A $\frac{4}{3}$-Approximation Algorithm for Half-Integral Cycle Cut Instances of the TSP

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Joint work with

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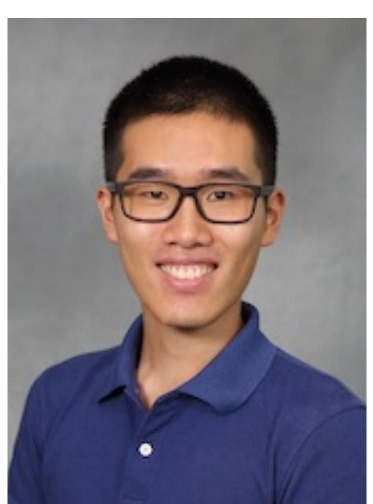
Slides by Billy



BIRS 2023

20 Sept

Outline of This Talk

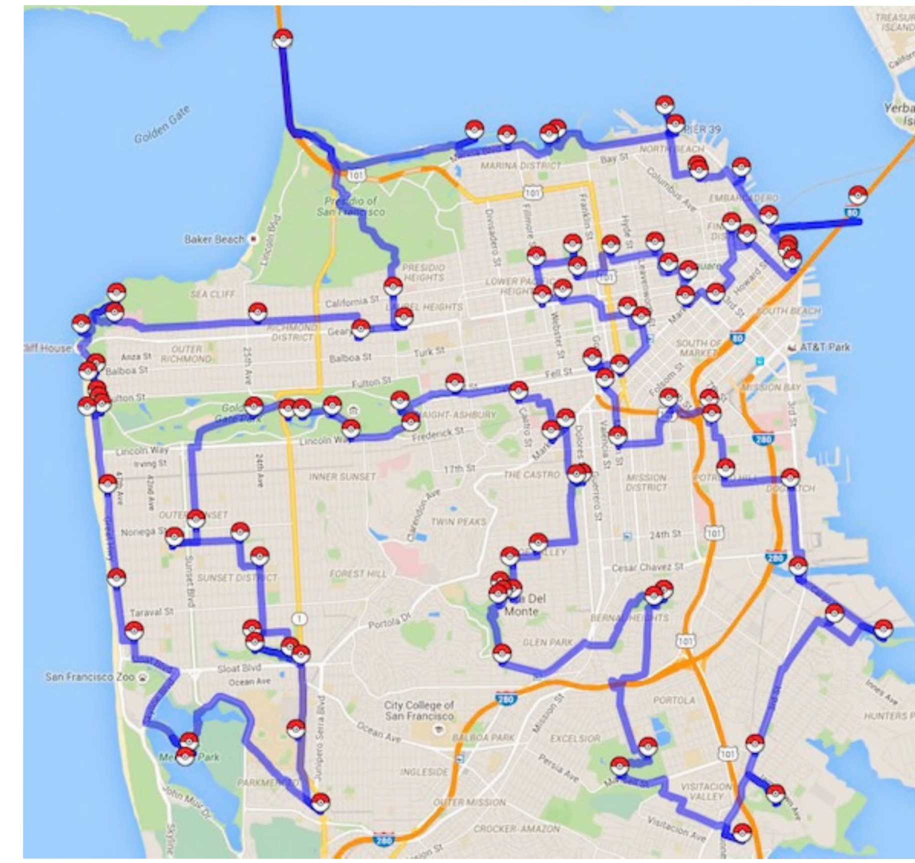
1. TSP preliminaries

2. What is the class of instances we study and why are they interesting?

3. Sketch of the approximation algorithm

Traveling Salesman Problem (TSP)

<u>Input</u>: Complete graph $G=(V,E)$ with edge costs $(c_e : e \in E)$ satisfying triangle inequality.



99 Pokestops in SF
(credit: Bill Cook)

<u>Output</u>: Minimum-cost Hamiltonian cycle.

- One of the most basic examples of a vehicle routing problem
- NP-hard [Karp '72]

What is known about approximation algorithms for the TSP?

For 45 years, best-known approximation was 1.5. [Christofides '76, Serdyukov '78]

Recent breakthrough reduced this to $\approx 1.5 - 10^{-36}$ [Karlin, Klein, Oveis Gharan '21]

NP-hard to approximate within a factor of $\frac{123}{122}$ [Karpinski, Lampis, Schmied '13]

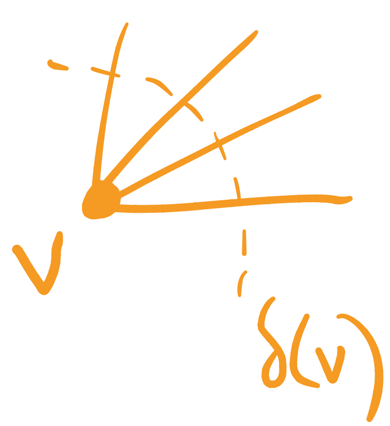
Subtour LP

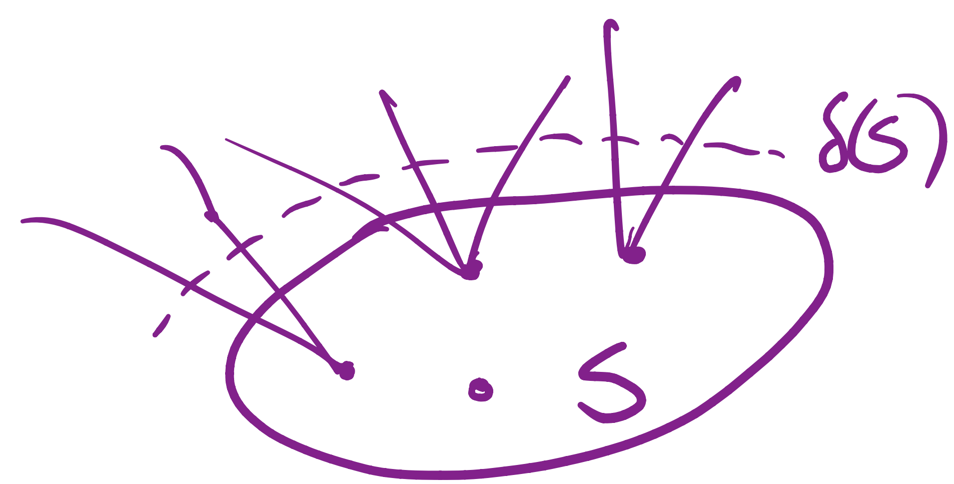
Dantzig, Fulkerson, Johnson '54

Held, Karp '71

$$\min \sum_e c_e x_e$$

$$\begin{aligned}
\text{s.t.}\quad x(\delta(v)) &= 2 && \forall\, v \in V \\
x(\delta(S)) &\geq 2 && \forall\, \emptyset \subsetneq S \subsetneq V \\
x_e &\geq 0 && \forall\, e \in E
\end{aligned}$$





S is <u>tight</u> if $x(\delta(S)) = 2$

Subtour LP

Dantzig, Fulkerson, Johnson '54
Held, Karp '71

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$$x(\delta(S)) \geq 2 \quad \forall \emptyset \subsetneq S \subsetneq V$$

$$x_e \geq 0 \quad \forall e \in E$$

Clearly, $LP(G) \leq OPT(G)$ $\forall$ graphs G.

$\therefore$ To get a bound against OPT, it suffices to bound against LP.

i.e. $ALG(G) \leq \alpha \cdot LP(G)$ $\forall G$ $\Rightarrow$ $ALG(G) \leq \alpha \cdot OPT(G)$ $\forall G$.

How different can LP be from OPT?

Def. Integrality gap is $\sup_G \frac{OPT(G)}{LP(G)}$.

Subtour LP

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Integrality gap of subtour LP is

- $\leq \frac{3}{2}$ [Wolsey '80]
- $\leq \frac{3}{2} - \varepsilon$ [Karlin, Klein, Oveis Gharan '22]
- $\geq \frac{4}{3}$ [folklore]

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$\frac{4}{3}$-conjecture: The integrality gap of the subtour LP is $\frac{4}{3}$.

We prove the $\frac{4}{3}$-conjecture for a class of TSP instances.

Our Result

> The $\frac{4}{3}$-conjecture holds for half-integral cycle cut instances of the TSP.

<u>Half-integral</u>: Solution to LP has $x_e \in \{0, \frac{1}{2}, 1\}$ $\forall e \in E$.

<u>Cycle cut instance</u>: $\underbrace{\text{Tight cuts}}_{S:\, x(\delta(S))=2}$ have a specific structure.

> These capture all known worst-case instances for the $4/3$-conjecture.

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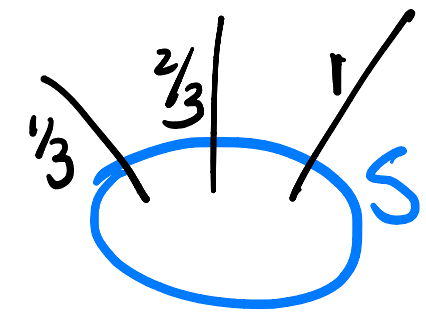
Why are half-integral instances interesting?

- Conjecture. [Schalekamp, W, van Zuylen '14]
  Half-integral instances are the worst-case instances for the integrality gap.

- Current-best approximation for TSP $(1.5-\varepsilon)$ [Karlin, Klein, Oveis Gharan '21]
  built on ideas from half-integral case [KKO '20]

- Currently, best approximation for half-integral TSP is 1.4983
  [Gupta, Lee, Li, Mucha, Newman, Sarkar '22]

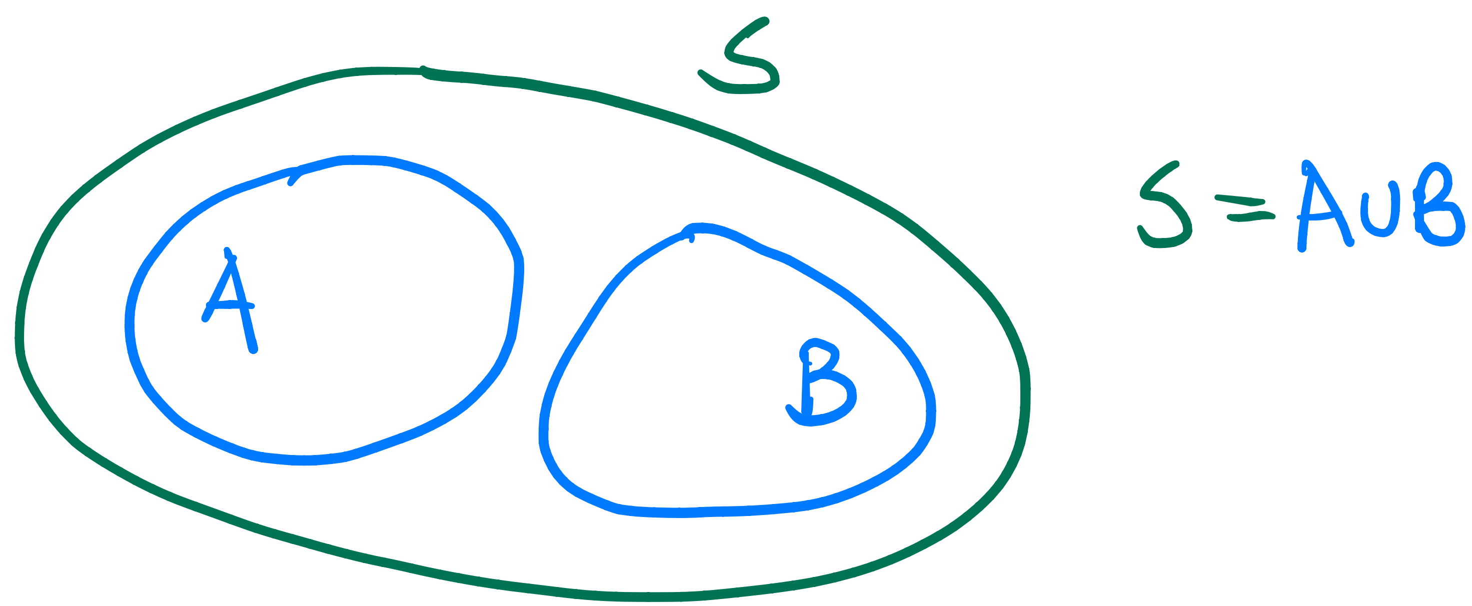
What are cycle cut instances?



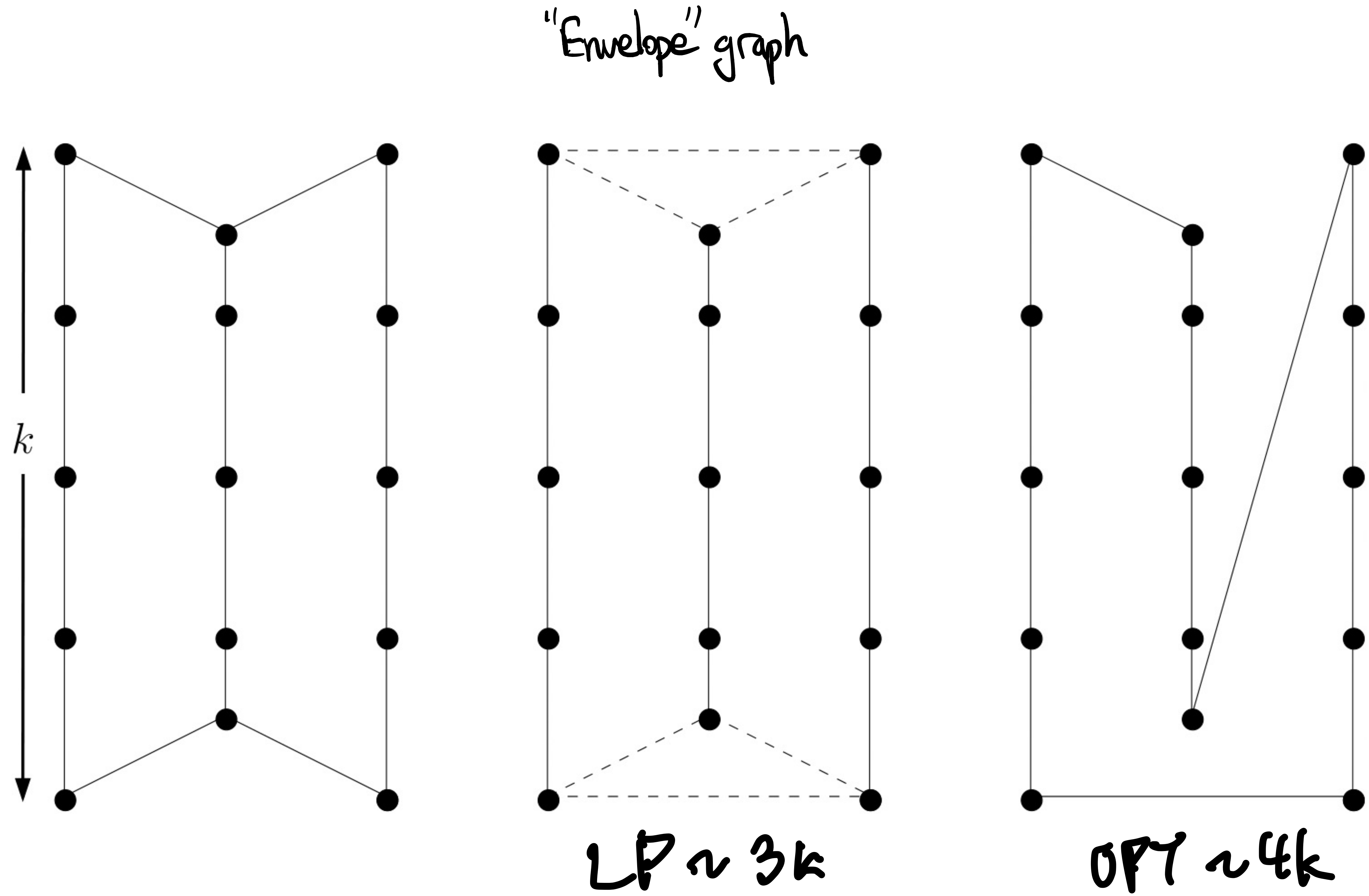
- A tight cut is $S \subseteq V$ s.t. $x(\delta(S)) = 2$
- A cycle cut instance is one where

  $\forall$ tight cuts S with $|S| \geq 2$,

  $\exists$ tight cuts $A, B \neq S$ s.t. $A \cup B = S$.

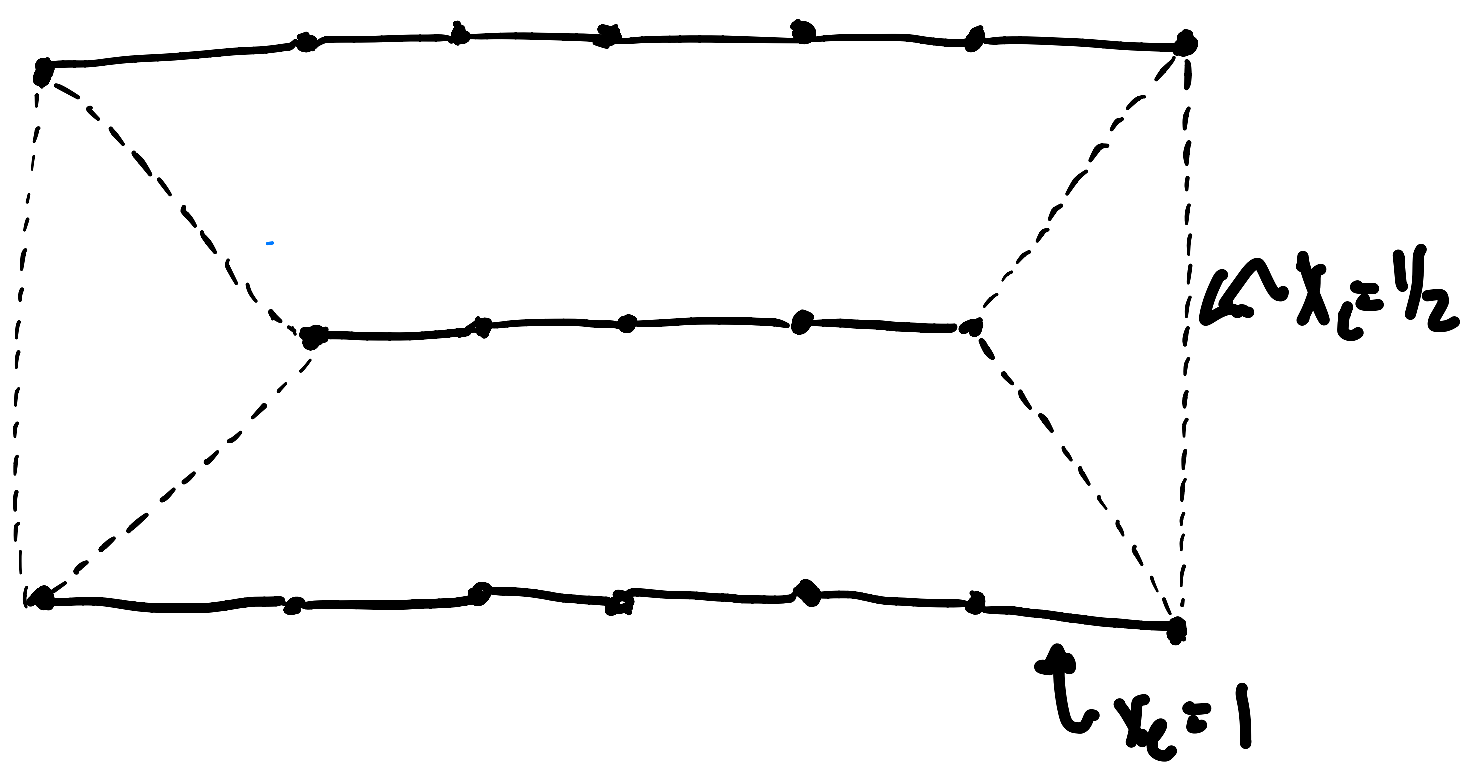


Half-integral cycle cut instances capture the known cases where the 4/3-conjecture is tight

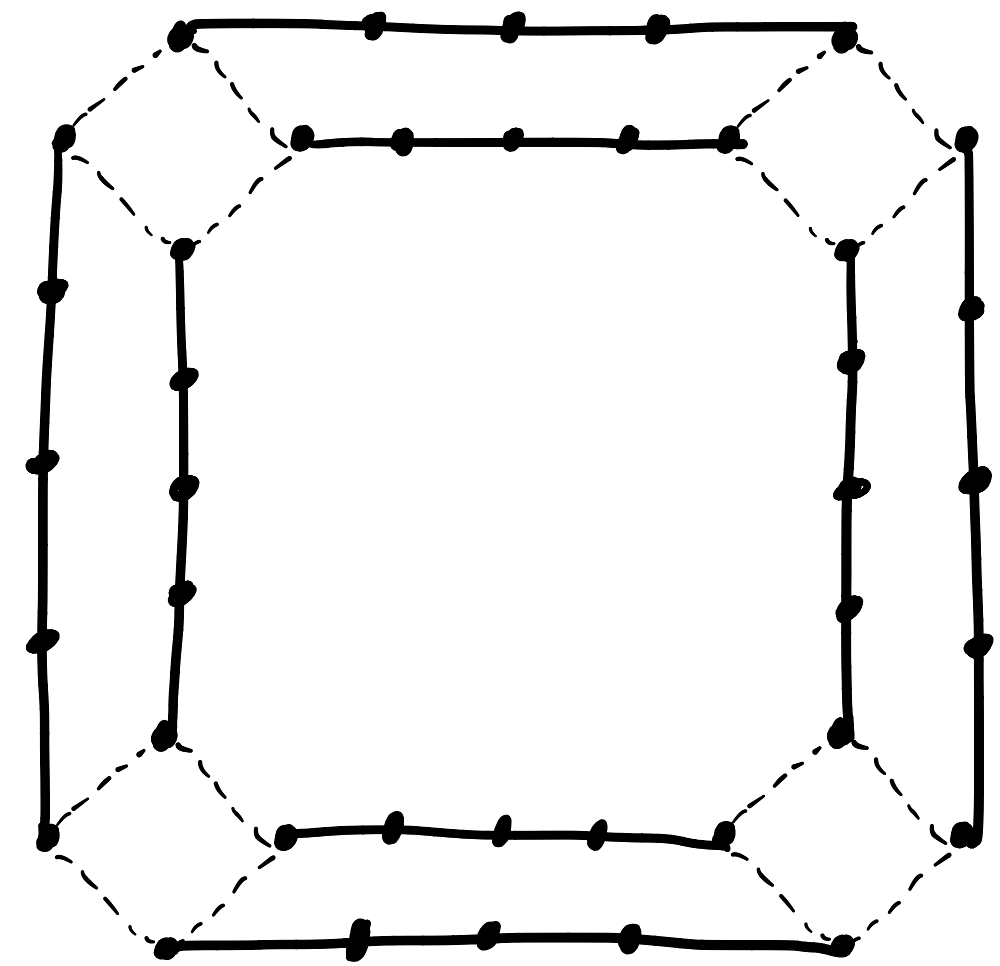


Half-integral cycle cut instances capture the known cases where the 4/3-conjecture is tight

"Envelope" graph



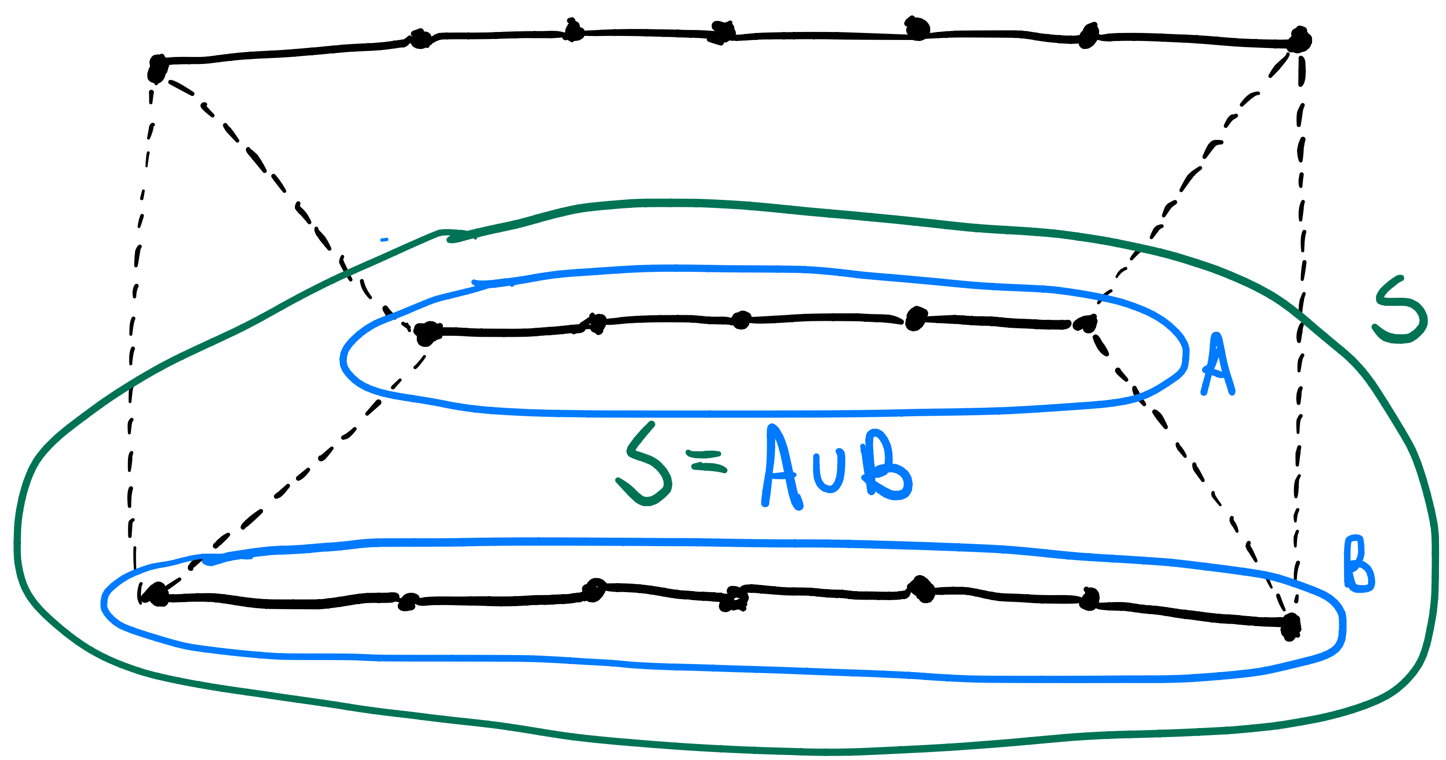
K-donuts [Boyd, Sebő '17]



K-donut where k=4

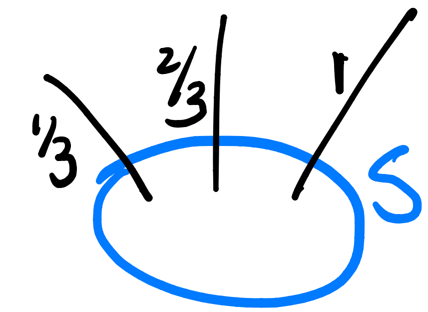
Half-integral cycle cut instances capture the known cases where the 4/3-conjecture is tight

"Envelope" graph

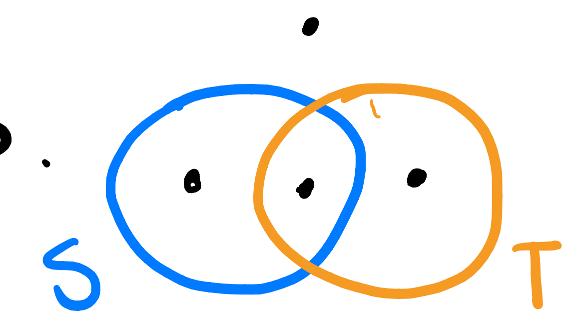


A more useful view of cycle cut instances

- A tight cut is $S \subseteq V$ s.t. $x(\delta(S)) = 2$



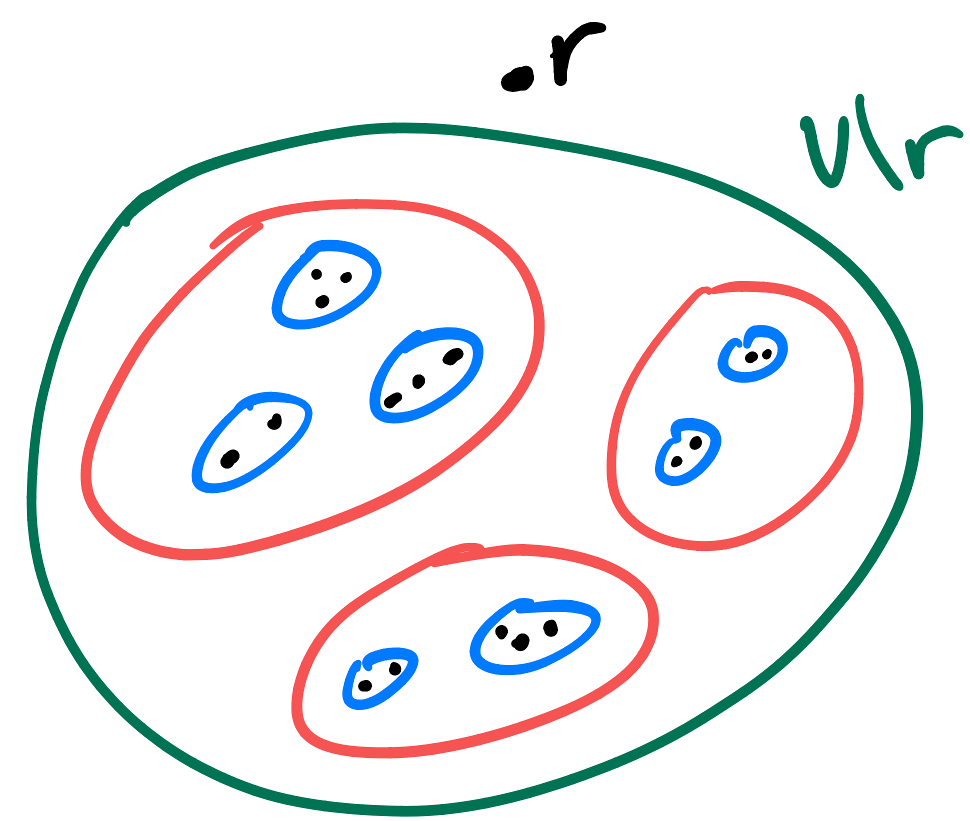
- Two cuts $S, T \subseteq V$ cross if $S \cap T, \bar{S} \cap T, S \cap \bar{T}, \bar{S} \cap \bar{T} \neq \phi$.



- A critical cut is a tight cut that does not cross any other tight cut.
- Fix arbitrary root vertex $r \in V$.
- Define hierarchy $\mathcal{H} = \{S \subseteq V \setminus r : S \text{ is a critical cut}\}$.

A more useful view of cycle cut instances

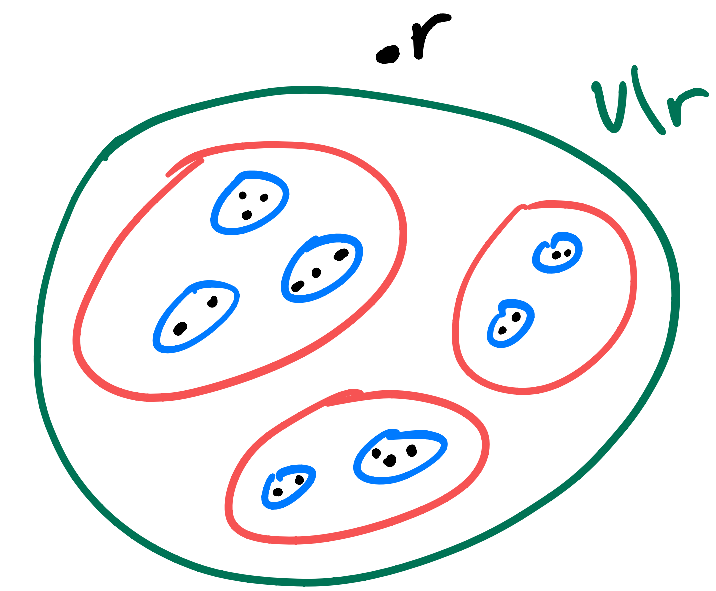
- Define hierarchy $\mathcal{H} = \{S \subseteq V \setminus r : S \text{ is a } \underbrace{\text{critical cut}}_{\text{tight cut that does not cross any other tight cut}}\}$.
- $\mathcal{H}$ is a laminar family
  - Topmost element of $\mathcal{H}$ is $V \setminus r$
  - Bottommost elements are singleton vertices in $V \setminus r$.
- $S \in \mathcal{H}$ is a cycle cut if
  (1) $|S| \geq 2$
  (2) After contracting $V \setminus S$ and the children of S, resulting graph is a cycle.



Hierarchy of critical cuts

A more useful view of cycle cut instances

- $\mathcal{H} = \{S \subseteq V \setminus r : S \text{ is a critical cut}\}$



Hierarchy of critical cuts

- $S \in \mathcal{H}$ is a **cycle cut** if
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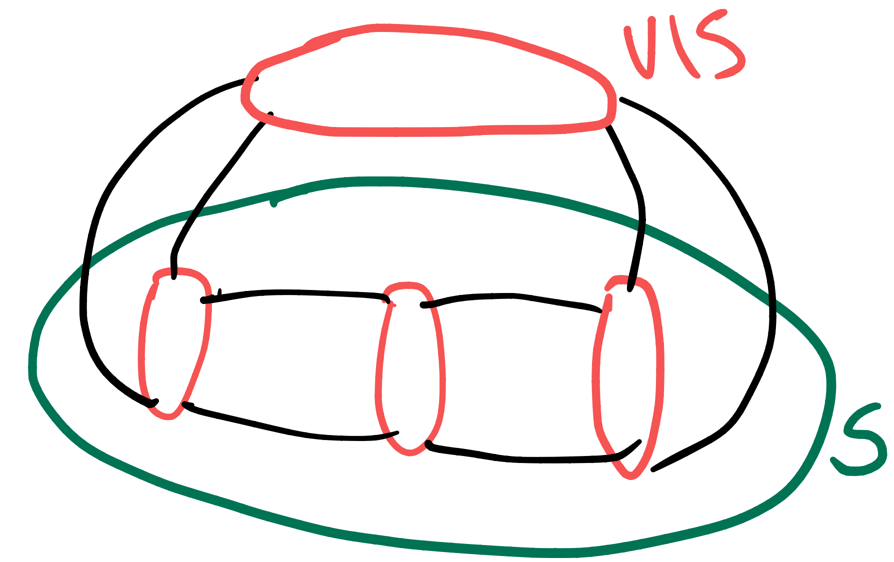
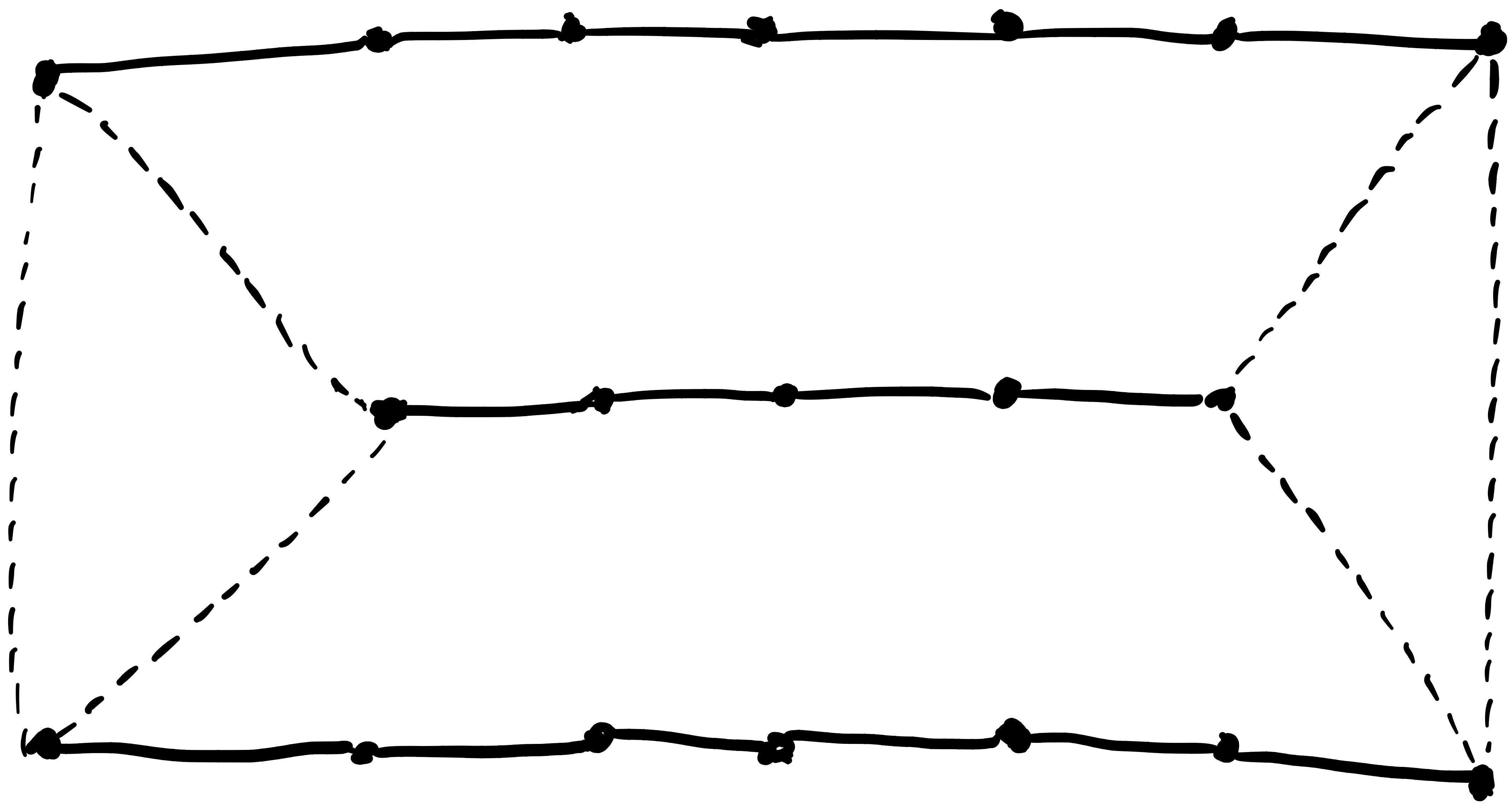


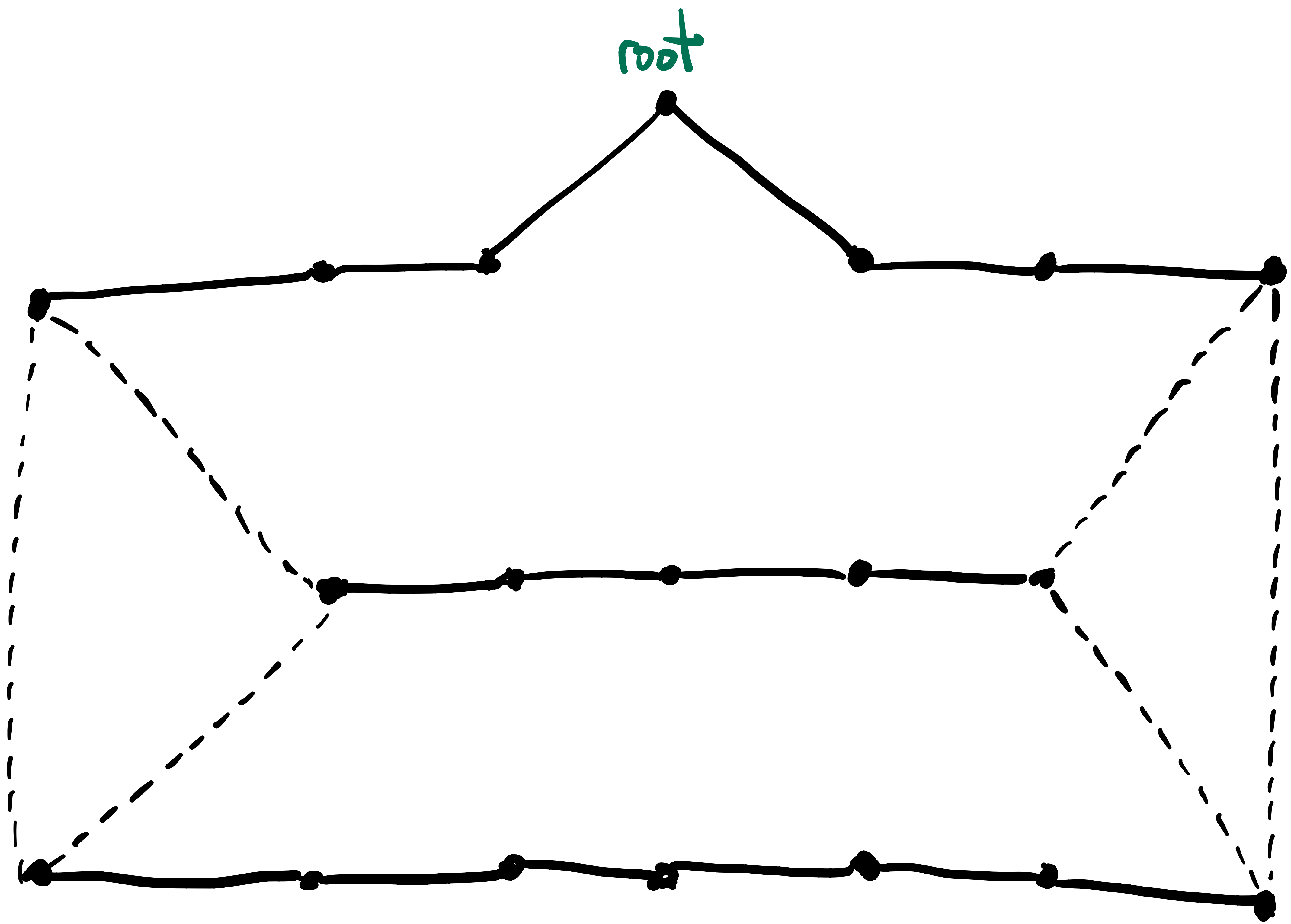
A cycle cut S

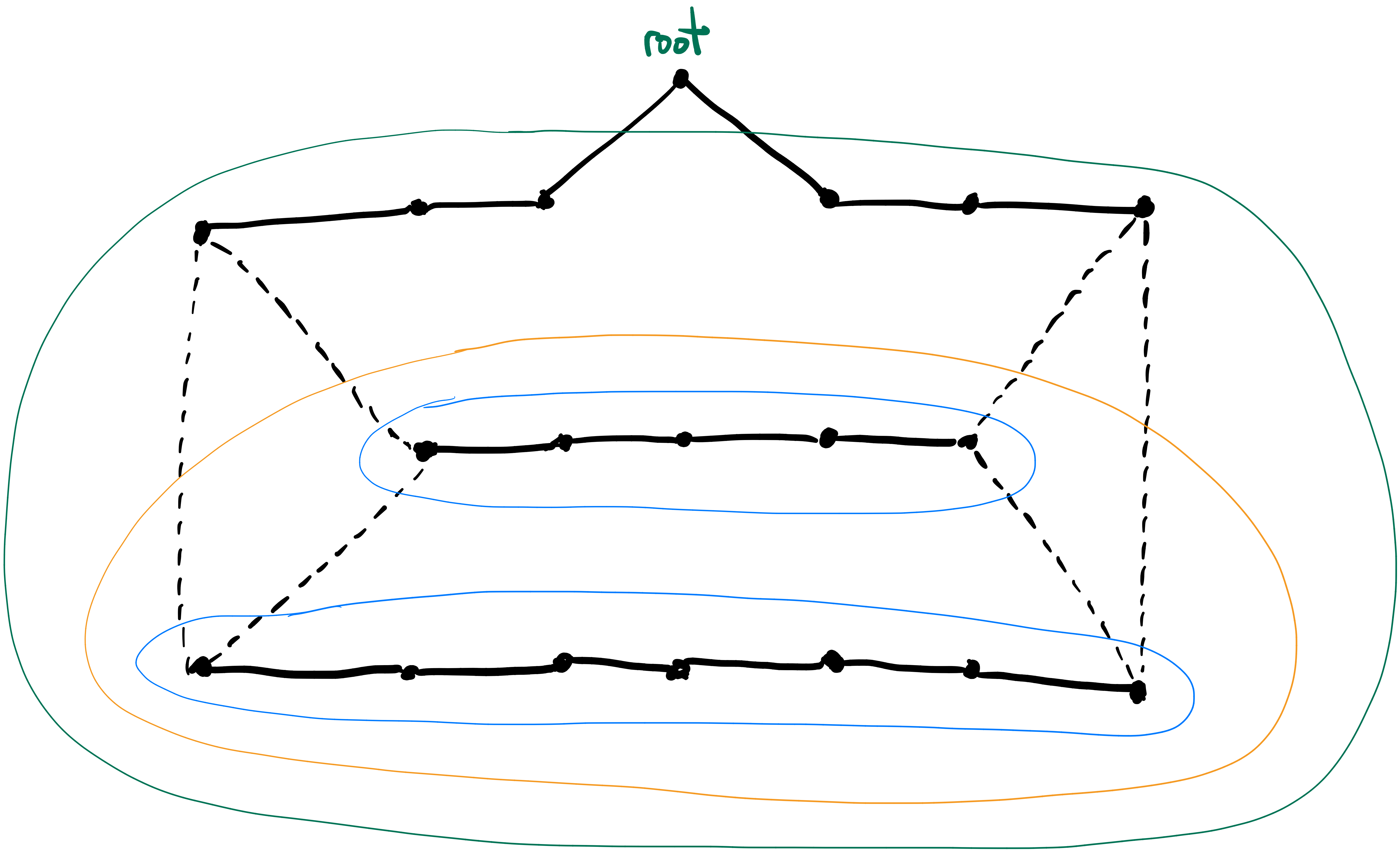
<u>Fact</u>. If G is a cycle cut instance, all cuts in the hierarchy are cycle cuts (for any choice of r).

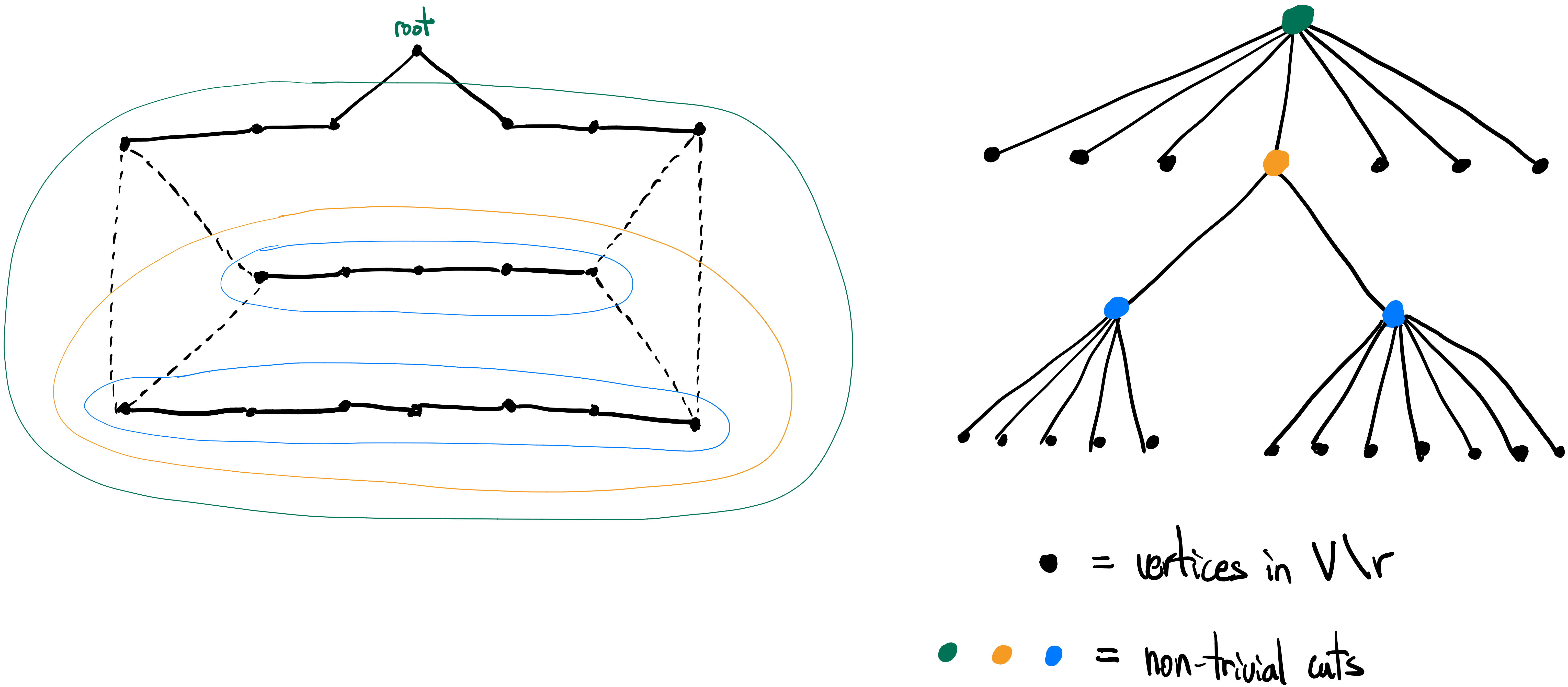
<u>Fact</u>. If for some choice of r, $\mathcal{H}$ consists only of cycle cuts, G is a cycle cut instance.

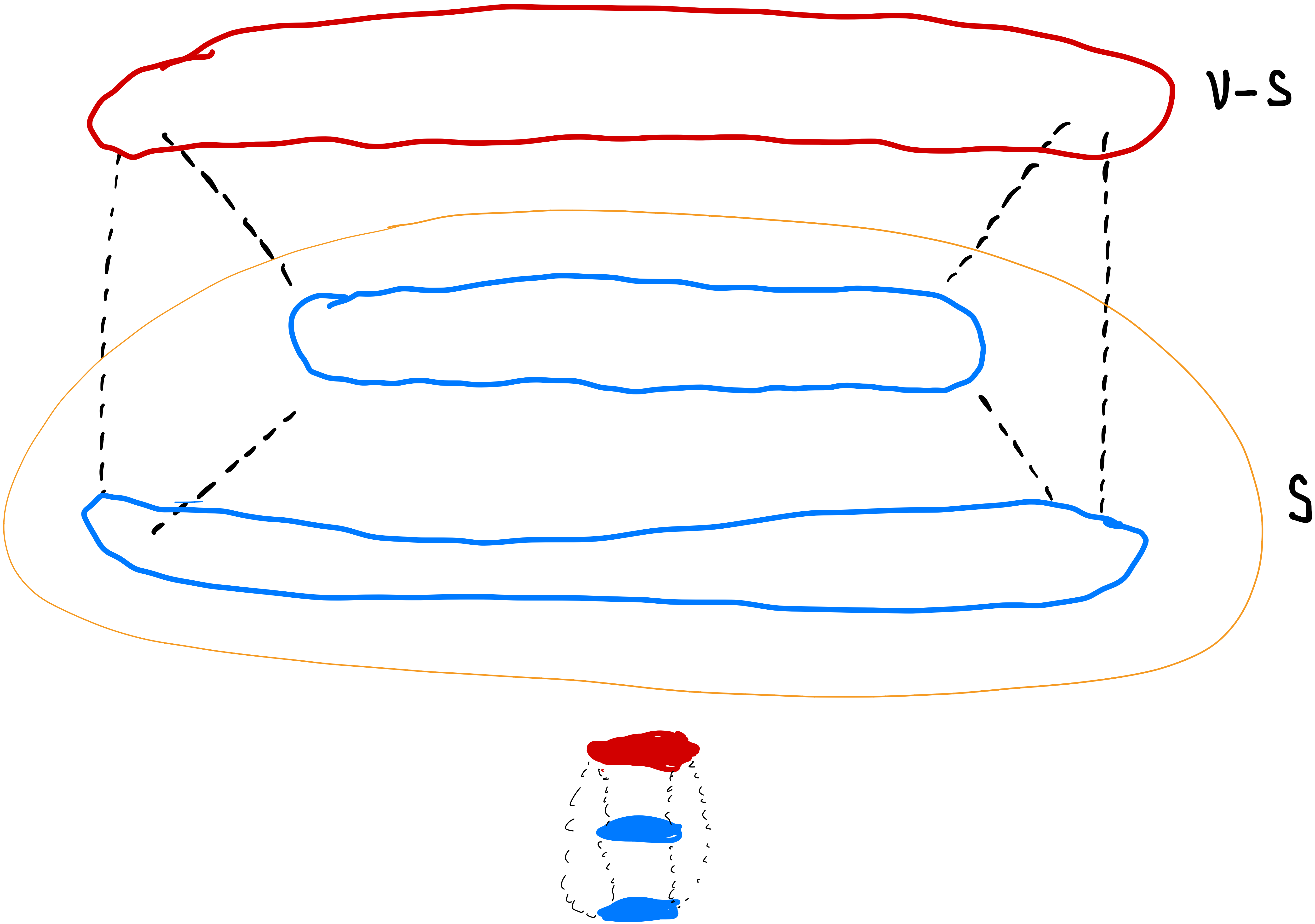
Illustration of Hierarchy




root


root


root
= vertices in V\r
= non-trivial cuts


$V-S$
S

To sum up,

a half-integral cycle cut instance of the TSP

is one where

① Solution x to subtour LP has $x_e \in \{0, \frac{1}{2}, 1\}$ $\forall$ edges e

② All cuts in the hierarchy are cycle cuts.

All known hard instances for the $\frac{4}{3}$-conjecture are half-integral cycle cut instances.

Our result is ...

An algorithm that outputs a tour T with

$$\mathbb{E}[\mathrm{cost}(T)]^{*} \leq \frac{4}{3} \sum_e c_e x_e$$

for any half-integral cycle cut instance of the TSP.

* $\mathbb{E}$ over randomness in algorithm.

Can be derandomized.

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Our Approach

- Triangle inequality $\Rightarrow$ it suffices to find Eulerian tour T s.t. $\text{cost}(T) \le \frac{4}{3} \cdot LP$.

  connected, every vertex even degree

- We'll construct a distribution of Eulerian tours such that each edge e is used at most $\frac{4}{3} x_e$ of the time in expectation

- Sampling from this distribution gives the result

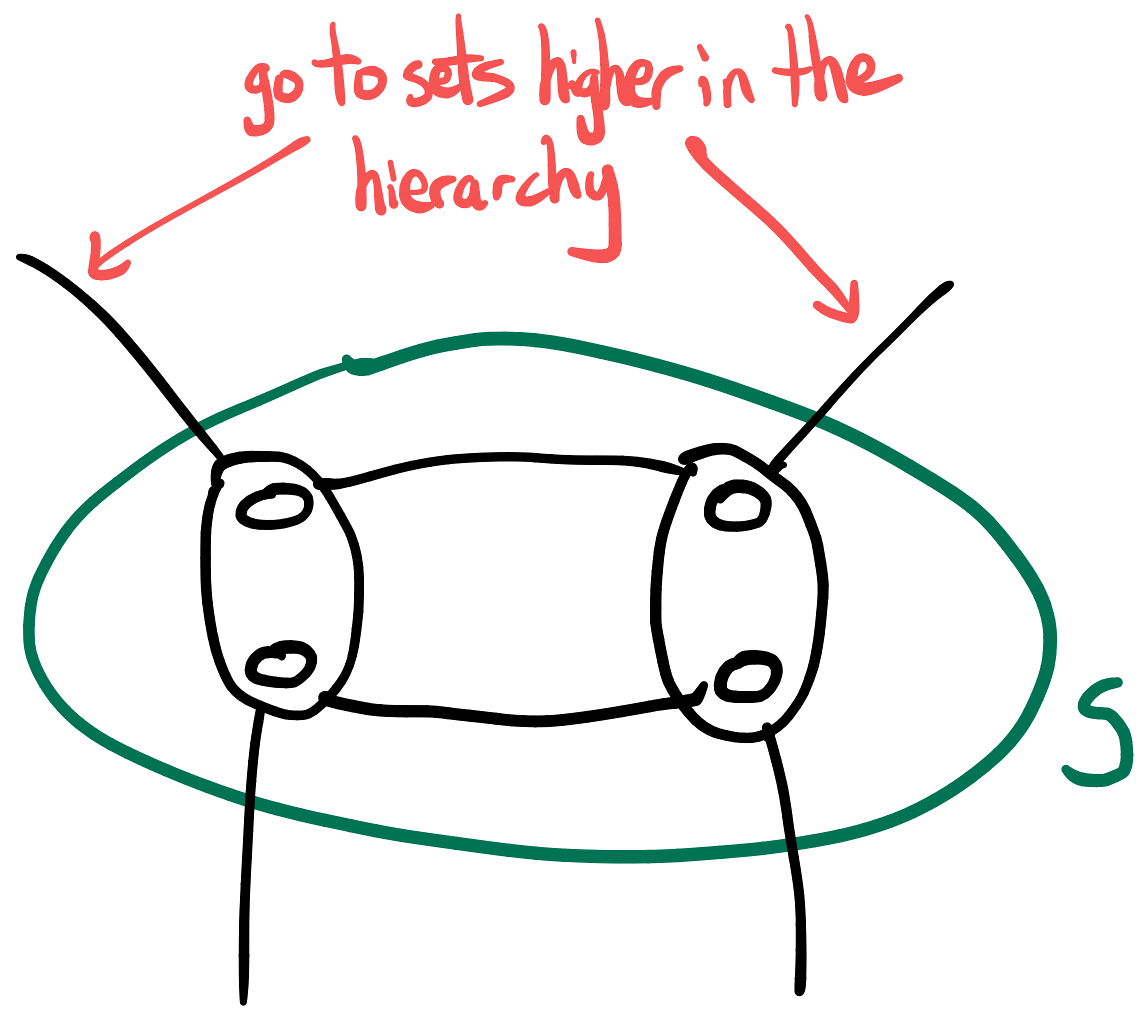
- Work on the hierarchy top-down

- Inductively specify the distribution of edges entering each cut

- Give rules for how to connect children given edges entering parent

Proof Sketch

- Simplifying assumptions: (1) Each $S \in \mathcal{H}$ has exactly 2 children

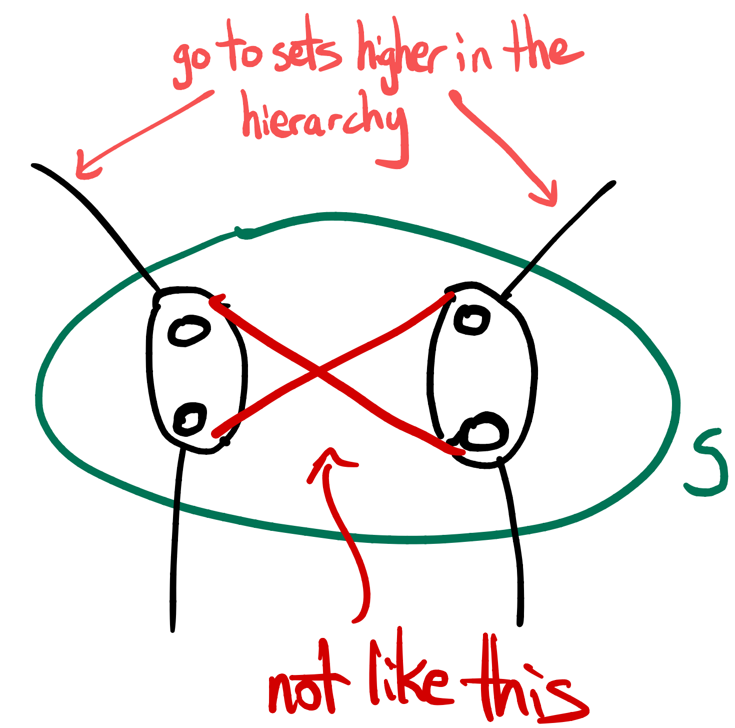


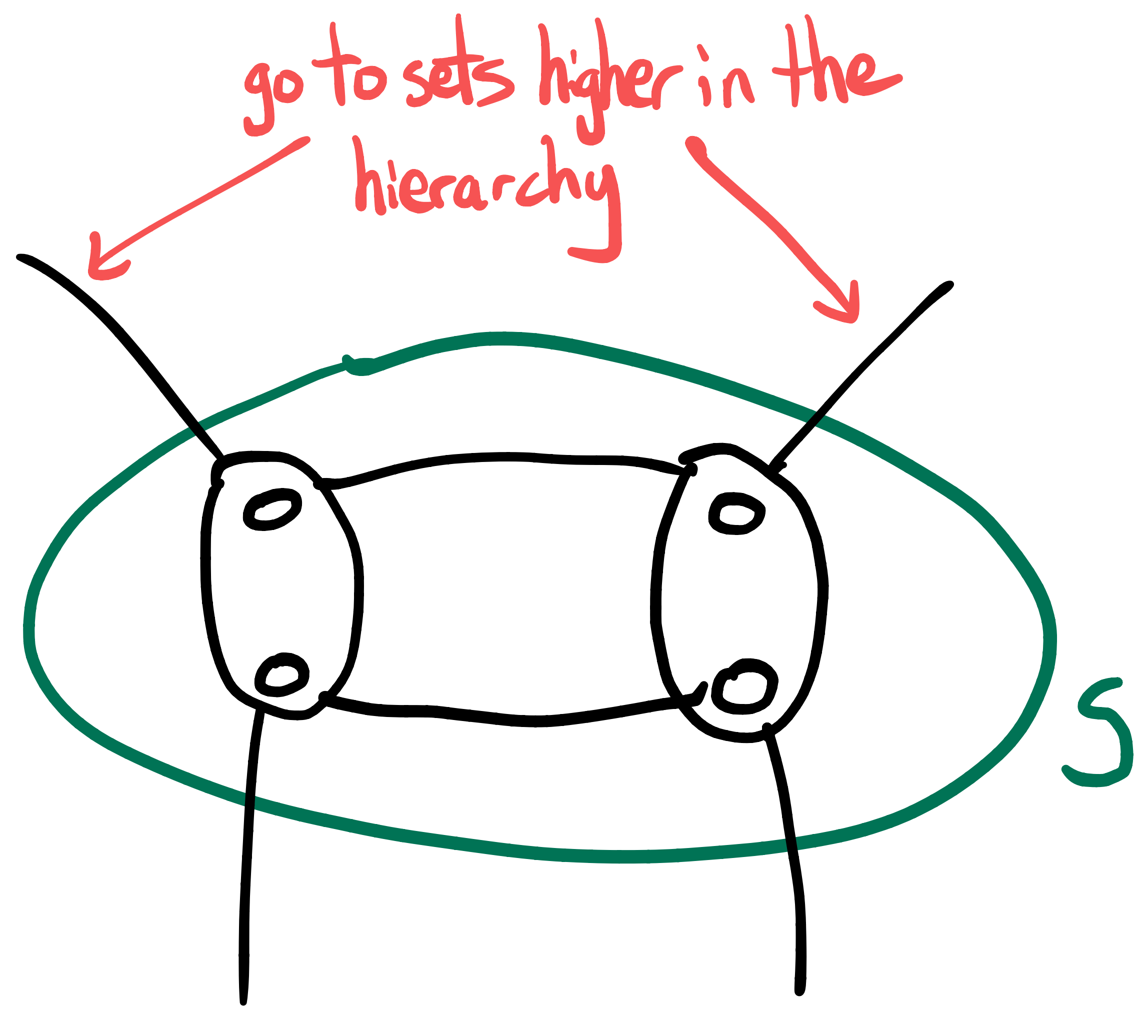
— : Edge with $x_e = \frac{1}{2}$

Proof Sketch

- Simplifying assumptions: (1) Each $S \in \mathcal{H}$ has exactly 2 children,

  (2) Edges in S are "straight"





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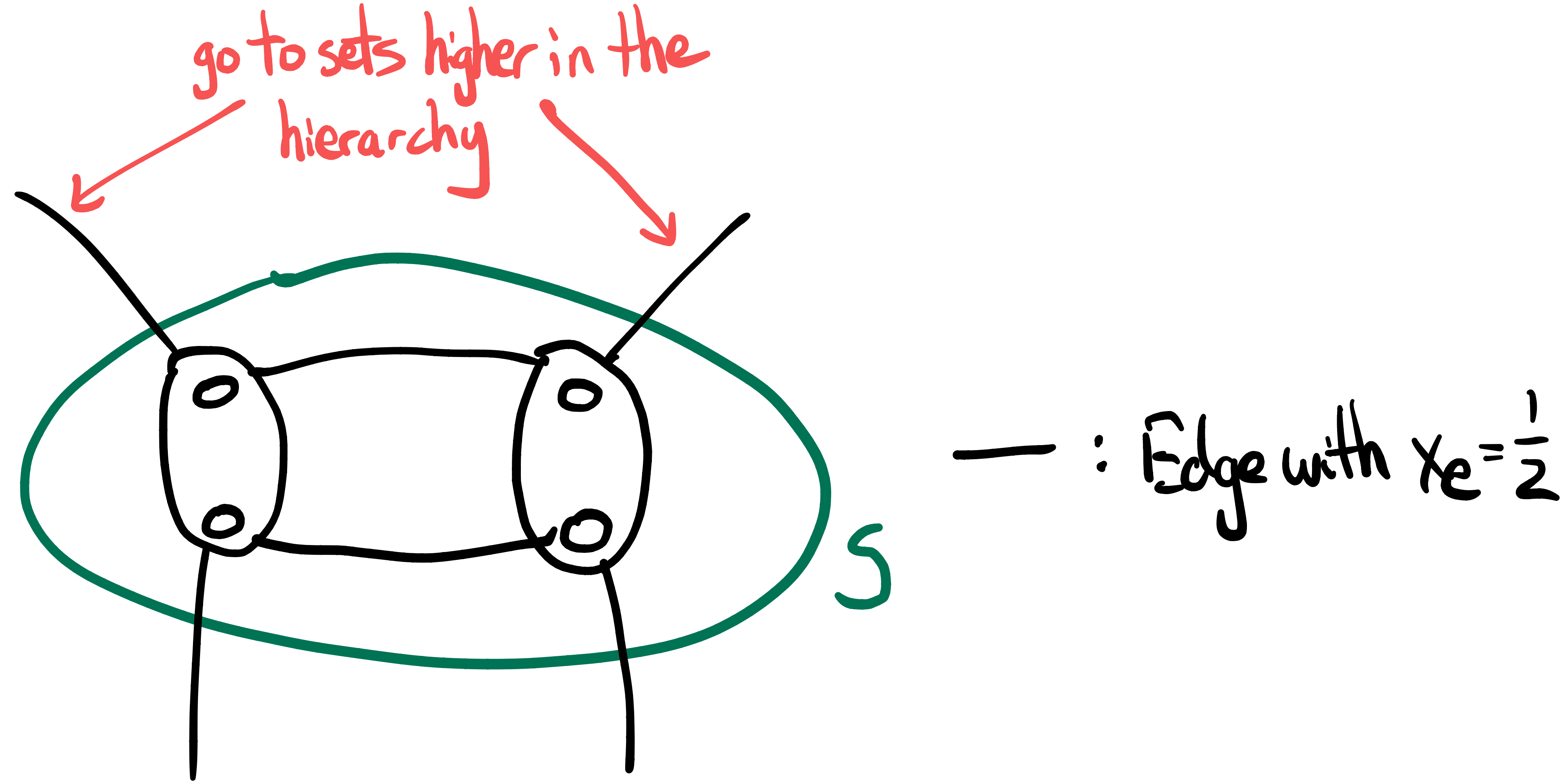
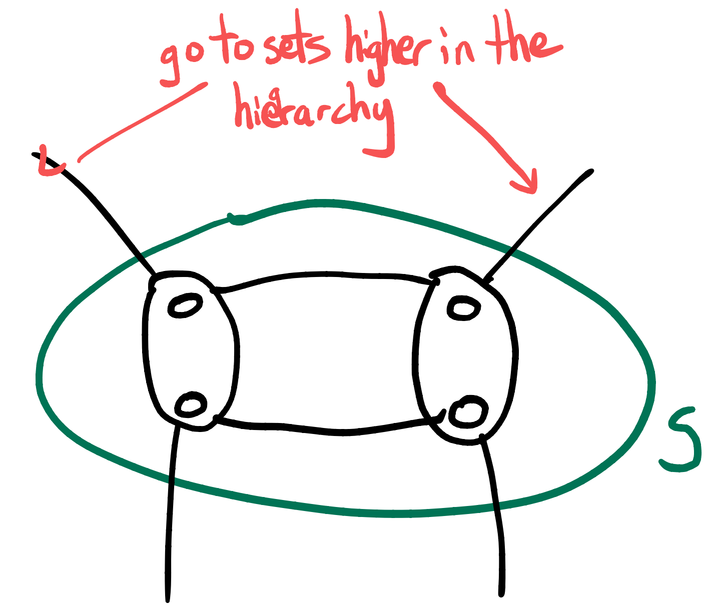
Proof Sketch

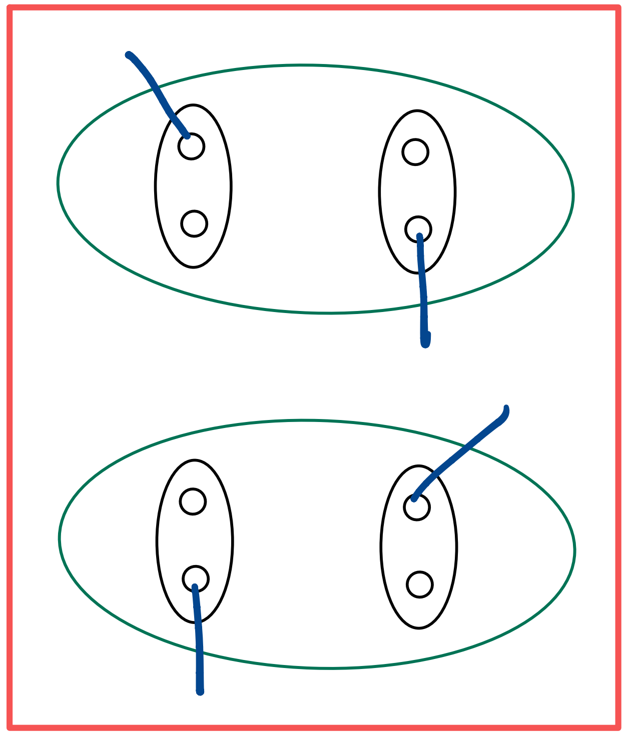


- For Eulerian tour, need to select an even # of edges entering each set
- Take 0, 1, or 2 copies of each edge
- Focus on edges with 1 copy and group by type

The Four States



1



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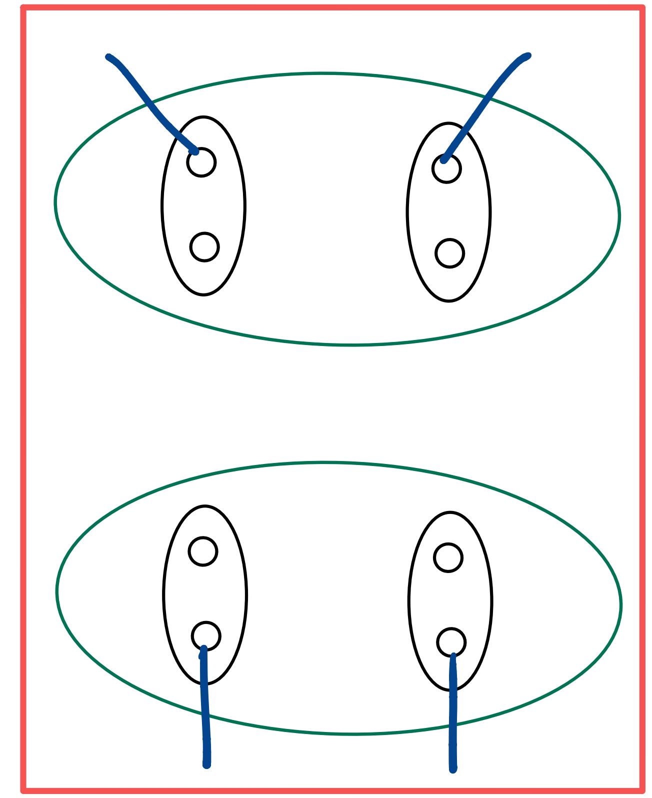

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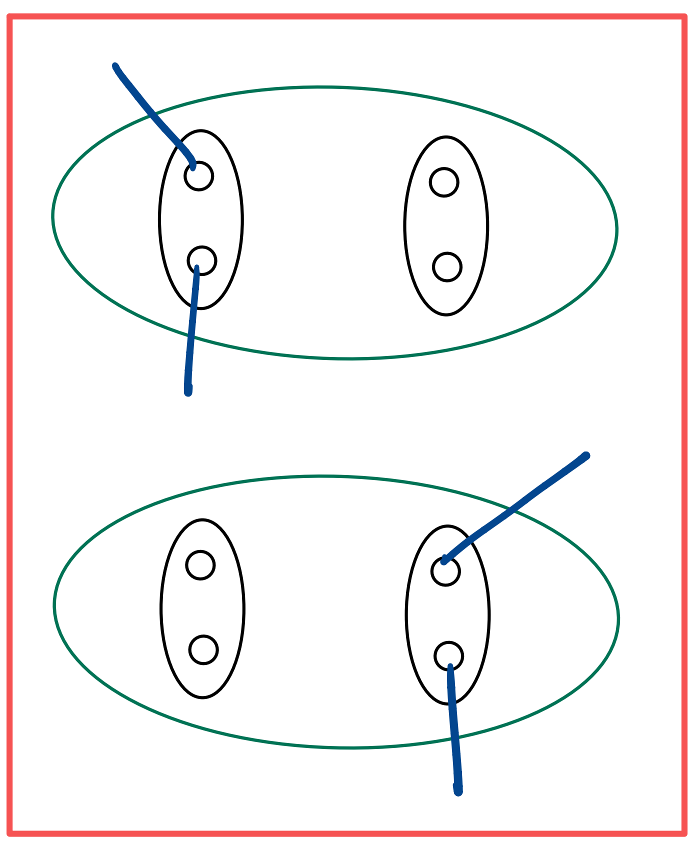
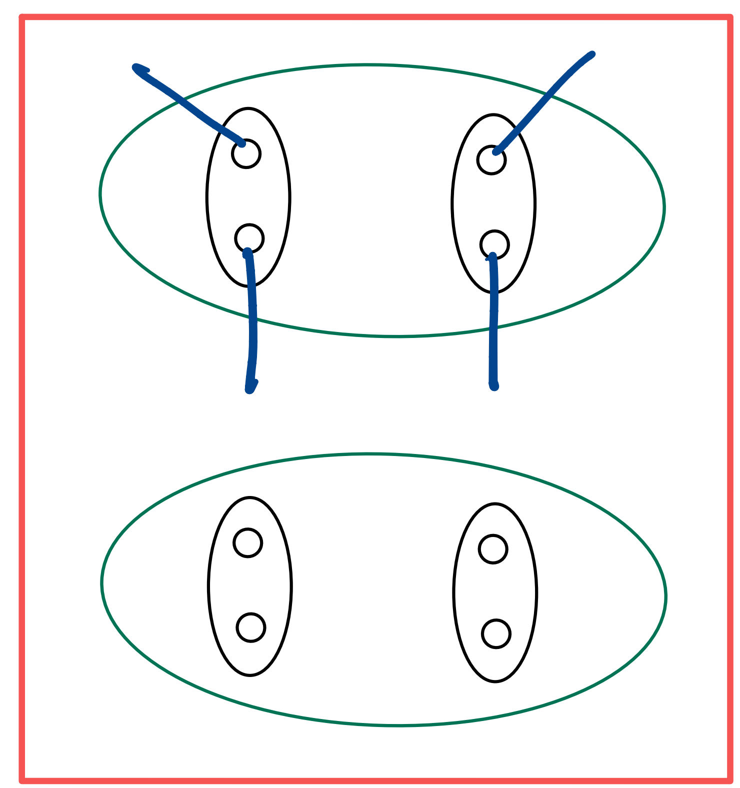
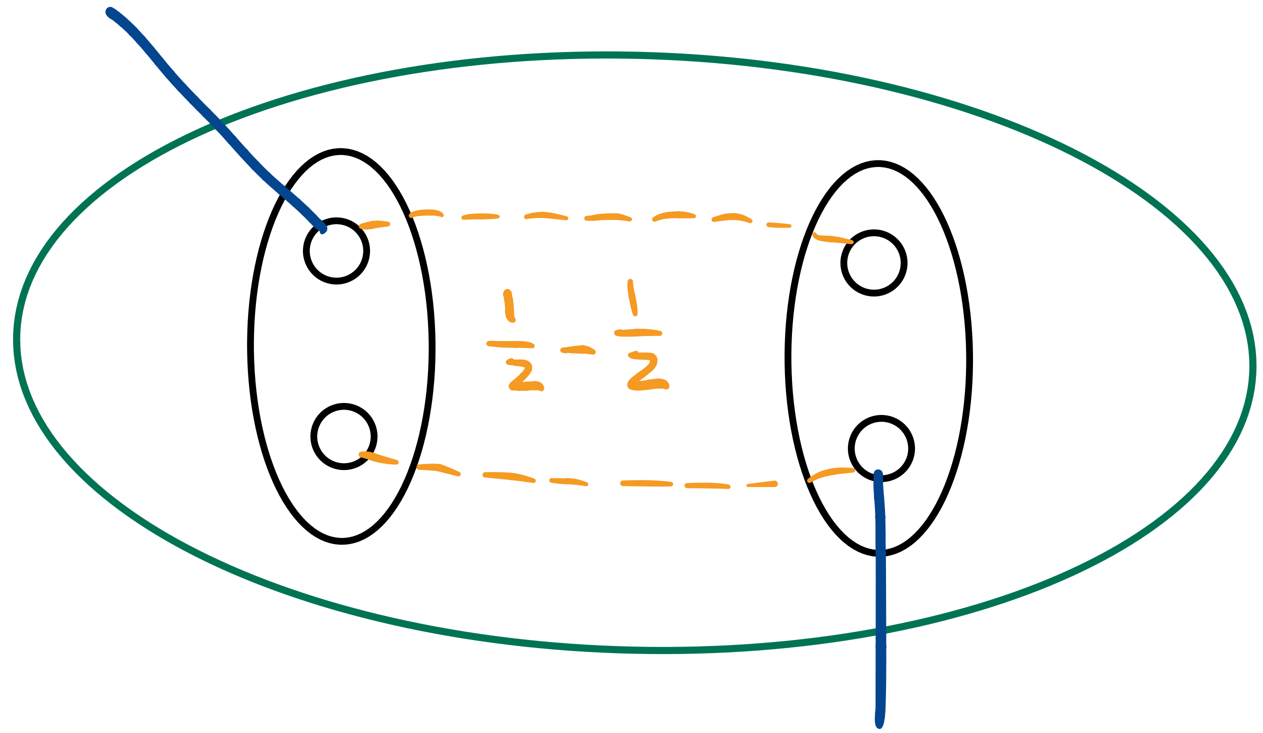


4

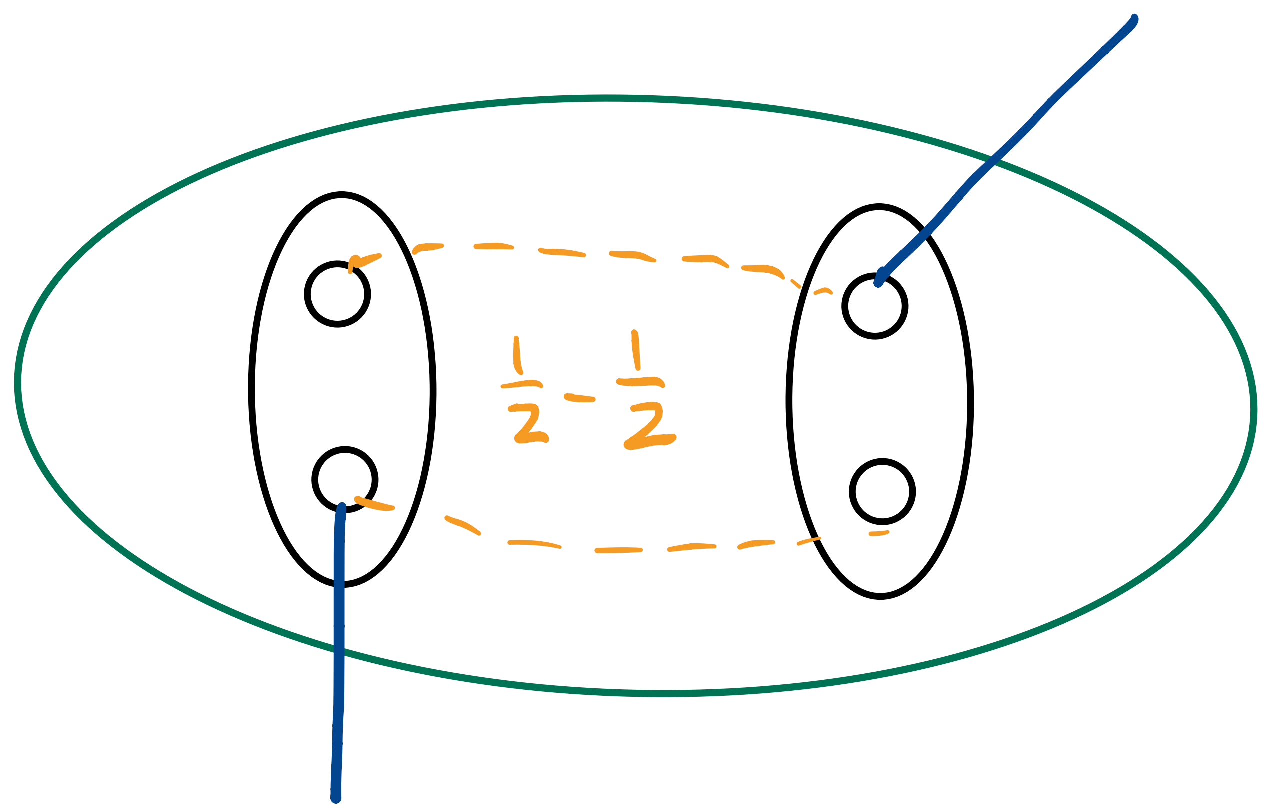


* Blue edges represent parity of edges entering the cut.

State 1



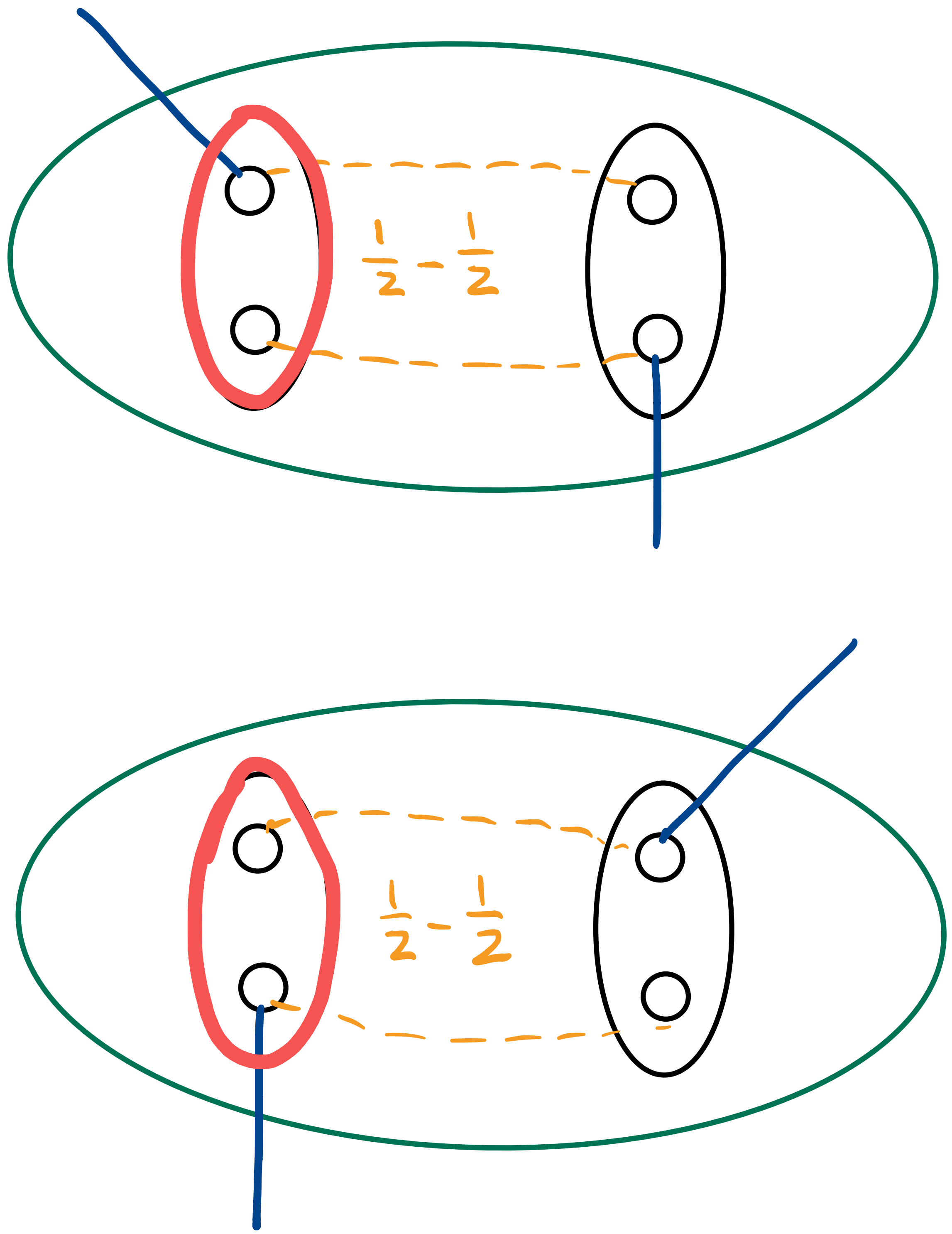
* Blue edges represent <u>parity</u> of edges entering the cut.

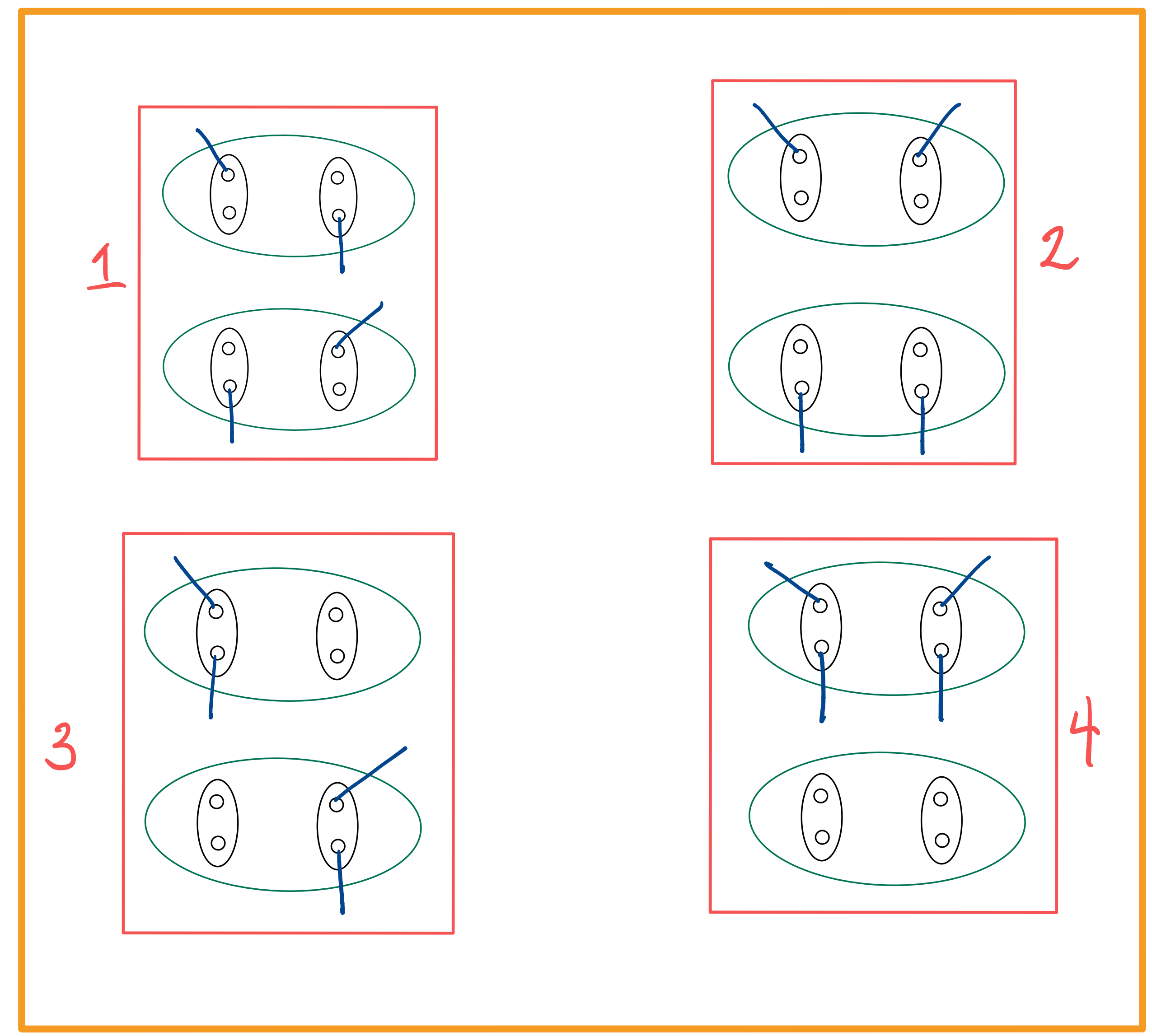


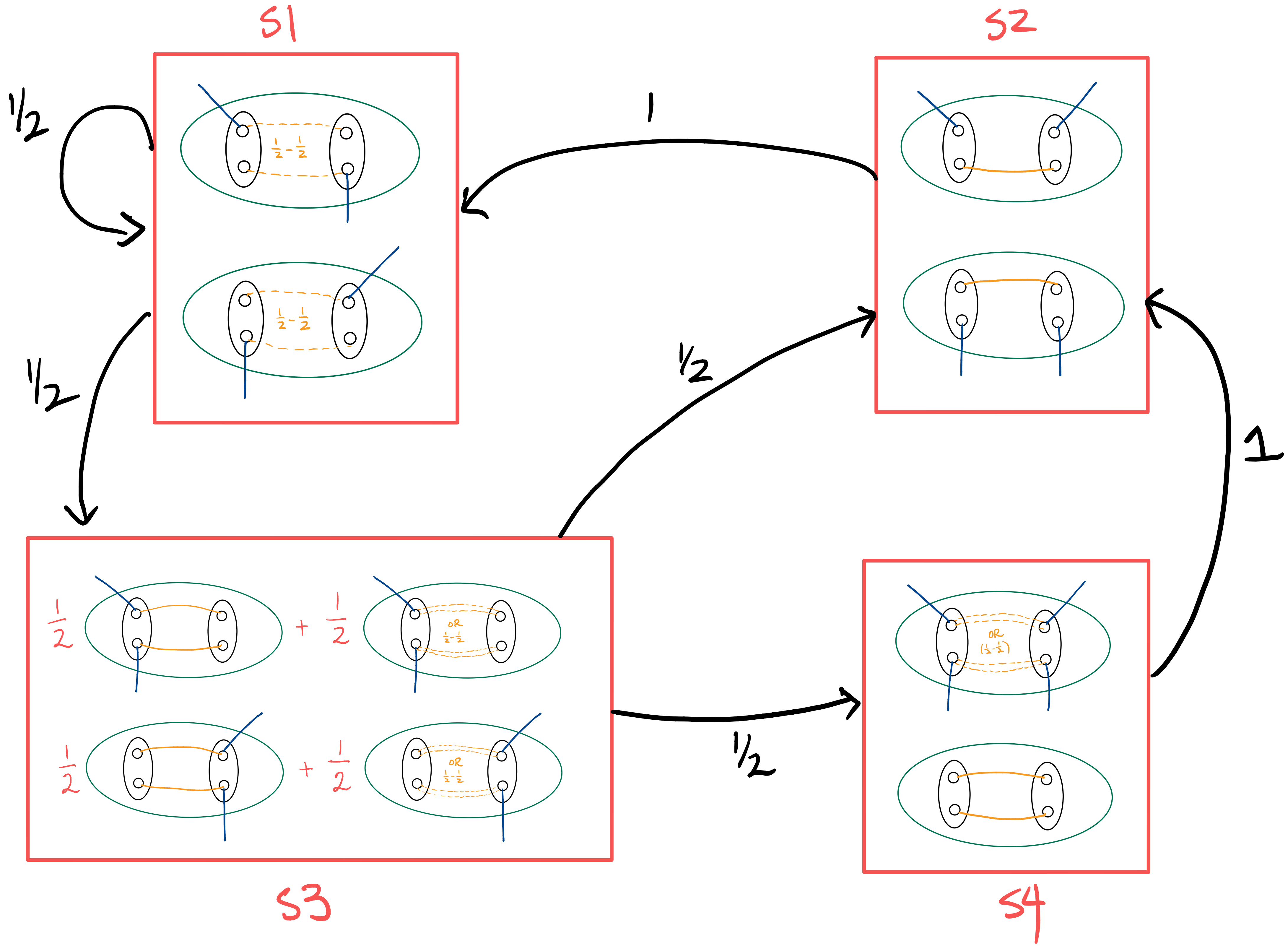
Edges connecting children used $\frac{1}{2}$ the time in expectation.

These rules induce a distribution over states for each child.

e.g. If parent is in state 1, children are in $\begin{cases} \text{state } 1 & \text{w.p. } \frac{1}{2}, \\ \text{state } 3 & \text{w.p. } \frac{1}{2}. \end{cases}$



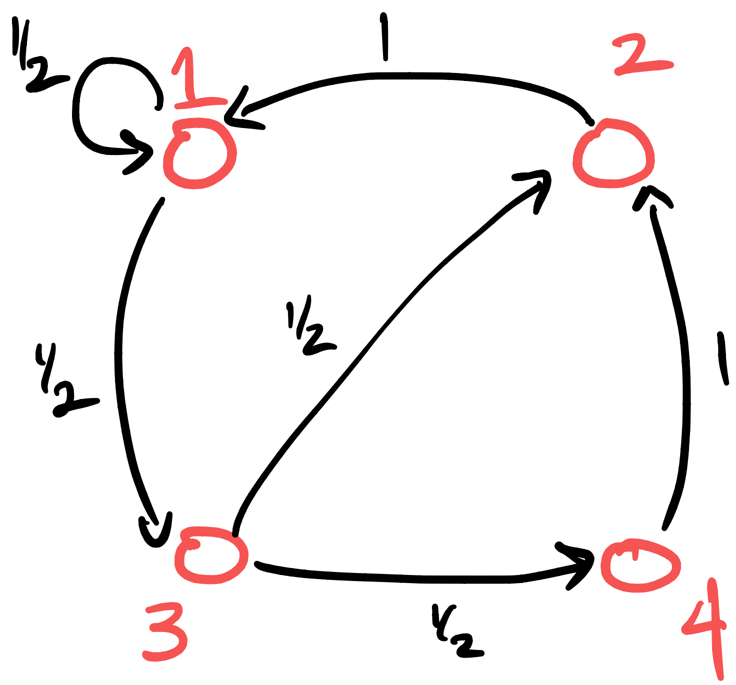




Markov chain mapping distribution of patterns on the parent to distribution on the children.

* Being in a state means equally likely to be in top picture vs bottom picture.

The Fixed Point



$$\pi = \left(\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}\right).$$

- Can check that states 1, 2, 3, 4 use each edge $\frac{1}{2}, \frac{1}{2}, 1, 1$ of the time, resp.

$\therefore$ Under π, each edge is used $\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \pi_3 + \pi_4 = \frac{2}{3} = \frac{4}{3} x_e$ of the time.

* $\left(\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}\right)$ is fixed point even in the general case!

Algorithm Recap

- Algorithm inducts on the hierarchy top-down
- At top level, sample edges according to fixed point $p = (\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9})$
  - state 1 w.p. $\frac{4}{9}$, state 2 w.p. $\frac{2}{9}$, etc.
- For each cut in $\mathcal{H}$, given its state, connect its children according to the rules.
- p is fixed point $\Rightarrow$ for every $S \in \mathcal{H}$, $\Pr[S \text{ is in state } i] = p_i$
- Under p, each edge is used $\frac{4}{3} x_e$ of the time (in expectation)
- Resulting set of edges is Eulerian, with expected cost $= \frac{4}{3} \sum_e c_e x_e$
- Can be derandomized using method of conditional expectations

Future Directions

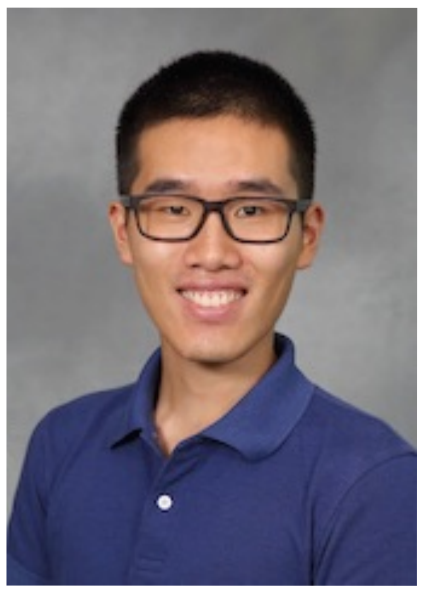
- $4/3$ for cycle cut instances that are not half-integral?

- What about the degree cut* case?

  * Degree cut ≡ critical cut that is not a cycle cut.

- (In progress) Max Entropy is not a $4/3$-approx. alg. for these instances.

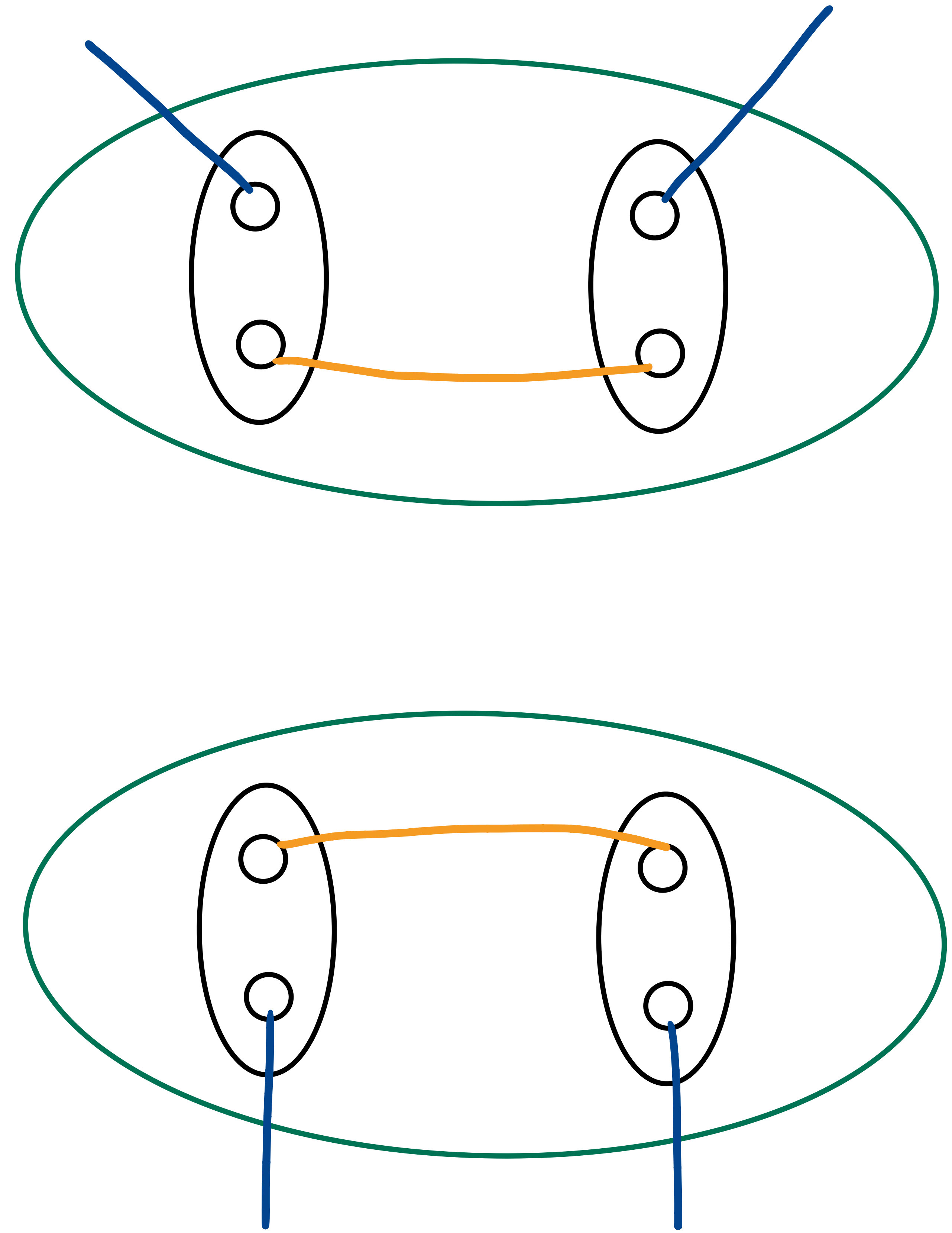
Thank you!



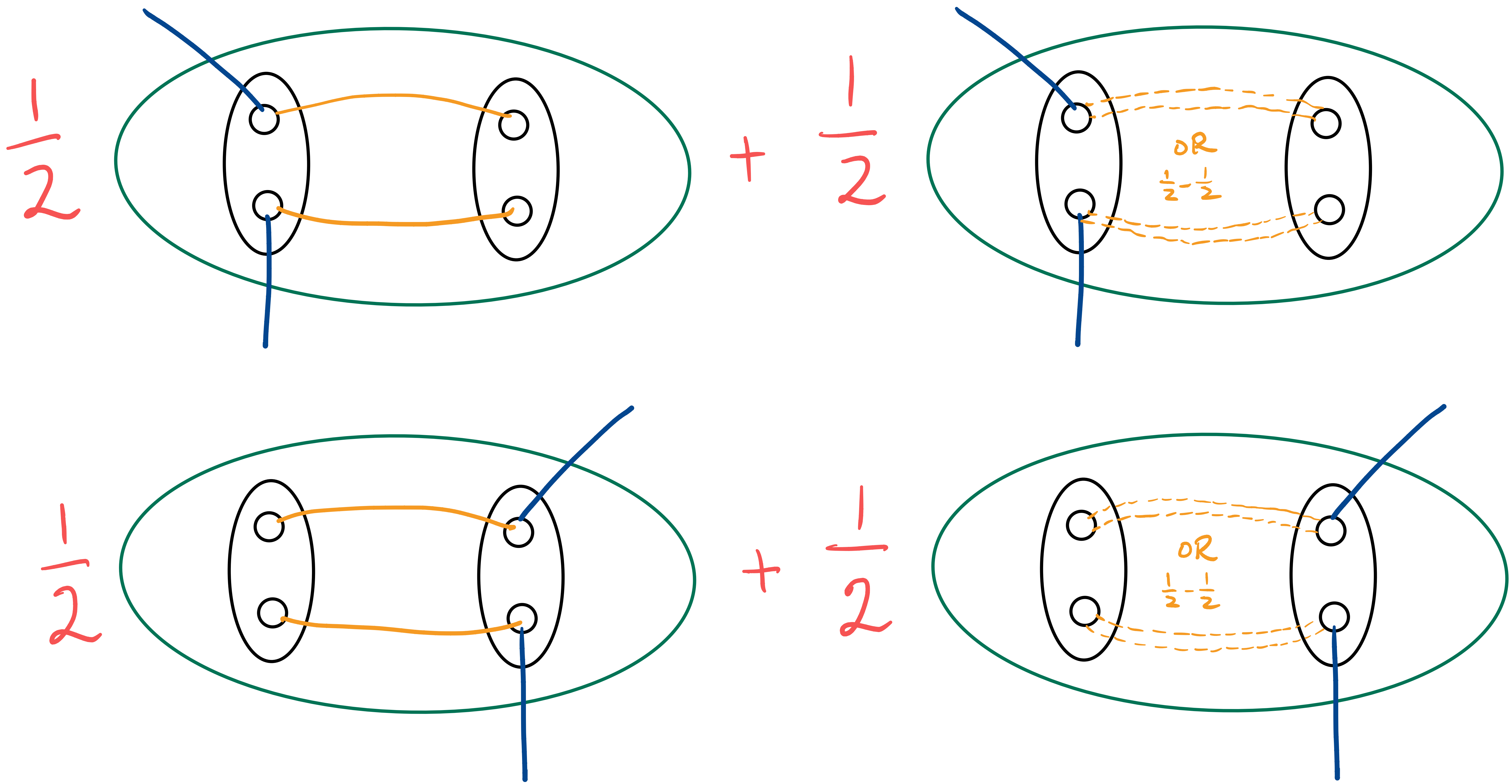
On the market this year!

Happy to discuss more!

State 2



State 3



State 4