quadratic pair on } \mathcal{A} \end{array} \right\} \leftrightarrow \xi(S)^{-1}(1_{\mathcal{A}}) \subset (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S).$$

Solution 1

Lemma [GNR]

Given (\mathcal{A}, σ, f) , if S is affine there exists $\ell \in \mathcal{A}(S)$ such that $f = \text{Trd}_{\mathcal{A}}(\ell \cdot _)$.

Fix (\mathcal{A}, σ) . Take affine cover $\{U_i \rightarrow S\}_{i \in I}$. Given f ,

$$\rightarrow \ell_i \in \mathcal{A}(U_i)$$

$$\rightarrow [\ell_i] \in (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(U_i), \text{ these glue}$$

$$\Rightarrow [\ell] \in (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S)$$

$$\Rightarrow \pi([\ell]) = 1 \text{ since } \ell_i + \sigma(\ell_i) = 1.$$

Given $[\ell] \in (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S)$ with $\pi([\ell]) = 1$,

$$\rightarrow \text{get } \ell_i \in \mathcal{A}(U_i) \text{ with } \ell_i + \sigma(\ell_i) = 1,$$

$$\rightarrow f_i = \text{Trd}_{\mathcal{A}|_{U_i}}(\ell_i \cdot _): \text{Sym}_{\mathcal{A}, \sigma}|_{U_i} \rightarrow \mathcal{O}|_{U_i}, \text{ these glue}$$

$$\Rightarrow f: \text{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}.$$

Problem 2

Given (\mathcal{A}, σ, f) , there may not exist $a \in \mathcal{A}(S)$ with $\text{Trd}_{\mathcal{A}}(a) = 1$.

Example [GNR]

Take \mathcal{Q} as above. $\mathcal{Q}(E) = \mathcal{O}(E) \cong \mathbb{F}$. Whenever $\mathcal{Q}(T) \cong M_2(\mathcal{O}(T))$,

$$\mathcal{Q}(E) \rightarrow M_2(\mathcal{O}(T))$$

$$c \mapsto \begin{bmatrix} c|_T & 0 \\ 0 & c|_T \end{bmatrix}$$

So $\text{Trd}_{\mathcal{A}}(c) = 0$.

Solution 2

$\text{Trd}_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{O}$ is a surjective map of sheaves. $\ker(\text{Trd}_{\mathcal{A}}) = \mathfrak{sl}_{\mathcal{A}}$.

Lemma

If $\deg(\mathcal{A}) \geq 6$ there is a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathfrak{sl}_{\mathcal{A}} & \longrightarrow & \mathcal{A} & \xrightarrow{\text{Trd}_{\mathcal{A}}} & \mathcal{O} \longrightarrow 0 \\
 & & \downarrow c & & \downarrow c & & \downarrow \exists \rho \\
 0 & \longrightarrow & \text{Alt}_{\mathcal{C}\ell, \underline{\sigma}} & \longrightarrow & \mathcal{C}\ell(\mathcal{A}, \sigma, f) & \longrightarrow & \mathcal{C}\ell(\mathcal{A}, \sigma, f) / \text{Alt}_{\mathcal{C}\ell, \underline{\sigma}} \longrightarrow 0 \\
 & & & & \downarrow \text{Id} + \underline{\sigma} & & \downarrow \pi \\
 & & & & \text{Symd}_{\mathcal{C}\ell, \underline{\sigma}} & \xlongequal{\quad\quad\quad} & \text{Symd}_{\mathcal{C}\ell, \underline{\sigma}}
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathcal{O} & \xrightarrow{\rho} & \mathcal{C}\ell(\mathcal{A}, \sigma, f) / \text{Alt}_{\mathcal{C}\ell, \underline{\sigma}} \xrightarrow{\pi} \text{Symd}_{\mathcal{C}\ell, \underline{\sigma}} \\
 a & \longmapsto & a \cdot 1_{\mathcal{C}\ell}
 \end{array}$$

The Canonical Pair $\deg \geq 8$

$$\rho: \mathcal{O} \rightarrow \mathcal{Cl}(\mathcal{A}, \sigma, f) / \text{Alt}_{\mathcal{Cl}, \underline{\sigma}}$$

Definition

Let (\mathcal{A}, σ, f) be such that $(\mathcal{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma})$ is orthogonal, $\deg(\mathcal{A}) \geq 8$. Call the semi-trace \underline{f} corresponding to $\rho(1)$ the canonical semi-trace.

Theorem

We have a commutative diagram

$$\begin{array}{ccc} \mathbf{PGO}(\mathcal{A}, \sigma, f) & \xrightarrow{\quad\quad\quad} & \mathbf{Aut}(\mathcal{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma}) \\ & \searrow & \nearrow \\ & \mathbf{PGO}(\mathcal{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma}, \underline{f}) & \end{array}$$

If $S = \text{Spec}(\mathbb{F})$ this recovers [DQ21].

Twisting

Let $\mathcal{V} = \text{Span}_{\mathcal{O}}(\{v_1, \dots, v_n\})$, $\mathbb{H}(\mathcal{V}) = \text{Span}_{\mathcal{O}}(\{v_1, \dots, v_n, v_n^*, \dots, v_1^*\})$
with hyperbolic quadratic form

$$q_{2n}(a_1 v_1 + \dots + a_n v_n + a_n^* v_n^* + \dots + a_1^* v_1^*) = \sum_{i=1}^n a_i a_i^*$$

$\rightarrow (\text{End}_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_{2n}, f_{2n})$

$\rightarrow \mathcal{C}_0 = (\text{Cl}(\text{End}_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_{2n}, f_{2n}), \underline{\sigma}_{2n}, \underline{f}_{2n})$.

For (\mathcal{A}, σ, f) of degree $2n$, let $\mathcal{P} = \text{Isom}((\text{End}_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_{2n}, f_{2n}), (\mathcal{A}, \sigma, f))$

$$(\text{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma}, \underline{f}) \cong \mathcal{P} \wedge^{\text{PGO}_{2n}} \mathcal{C}_0$$

No Canonical Pair when $\deg = 4$

Only excluded case when $\underline{\sigma}$ is orthogonal is $\deg(\mathcal{A}) = 4$ and $2 = 0 \in \mathcal{O}(S)$.

Theorem

Let (\mathcal{A}, σ, f) be degree 4 and $2 = 0 \in \mathcal{O}(S)$. Then there is no canonical semi-trace on $(\mathcal{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma})$. In particular, for all $f' : \text{Sym}_{\mathcal{Cl}, \underline{\sigma}} \rightarrow \mathcal{O}$,

$$\begin{array}{ccc} \mathbf{PGO}(\mathcal{A}, \sigma, f) & \xrightarrow{\hspace{15em}} & \mathbf{Aut}(\mathcal{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma}) \\ & \searrow \text{\scriptsize } \not\exists & \nearrow \\ & \mathbf{PGO}(\mathcal{Cl}(\mathcal{A}, \sigma, f), \underline{\sigma}, f') & \end{array}$$

Problem: $\text{Alt}_{\mathcal{Cl}(\text{End}_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \underline{\sigma}_4)}$ is too small.

Problem 3

Assume S affine. For $(\mathcal{Cl}(\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V}))), \sigma_4, f_4), \underline{\sigma}_4, f')$, $f' = \text{Trd}_{\mathcal{Cl}}(\ell \cdot _)$.
 $\varphi \in \mathbf{PGO}(\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V}))), \sigma_4, f_4)$ respects f' if and only if $\ell - \varphi(\ell) \in \mathcal{A}lt_{\mathcal{Cl}, \underline{\sigma}_4}$.

$$\Rightarrow \ell = 1 + e_1 e_1^* + e_2 e_2^*$$

but this is only fixed if $t^2 = t$ for all $t \in \mathcal{O}(S)$.

Example

For a boolean ring R and $(M_2(R), \sigma_4, f_4)$, there exists
 $\ell \in \text{Cl}(M_2(R), \sigma_4, f_4)$ such that

$$f' = \text{Trd}_{\mathcal{Cl}}(\ell \cdot _)$$

is a semi-trace for $\underline{\sigma}_4$ fixed by all $\varphi \in \mathbf{PGO}_4(R)$.

Thank You