Maximal stable quotients of invariant types in NIP theories

Krzysztof Krupiński
(joint work with Adrián Portillo)

Instytut Matematyczny
Uniwersytet Wrocławski

Banff
19-24 February, 2023

Stable hyperdefinable sets

Definition

An *(A-)hyperdefinable* set is a quotient X/E, where X is (A-)type-definable and E is an (A-)type-definable equivalence relation on X.

Definition

An A-hyperdefinable set X/E is *stable* if for every A-indiscernible sequence $(a_i, b_i)_{i<\omega}$ with $a_i \in X/E$ for all (equivalently, some) $i < \omega$, we have

$$\mathrm{tp}(a_i, b_j/A) = \mathrm{tp}(a_j, b_i/A)$$

for all (some) $i \neq j < \omega$.

Characterizations of stability

Let $\mathcal{F}_{X/E}$ be the family of all functions $f : X \times \mathfrak{C}^m \to \mathbb{R}$ which factor through $X/E \times \mathfrak{C}^m$ and can be extended to a CL-formula $\mathfrak{C}^\lambda \times \mathfrak{C}^m \to \mathbb{R}$ over $\emptyset$, where m ranges over ω.

Proposition

X/E is stable as a hyperdefinable set if and only if every $f \in \mathcal{F}_{X/E}$ is stable.

Using this and various results of Ben-Yaacov and Usvyatsov on stability in continuous logic, we deduced:

Characterizations of stability — cont.

Theorem

Let X/E be a 0-hyperdefinable set. The following conditions are equivalent:

1. X/E is stable.
2. $\forall\ M \models T\ \forall f \in \mathcal{F}_{X/E}\ \forall p \in S_f(M)$ (p is definable).
3. $\exists \lambda \geq |T|\ \forall M \models T\ (|M| \leq \lambda \implies |S_{X/E}(M)| \leq \lambda)$.
4. Any global invariant (over some A) type $p \in S_{X/E}(\mathfrak{C})$ is generically stable.

Moreover, they imply:

5. Any indiscernible sequence of elements of X/E is totally indiscernible.

And, under NIP, they are equivalent to (5).

Characterizations of stability — cont.

Using item (5) of the last theorem, we easily get an extension of a result of Onshuus and Peterzil to a hyperdefinable context.

Proposition

Assume NIP. An A-hyperdefinable set X/E is stable if and only if it is *weakly stable* in the sense that for every A-indiscernible sequence $(a_i, b_i, c)_{i<\omega}$ with $a_i, b_i \in X/E$ for all (equivalently, some) $i < \omega$, we have

$$\mathrm{tp}(a_i, b_j, c/A) = \mathrm{tp}(a_j, b_i, c/A)$$

for all (some) $i \neq j < \omega$.

Fact (Haskel and Pillay)

Stable hyperdefinable sets are closed under taking Cartesian products, type-definable quotients, and hyperdefinable subsets.

Maximal stable quotients of NIP groups

Theorem (Haskel and Pillay)

Let G be a group 0-type-definable in a NIP theory. Then there exists a smallest type-definable (over a small set of parameters) subgroup G^{st} of G with stable quotient G/G^{st}. Moreover, G^{st} is 0-type-definable and normal. Similarly, there is also a 0-$\bigwedge$-definable subgroup $G^{st,0}$ which is defined as the intersection of all relatively definable (with parameters) subgroups H of G such that G/H is stable.

Example

Consider a monster model K of ACVF, and $G := (V, +)$, where V is the valuation ring of K. Then $G^{st} = G^{st,0}$ is precisely the additive group of the maximal ideal of V, and G/G^{st} is the additive group of the residue field.

Maximal stable quotients of NIP groups — cont.

Assume NIP.

Definition

- A dense indiscernible sequence I is *distal* if for any distinct Dedekind cuts $\mathfrak{c}_1, \mathfrak{c}_2$, if a fills $\mathfrak{c}_1$ and b fills $\mathfrak{c}_2$, then $I \cup \{a\} \cup \{b\}$ is indiscernible.
- The theory T is *distal* if all dense indiscernible sequences (of tuples from the home sort) are distal.
- We say that T^{heq} *is distal* if all dense indiscernible sequences $(a_i/E)_{i \in \mathcal{I}}$ of hyperimaginaries (where E is 0-$\bigwedge$-definable) are distal.

Theorem

If $(a_i)_{i \in \mathcal{I}}$ is a (dense) distal sequence, then $(a_i/E)_{i \in \mathcal{I}}$ is a distal sequence of hyperimaginaries. Thus, if T is distal, then T^{heq} is distal.

Maximal stable quotients of NIP groups — cont.

Corollary

For a distal theory T, a hyperdefinable set X/E is stable if and only if E is a bounded equivalence relation. In particular, for a group G $\bigwedge$-definable in a distal theory, $G^{st} = G^{00}$.

Let $\mathcal{M} := (\mathbb{R}, +, I)$, where $I := [0, 1]$. Let $T := \mathrm{Th}(\mathcal{M})$. Let $\mathcal{N} := (\mathbb{R}, +, -, R_r)_{r \in \mathbb{N}^+}$, where $R_r(x, y)$ holds if and only if $0 \leq y - x \leq r$. Then $\mathcal{M}$ and $\mathcal{N}$ are interdefinable over $\emptyset$. Let $G := (\mathbb{R}, +)$, and G^* the interpretation of G in the monster model.

Proposition

G^{*st} is precisely the subgroup of all infinitesimals, and $G^* = G^{*st,0} \neq G^{*st} \neq G^{*00} = G^*$.

When we lack a group structure, a counterpart of quotienting by a subgroup is taking a quotient by an equivalence relation. However, a naive counterpart of Haskell and Pillay's theorem does not work, i.e. for any non-stable type-definable set X a finest type-definable equivalence relation E on X with stable quotient X/E does not exist. Indeed, suppose for a contradiction that it exists. By assumption, there is a non-trivial E-class $[a]_E$. Then $E \cap \equiv_a$ is a type-definable equivalence relation on X with stable quotient $X/(E \cap \equiv_a)$ and $E \cap \equiv_a$ is strictly finer than E, a contradiction.

The main theorem

$\mathfrak{C}$ always denotes a monster model of the theory in question.

Theorem

Assume NIP. Let $p(x) \in S(\mathfrak{C})$ be an A-invariant type. Assume that $\mathfrak{C}$ is at least $\beth_{(2^{2^{|T|+|A|+|x|}})^+}$-saturated. Then, there exists a finest equivalence relation E^{st} on $p(\mathfrak{C}')$ (the set of realizations of p in the bigger monster model $\mathfrak{C}'$) relatively type-definable over a small set of parameters of $\mathfrak{C}$ and with stable quotient $p(\mathfrak{C}')/E^{st}$.

Question

Is E^{st} type-definable over A?

If yes, we could drop the specific high degree of saturation assumption in the above theorem.

Examples

Example

Let $T := \mathrm{Th}((\mathbb{R}, R_r, f_s)_{r\in\mathbb{Q}^+, s\in\mathbb{Q}})$, where $f_s(x) := x + s$ and $R_r(x, y)$ holds if and only if $0 \leq y - x \leq r$.

Let $p \in S_x(\mathfrak{C})$ be the complete global type determined by

$$\bigwedge_{c\in\mathfrak{C}} \bigwedge_{n\in\omega} (\neg R_n(x, c) \wedge \neg R_n(c, x)).$$

Then E^{st} is the equivalence relation on $p(\mathfrak{C}')$ defined by

$$\bigwedge_{r\in\mathbb{Q}^+} (R_r(x, y) \vee R_r(y, x)).$$

Example

The same is true for $T := \mathrm{Th}((\mathbb{R}, +, -, 1, R_r(x, y))_{r\in\mathbb{Q}^+})$.

On the proof of the main theorem

The proof of the main theorem is a non-trivial adaptation of the proof of Haskel and Pillay's result on the existence of G^{st} (which in turn is based on the proof of the existence of G^{00}). The key new idea is to use relatively type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$ which were studied and used by Hrushovski, Pillay, and myself to prove G-compactness of all amenable theories; another point is an application of strong heir extensions.

Relatively type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$

Definition

By a *relatively type-definable* subset of $\mathrm{Aut}(\mathfrak{C})$, we mean a subset of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(a), b))\}$ for some partial type $\pi(x, y)$ (without parameters), where x and y are short tuples of variables, and a, b are corresponding tuples from $\mathfrak{C}$.

In particular, given a partial type $\pi(x, y, z)$ over the empty set and (short) tuples a, b, c in $\mathfrak{C}$ corresponding to x, y, z, respectively, we have a relatively type-definable subset of $\mathrm{Aut}(\mathfrak{C})$ of the form

$$A_{\pi(x;y,z),a,b,c} := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(a), b, c)\}.$$

When $|x| = |y|$,

$$A_{\pi,a,c} := A_{\pi(x;y,z),a,a,c}.$$

Some ingredients of the proof of the main theorem

From now on, $\mathfrak{C} \prec \mathfrak{C}'$ are monster model such that $\mathfrak{C}$ is small in $\mathfrak{C}'$ and $p \in S(\mathfrak{C})$ is a global A-invariant type, where $A \subseteq \mathfrak{C}$ is small. Relatively type-definable subsets of $\mathrm{Aut}(\mathfrak{C}')$ are used to prove:

Lemma 1

Let $a \in \mathfrak{C}'$ and $(a_i)_{i<\omega} \subseteq \mathfrak{C}'$ be such that $a_0 \underset{a}{\equiv} a_i$ for all $i < \omega$ and $a \models p\restriction_{a_{<\omega}}$. Let $\pi(x, y, z)$ be a partial type over the empty set such that for every $i < \omega$ the partial type $\pi(x, y, a_i)$ defines an equivalence relation on $p\restriction_{a_i}(\mathfrak{C}')$. Assume that there is a formula $\varphi(x, y, z)$ implied by $\pi(x, y, z)$ such that for every $i < \omega$

$$\bigcap_{i \neq j} \pi(\mathfrak{C}', \mathfrak{C}', a_j) \cap (p\restriction_{a_{<\omega}}(\mathfrak{C}'))^2 \not\subseteq \varphi(\mathfrak{C}', \mathfrak{C}', a_i).$$

Then, T has IP.

Some ingredients of the proof of the main theorem

Proof.

[On the proof of Lemma 1] Since $p|_{a_{<\omega}}$ is complete, the non-inlcusion in the lemma is equivalent to

$$\bigcap_{j\neq i} \pi(\mathfrak{C}', a, a_j) \cap p\restriction_{a_{<\omega}}(\mathfrak{C}') \not\subseteq \varphi(\mathfrak{C}', a, a_i),$$

which in turn can be expressed as

$$\mathrm{Aut}(\mathfrak{C}'/a_{<\omega}) \cap A_{\bigwedge_{j\neq i}\pi(x;y,z_j),a,a,(a_j)_{j\neq i}} \not\subseteq A_{\varphi(x;y,z_i),a,a,a_i}.$$

Then, we work with relatively type-definable subsets to get IP.

Some ingredients of the proof of the main theorem

On the proof of Lemma 1 — cont.

For every $i < \omega$, choose some

$$\sigma_i \in \mathrm{Aut}(\mathfrak{C}'/a_{<\omega}) \cap A_{\bigwedge_{j \neq i} \pi(x;y,z_j),a,a,(a_j)_{j \neq i}} \setminus A_{\varphi(x;y,z_i),a,a,a_i},$$

and let σ_I denote the composition $\prod_{i \in I} \sigma_i$, for any finite $I \subseteq \omega$.

We prove that there is a formula $\theta(x,y,z)$ implied by $\pi(x,y,z)$ such that for all $i < \omega$ the set $(\mathrm{Aut}(\mathfrak{C}'/a_i) \cap A_{\pi,a,a_i}) \cdot (\mathrm{Aut}(\mathfrak{C}'/a_i) \cap A_{\theta,a,a_i}) \cdot (\mathrm{Aut}(\mathfrak{C}'/a_i) \cap A_{\pi,a,a_i})$ is contained in $\mathrm{Aut}(\mathfrak{C}'/a_i) \cap A_{\varphi,a,a_i}$.

Claim For any finite $I \subseteq \omega$, $\models \theta(\sigma_I(a), a, a_i) \iff i \notin I$.

Thus, θ has IP. □

Some ingredients of the proof of the main theorem

Lemma 2

Assume NIP. Let $p(x) \in S(\mathfrak{C})$ be an A-invariant type, let $\pi(x, y, z)$ be a partial type over the empty set, and let $a_0 \subseteq \mathfrak{C}'$ be such that $\pi(x, y, a_0)$ defines an equivalence relation on $p\restriction_{a_0}(\mathfrak{C}')$. Then, for any $(a_i)_{i<\lambda}$, where $\lambda \geq \beth_{(2^{(|a_0|+|x|+|T|+|A|)})^+}$, satisfying $a_i \underset{A}{\equiv} a_0$ for all $i < \lambda$, there exists $i < \lambda$ such that

$$\bigcap_{j \neq i} \pi(\mathfrak{C}', \mathfrak{C}', a_j) \cap (p\restriction_{a_{i<\lambda}}(\mathfrak{C}'))^2 \subseteq \pi(\mathfrak{C}', \mathfrak{C}', a_i).$$

Some ingredients of the proof of the main theorem

On the proof of Lemma 2.

Suppose for a contradiction that it does not hold, and choose pairs $(b_i, c_i)_{i<\lambda}$ witnessing it. Extract an A-indiscernible sequence $(a'_i, b'_i, c'_i)_{i<\omega}$ from $(a_i, b_i, c_i)_{i<\lambda}$. Then

$$(b'_i, c'_i) \in \bigcap_{j \neq i} \pi(\mathfrak{C}', \mathfrak{C}', a'_j) \cap (p\restriction_{a'_{i<\omega}}(\mathfrak{C}'))^2,$$

and there is $\varphi(x, y, z)$ implied by $\pi(x, y, z)$ with $\models \neg\varphi(b'_i, c'_i, a'_i)$. Pick any $a \models p|_{a'_{<\lambda}}$. Since p is A-invariant and $a'_i \equiv_A a'_j$, we get $a'_i \equiv_a a'_j$. Then the sequence $(a'_i)_{i<\omega}$ together with a, $\pi(x, y, z)$, and $\varphi(x, y, z)$ satisfy the assumptions of Lemma 1, so we get IP. □

Some ingredients of the proof of the main theorem

Let $\nu := \beth_{(2^{2^{|T|+|A|+|x|}})^+}$.

Claim

If for every countable partial type $\pi(x, y, z)$ over the empty set and countable tuple a_0 from $\mathfrak{C}$ such that $\pi(x, y, a_0)$ defines an equivalence relation E_{a_0} on $p(\mathfrak{C}')$ with stable quotient there is no sequence $(a_i)_{i<\nu}$ of (countable) tuples a_i in $\mathfrak{C}$ such that for all $i < \nu$ we have $a_i \underset{A}{\equiv} a_0$ and $\bigcap_{j<i} E_{a_j} \not\subseteq E_{a_i}$, then the theorem holds.

Proof of Claim.

Let E_i, $i \in I$, be all relatively type-definable over small subsets of $\mathfrak{C}$ equivalence relations on $p(\mathfrak{C}')$ with stable $p(\mathfrak{C}')/E_i$. Each $E_i = \bigcap_{j \in I_j} E_i^j$, where $E_i^j := \pi_i^j(\mathfrak{C}', \mathfrak{C}', a_i^j) \cap p(\mathfrak{C}')^2$ for some countable $\pi_i^j(x, y, z)$ and a_i^j. Since the number of possible π_i^j's and $\text{tp}(a_i^j/A)$'s is $\leq 2^{|T|+|A|+|x|} < \text{cf}(\nu)$, by assumption, we get that $\bigcap_{i \in I} E_i$ is an intersection of $< \nu$ relations of the form E_i^j. □

Some ingredients of the proof of the main theorem

On the proof of the main theorem.

Suppose it fails. By the claim, there exists a countable type $\pi(x, y, z)$ over $\emptyset$ and a countable tuple a_0 in $\mathfrak{C}$ such that $\pi(x, y, a_0)$ defines an equivalence relation on $p(\mathfrak{C}')$ with $p(\mathfrak{C}') \big/ \pi(\mathfrak{C}', \mathfrak{C}', a_0) \cap p(\mathfrak{C}')^2$ stable and there is $(a_i)_{i<\nu} \subseteq \mathfrak{C}$ such that for all $i < \nu$, $a_i \equiv_A a_0$ and $\bigcap_{j<i} \pi(\mathfrak{C}', \mathfrak{C}', a_j) \cap p(\mathfrak{C}')^2 \not\subseteq \pi(\mathfrak{C}', \mathfrak{C}', a_i)$.

Enlarging a_0, we can assume that a_0 enumerates an $\aleph_0$-saturated model in L_A of size at most $2^{|T|+|A|}$ and $\pi(x, y, a_0)$ defines an equivalence relation on $p\restriction_{a_0}(\mathfrak{C}')$. Using strong heirs, this relation also yields stable quotient.

Then we use extracting indiscernibles, Lemma 2, and stability of the quotient to produce a data satisfying the assumption of Lemma 1, which yields IP. □