An $O(\log \log n)$-Approximation for Submodular Facility Location

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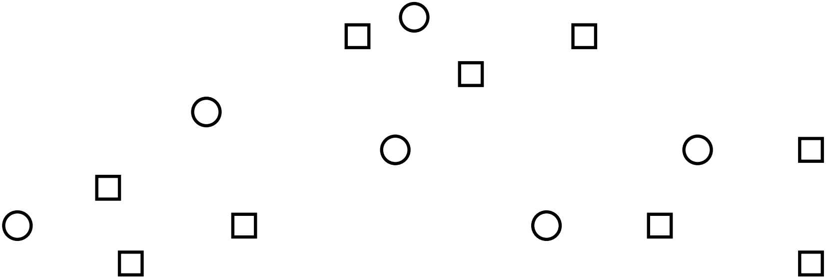
Facility Location

Given:

- set of Clients C, set of Facilities F
- opening facility cost o_f

Goal:

- minimize $\sum_{c \in C} d(c, F'(c)) + \sum_{f \in F'} o_f$



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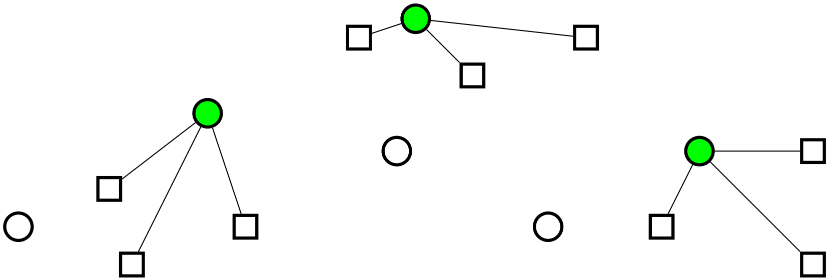

Submodular Set Functions

$g : 2^V \to \mathbb{R}$ is **submodular** if:

$$\forall A, B \subseteq V, \quad g(A) + g(B) \geq g(A \cup B) + g(A \cap B)$$

$$\forall A \subseteq B \subseteq V, \forall x \in V / B, \quad g(A \cup \{x\}) - g(A) \geq g(B \cup \{x\}) - g(B)$$

Submodular Facility Location

Given:

- n clients C and m facilities F,
- $d : (C \cup F) \times (C \cup F) \to \mathbb{R}_{\geq 0}$
- a monotone submodular opening cost $g(.)$

Goal:

$$Minimize \sum_{c \in C} d(c, \varphi(c)) + \sum_{f \in F} g(\varphi^{-1}(f))$$

Where $\varphi : C \to F$ is assignment of each client to some facility

Known Results

SFL is APX-hard [Guha, Khuller,1999]

Svitkina and Tardos show [2010] that:

- There is $O(\log n)$ approximation for general SFL with multiple submodular function. The result is tight, becuase of reduction from Set Cover Problem.

- $(4.237 + \epsilon)$approximation for special case of SFL where sudmodular function $g(.)$ is specified by a rooted cost tree T

Our Results

Main Contribution

- There is a polynomial-time $O(\log \log n)$-approximation algorithm for SFL.

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Generalizations and Variants

- There is a polynomial-time $O(\log \log n)$-approximation algorithm for MULTSFL.
- There is a polynomial-time $O(\log \log n)$-approximation algorithm for ADDSFL.
- There is a polynomial-time $O(\log \log \frac{n}{\pi_{\min}})$-approximation algorithm for the UNIVERSAL STOCHASTIC FACILITY LOCATION problem.

LP Relaxation

Conf-LP:

$$
\begin{aligned}
\min \sum_{f \in F} \sum_{R \subseteq C} g(R) \cdot x_R^f + \sum_{c \in C} \sum_{f \in F} \sum_{R \ni c} d(c, f) \cdot x_R^f \qquad & (1) \\
\text{s.t.} \sum_{f \in F} \sum_{R \ni c} x_R^f = 1 \qquad & \forall c \in C; \\
\sum_{R \subseteq C} x_R^f = 1 \qquad & \forall f \in F; \\
x_R^f \geq 0 \qquad & \forall R \subseteq C,\ \forall f \in F.
\end{aligned}
$$

Overview of the algorithm

1. Compute fractional solution to Conf-LP
2. Sample partial assignment S_1 to remove most of the clients from the remaining instance
3. Embed the remaining instance into a tree
4. Use filtering techniques to remove the connection cost from the picture and obtain Descendent-Leaf Assignment problem (DLA)
5. Approximately solve DLA via LP rounding

2) Reducing the connection cost by removing client

1. Let $\dot{x}$ be a solution to Conf-LP
2. For $\ln \ln N$ times, sample random partial assignments by selecting configuration (f, R) indep. with probability $\dot{x}_R^f$
3. let C_1 be the covered clients and S_1 the partial assignment, then:
   - Each client belongs to C_1 with **large enough** probability,

     $$\mathbb{P}[c \notin C_1] \leq e^{-\ln \ln N} = \frac{1}{\ln N}$$

   - The expected cost of S_1 is **small enough**

     $$\mathbb{E}[cost(S_1)] \leq \ln \ln N \cdot cost(\dot{x})$$

.

3) Embedding the remaining instance on an HST

Let $\ddot{x}$ be $\dot{x}$ restricted to $C_2 = C \setminus C_1$ we have:

- $open(\ddot{x}) \leq open(\dot{x})$ and $\mathbb{E}[conn(\ddot{x})] \leq \frac{1}{\ln N} conn(\dot{x})$.

We map the input metric (M, d) into a metric on a Hierarchically well-Separated Tree (HST), (M', d_T) and obtain:

1. Every $a \in M$ is mapped to some leaf $v(a)$ of T
2. $\mathbb{E}\left[d_T(v(a), v(b))\right] \leq 8 \log |M| \cdot d(a, b)$;
3. T has depth $O(\log d_{\max})$.

Therefore,

$$\mathbb{E}[conn_{d_t}(\ddot{x})] = O(1) \cdot conn(\dot{x})$$

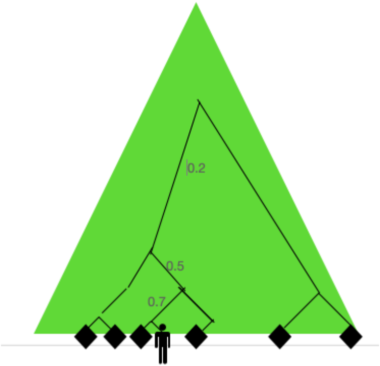
.

4) Filtering

Focus on a single client:

- Think of fractional connection as of a unit flow.
- Focus on the path from the client to the root of the tree.
- Find the lowest edge on the path on which the flow has value < 0.5.

Figure 1: A client fractionally connected to facilities.



4) Filtering and Descendent-Leaf Assignment(DLA)

DLA

Goal: Find an assignment of each $c \in \tilde{C}$ to some $f \in \tilde{F}_c$ so that the total opening cost is minimized.

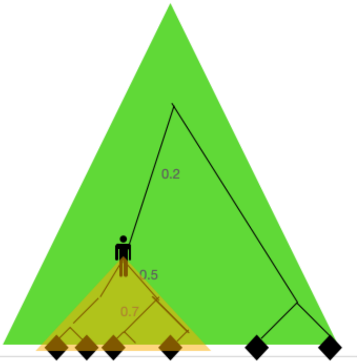
Convex-programming (CP) relaxation for DLA:

$$\min \sum_{f \in \tilde{F}} \hat{h}(z^f)$$

$$\text{s.t.} \quad \sum_{f \in \tilde{F}_c} z_c^f = 1 \quad \forall c \in \tilde{C}.$$

$$z_c^f \geq 0 \quad \forall c \in \tilde{C},\ \forall f \in \tilde{F}.$$

Figure 2: A client restricted to a subtree.

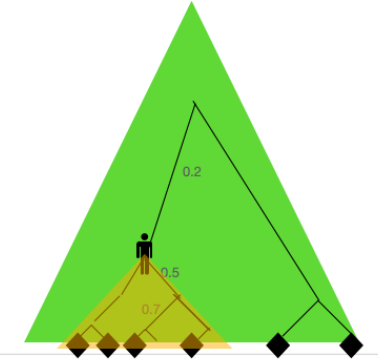


5) Solving DLA via LP rounding

We adapt method of [Bosman, Olver 2020]

- proceed by levels bottom-up.
- feature of the relaxation: in extreme solutions, subsets of clients served by a facility form a chain.
- when processing a node: select a subset of clients (from the chain) to be integrally served via threshold.
- merge nodes at the bottom.

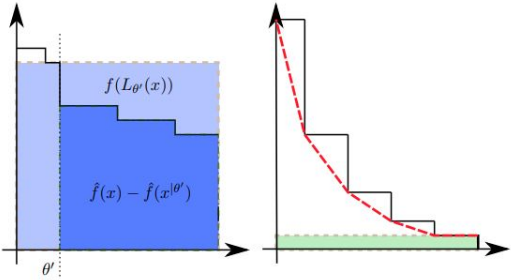
Figure 3: A client restricted to a subtree.



Lemma (Bosman, Olver, 2020)

Given $x \in [0,1]^V$ *and* $\alpha \in (0,1]$, *at least one of the following holds:*

1. *there exists* $\theta \in [0,1]$, *which can be computed in polynomial time, such that* $L_\theta(x)$ *is* $\frac{\alpha}{32}$*-supported;*
2. $2^{1/\alpha} f(L_1(x)) \leq \hat{f}(x)$.



Summery: $(\log\log N)$ approximation for SFL

- Compute a random partial assignment S_1, $\mathbb{E}(cost(S_1)) \leq O(\log\log N).cost(\dot{x})$
- Obtain residual fractional solution $\ddot{x}$ restricted to $S_2 = C \setminus S_1$ and embed it on the HST-type instance, we have:

$$\begin{aligned}\mathbb{E}[cost_T(\ddot{x})] &= open(\ddot{x}) + \mathbb{E}[conn_T(\ddot{x})] \\ &\leq open(\dot{x}) + O(\log N) \cdot \mathbb{E}[conn(\ddot{x})] \\ &\leq O(cost(\dot{x})).\end{aligned}$$

- Randomly round $\ddot{x}$ to an assignment of S_2 via a red. to DLA
- Obtain S_2 of cost at most $O(\log\log N) cost(\dot{x})$
- Return $S_1 + S_2$ as a feasible solution to SFL so that:

$$\begin{aligned}\mathbb{E}(S_1 + S_2) &\leq O(\log\log N) \cdot cost(\dot{x}) \\ &\leq O(\log\log N) \cdot cost(opt)\end{aligned}$$

Open Questions

- Is there any constant approximating for SFL Problem?
- Is there any constant factor approximation over tree instance for SFL problem?
- Is there $O(\log \log N)$ approximation for AFFINE SFL problem over tree instance, where the opening costs are sobmodular functions of form $g_f(S^f) = o_f + w_f \cdot g(S^f)$?

IPCO'24 in Wrocław, Poland

ipco2024.ii.uni.wroc.pl

6 November 2023: submission deadline

1-2 July 2024: summer school

3-5 July 2024: IPCO conference

PC chair: Jens Vygen

Local chair: Jarek Byrka

Summer school speakers:

- **Sophie Huiberts**
- **Neil Olver**
- **Vera Traub**