Convexity in Several Complex Variables

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April 11, 2023

Equivalent Formulations of Geometric Convexity

Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain (open and connected). That is, there exists a smooth function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\} \qquad \text{and} \qquad \nabla\rho(x) \neq 0 \text{ on } \partial\Omega.$$

The function ρ is a called a *defining function* for Ω.

Classical Theorem

The following are equivalent to the geometric convexity of Ω.

1. $\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0, \qquad \forall w \in T_P(\partial\Omega)$ and $\forall P \in \partial\Omega$.
2. Ω admits a smooth strictly convex exhaustion function $\psi : \Omega \to \mathbb{R}$.
3. For every compact $K \subset \Omega$ the set

$$\hat{K} := \left\{ x \in \Omega : f(x) \leq \sup_{t \in K} f(t) \text{ for all linear functions } f \right\}$$

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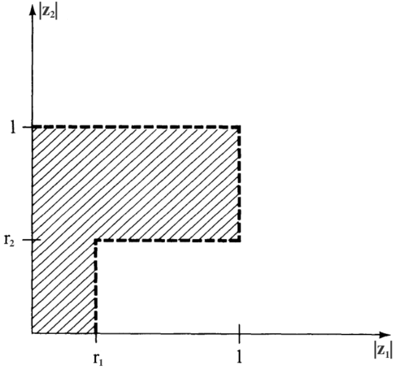
Hartogs Extension Phenomenon

Consider the domain in $\mathbb{C}^2$ with coordinates (z_1, z_2):

$$H = \left\{z \in \mathbb{C}^2 : |z_1| < r_1, |z_2| < 1\right\} \bigcup \left\{z \in \mathbb{C}^2 : |z_1| < 1,\ r_2 < |z_2| < 1\right\}$$



Any holomorphic function f on H has a unique extension $\hat{f}$ to the unit bidisk $\mathbb{D}^2$ given by

$$\hat{f}(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{f(z_1, \zeta)}{\zeta - z_2} d\zeta,$$

where $\delta \in (r_2, 1)$.

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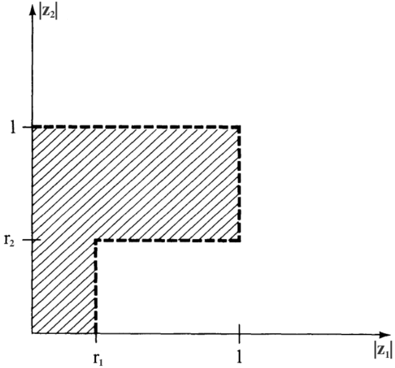

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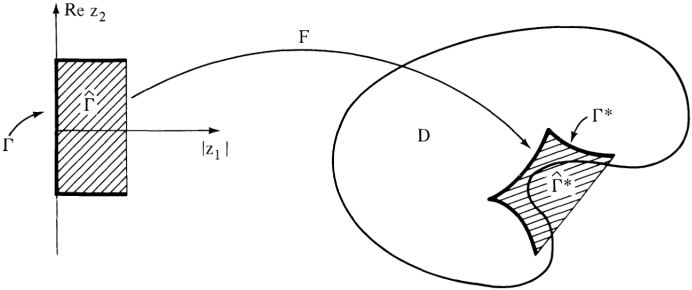

- A domain $\Omega \subseteq \mathbb{C}^n$ is a called *pseudoconvex* if for every $P \in \partial\Omega$ there exists a neighbourhood U of P and a $f \in \mathcal{O}(U \cap \Omega)$ that cannot be extended holomorphically to P.
- The *Levi problem*: is this notion equivalent to the existence of a $f \in \mathcal{O}(\Omega)$ which cannot be extended to *any* larger domain?
- How do we characterize pseudoconvex domains in $\mathbb{C}^n$?

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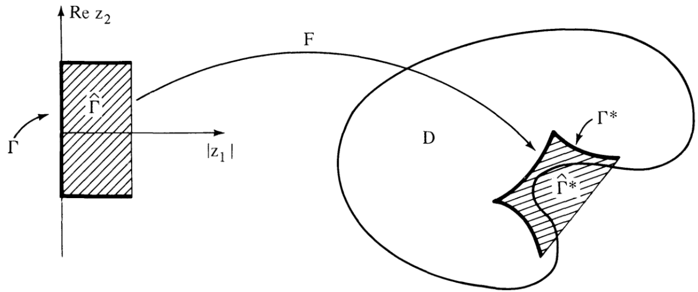

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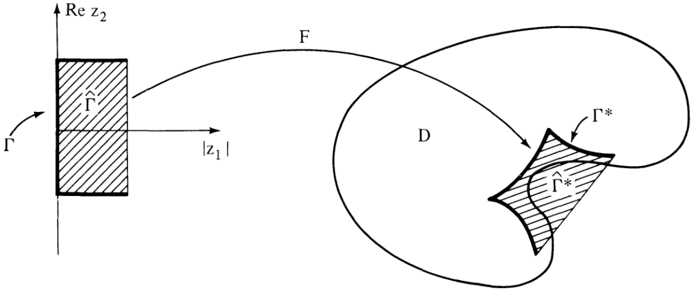

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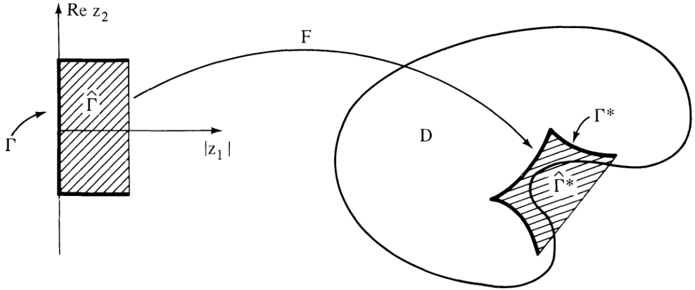

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Holomorphic Generalizations of Convexity

Definition

Let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^n$. Ω is called *$\mathcal{F}$-convex* if for every compact $K \subset \Omega$ the set

$$\hat{K}_{\mathcal{F}} := \left\{ z \in \Omega \,:\, |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathcal{F} \right\}$$

is compact in Ω as well. A compact $K \subset \Omega$ is $\mathcal{F}$-convex if $\hat{K}_{\mathcal{F}} = K$.

Examples

- Polynomial Convexity: $\Omega = \mathbb{C}^n$ and $\mathcal{F} =$ Polynomials on $\mathbb{C}^n$.
- Rational Convexity: $\Omega = \mathbb{C}^n$ and $\mathcal{F} =$ Rational functions.
- Holomorphic Convexity: $\mathcal{F} = \mathcal{O}(\Omega)$.
- The closed unit ball $\overline{B}_1 \subseteq \mathbb{C}^n$ is $\mathcal{F}$-convex for all $\mathcal{F}$ above.
- Spherical shell $\overline{B}_1 \setminus B_{1/2}$ is not $\mathcal{F}$-convex for any $\mathcal{F}$ above.
- If $K \subset \mathbb{C}^1$ is compact, then K is rationally convex and the polynomially convex hull of K is $K \cup \text{holes}(K)$.

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Approximation: Motivation

One of the main reasons for the study of polynomial and rational convexity is its connection with approximation of holomorphic functions. The motivation comes from dimension 1.

Theorem (Runge)

Given $K \subset \mathbb{C}$ compact and $f \in \mathcal{O}(K)$, there exists a sequence of rational functions $r_j(z)$ with poles off K that approximates f uniformly on K, i.e., $\mathcal{R}(K) = C(K)$. If K has no 'holes', then $r_j(z)$ can be taken to be polynomials, that is, $\mathcal{P}(K) = C(K)$.

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Approximation: Oka–Weil

The multidimensional version is following:

Theorem (Oka–Weil)

If $K \subset \mathbb{C}^n$ is polynomially (resp. rationally) convex and $f \in \mathcal{O}(K)$, then given any $\varepsilon > 0$, there exists a polynomial P (resp. rational function with poles off K) with $||f - P||_K < \varepsilon$.

Likewise, If $\Omega \subset \mathbb{C}^n$ is $\mathcal{O}(\Omega)$-convex, then any holomorphic function in a neighbourhood of a $\mathcal{O}(\Omega)$-convex compact K is the uniform limit on K of functions in $\mathcal{O}(\Omega)$.

- A partial converse is also true: *if $K \subset \mathbb{C}^n$ is a compact such that $\mathcal{P}(K) = C(K)$ (resp. $\mathcal{R}(K) = C(K)$), then K is polynomially (resp. rationally) convex.* Clearly, such K must have $\mathrm{int}(K) = \varnothing$.
- Roth's swiss-cheese example shows: *there exists a compact $K \subset \mathbb{C}$ with empty interior such that $\mathcal{R}(K) \neq C(K)$.*
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Pseudoconvexity

An upper semicontinuous function $\psi : \Omega \to [-\infty, \infty)$ is called *plurisubharmonic* (psh) on Ω if for every $P \in \Omega$ and $v \in \mathbb{C}^n$ the function $\mathbb{C} \ni \lambda \mapsto \psi(P + \lambda v)$ is subharmonic or identically $-\infty$ for $|\lambda| << 1$. Equivalently, $dd^c\psi \geq 0$ for $\psi \in \mathcal{C}^2(\Omega)$. Likewise, $\psi \in \mathcal{C}^2(\Omega)$ is *strictly plurisubharmonic* (spsh) if $dd^c\psi > 0$.

Examples (f holomorphic):

(1) $\log|f|$; (2) $|f|^p$ for $p > 0$; (3) $\sup_\alpha u_\alpha(z)$, where u_α psh $\forall \alpha$.

Theorem

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with defining function $\rho : \mathbb{C}^n \to \mathbb{R}$. TFAE to the pseudoconvexity of Ω:

1. *Ω is $\mathcal{O}(\Omega)$-convex;*
2. *$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0$ for all complex tangent vectors w of $\partial\Omega$ at P for all boundary points P.*
3. *Ω admits a smooth spsh exhaustion function $\psi : \Omega \to \mathbb{R}$.*
4. *$\bar{\partial} u = f$ is solvable in the $\mathcal{C}^\infty$-category at all form levels.*

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(1) $\log|f|$; (2) $|f|^p$ for $p > 0$; (3) $\sup_\alpha u_\alpha(z)$, where u_α psh $\forall\alpha$.

Theorem

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with defining function $\rho : \mathbb{C}^n \to \mathbb{R}$. TFAE to the pseudoconvexity of Ω:

1. *Ω is $\mathcal{O}(\Omega)$-convex;*
2. *$\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0$ for all complex tangent vectors w of $\partial\Omega$ at P for all boundary points P.*
3. *Ω admits a smooth spsh exhaustion function $\psi : \Omega \to \mathbb{R}$.*
4. *$\bar{\partial}u = f$ is solvable in the $\mathcal{C}^\infty$-category at all form levels.*

What about ($1 \implies 3$)?

Theorem

The function $z \mapsto -\log(dist(z, \partial\Omega))$ is plurisubharmonic on a pseudoconvex domain.

The key is to write

$$\mathrm{dist}(z, \partial\Omega) = \inf\{\delta_{\Omega,u}(z) \,:\, u \in \mathbb{C}^n \text{ with } |u| = 1\},$$

where $\delta_{\Omega,u}(z)$ measures how large a disk in the "u-direction" with center at z is contained in D. Then show $z \mapsto -\log \delta_{\Omega,u}(z)$ is plurisubharmonic for each $u \in \mathbb{C}^n$ with $|u| = 1$.

Theorem (Diederich and Fornaess)

If Ω is pseudoconvex with $\mathcal{C}^2$-boundary, then $-dist(z, \Omega)^\eta e^{-K|z|}$ is spsh in neighbourhood of $\partial\Omega$ for small $\eta > 0$ and large $K > 0$.

From this, one can conclude that pseudoconvex domains with $\mathcal{C}^2$-boundary admit *bounded* spsh exhaustion functions.

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Properties of Polynomial and Rational Convexity

- The intersection of a family of pseudoconvex domains is pseudoconvex whenever its interior is nonempty.
- A compact $K \subset \mathbb{C}^n$ is rationally convex iff for all $z \notin K$, there exists a polynomial P with $P(z) = 0$ but $P^{-1}(0) \cap K = \varnothing$.

Proposition

If $\Omega \subset \mathbb{C}^n$ is a bounded domain and $\overline{\Omega}$ is rationally convex, then Ω is pseudoconvex. The analogue for polynomially convex compacts is also true.

The former can be shown by taking the intersection of domains of the form $\mathbb{C}^n \setminus \Sigma_p$, where $p \notin \overline{\Omega}$, and Σ_p is a complex hypersurface that passes through p and avoids $\overline{\Omega}$.

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Envelopes of Holomorphy

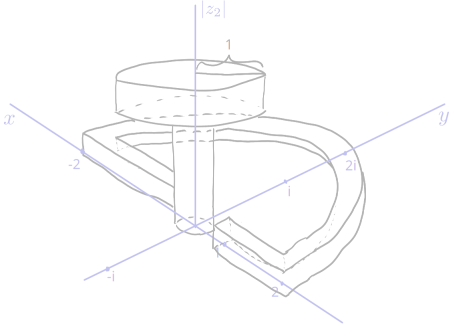
Given a domain $\Omega \subset \mathbb{C}^n$, the *envelope of holomorphy* $E(\Omega)$ is the largest "domain" to which all holomorphic fuctions on Ω extend.

Example

Consider the domain $H \cup W_1 \cup W_2 \subset \mathbb{C}^2$. Here,

$$H = \{\, |z_1| < \tfrac{1}{2}, |z_2| < 1 \} \bigcup \{\, |z_1| < 1,\ \tfrac{1}{2} < |z_2| < 1 \}$$

$$W_1 = \{|z_1 - 1| < \tfrac{1}{4}, |z_2| < \tfrac{1}{4}\}, \qquad W_2 = \{\mathrm{dist}(z, \gamma) < \tfrac{1}{4}\},$$



and γ is the curve in the z_1-plane given by

$$\gamma = \{x : 1 \leq x \leq 2\} \cup \{2e^{i\theta} : 0 \leq \theta \leq \pi\} \cup \{x : -2 \leq x \leq 0\}$$

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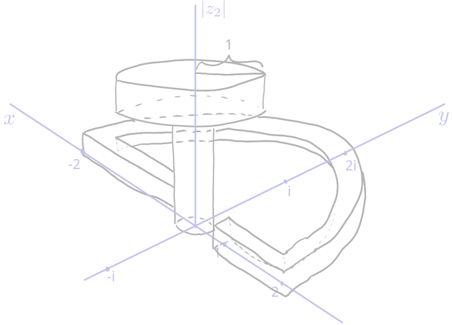

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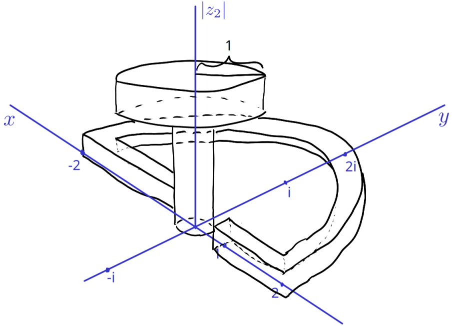

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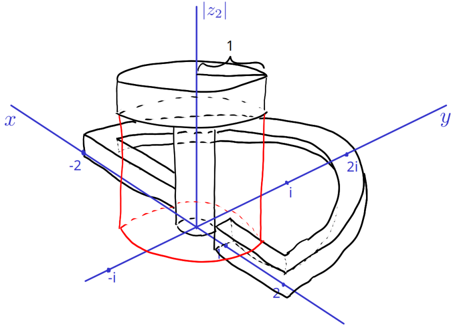

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Envelopes of Holomorphy and Stein Manifolds

The previous example shows that the notion of an "envelope of holomorphy" is not well-defined in the class of Euclidean domains.

Theorem

The envelope of holomorphy of a domain $\Omega \subset \mathbb{C}^n$ exists as a pair (X, π), where X connected complex manifold and π is a local homeomorphism $\pi : X \to \mathbb{C}^n$. Furthermore:

1. *X contains Ω in the sense that there exists a continuous map $\varphi : \Omega \to X$ with the property that $\pi \circ \varphi$ is the identity map on Ω, and*
2. *any holomorphic function $f \in \mathcal{O}(\Omega)$ extends to X in the sense that there exists a function $F \in \mathcal{O}(X)$ with $F \circ \varphi = f$.*

Such a manifold should be pseudoconvex in some sense.

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Stein Manifolds

Definition

We call a complex manifold X *Stein* if it satisfies the following:

1. X is $\mathcal{O}(X)$-convex;
2. There is a $f \in \mathcal{O}(X)$ with $f(P) \neq f(Q)$ whenever $P \neq Q$;
3. For every point $P \in X$ there exist functions $f_1, \ldots, f_n \in \mathcal{O}(X)$, $n = \dim X$, where $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are $\mathbb{C}$-linearly independent at P.

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The envelope of holomorphy of a domain $\Omega \subseteq \mathbb{C}^n$ is Stein.

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Every complex hypersurface (w/o boundary) of $\mathbb{C}^n$ is Stein.

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Embedding Stein Manifolds into Euclidean Spaces

Theorem

Every Stein manifold X admits a proper holomorphic embedding into some Euclidean space $\mathbb{C}^N$. Furthermore, N can be chosen to be $N = \left\lfloor \frac{3 \cdot \dim(X)}{2} \right\rfloor + 1$ whenever $\dim(X) > 1$. This choice of N is minimal.

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Every real analytic manifold admits a proper real analytic embedding into $\mathbb{R}^N$ for some N.

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