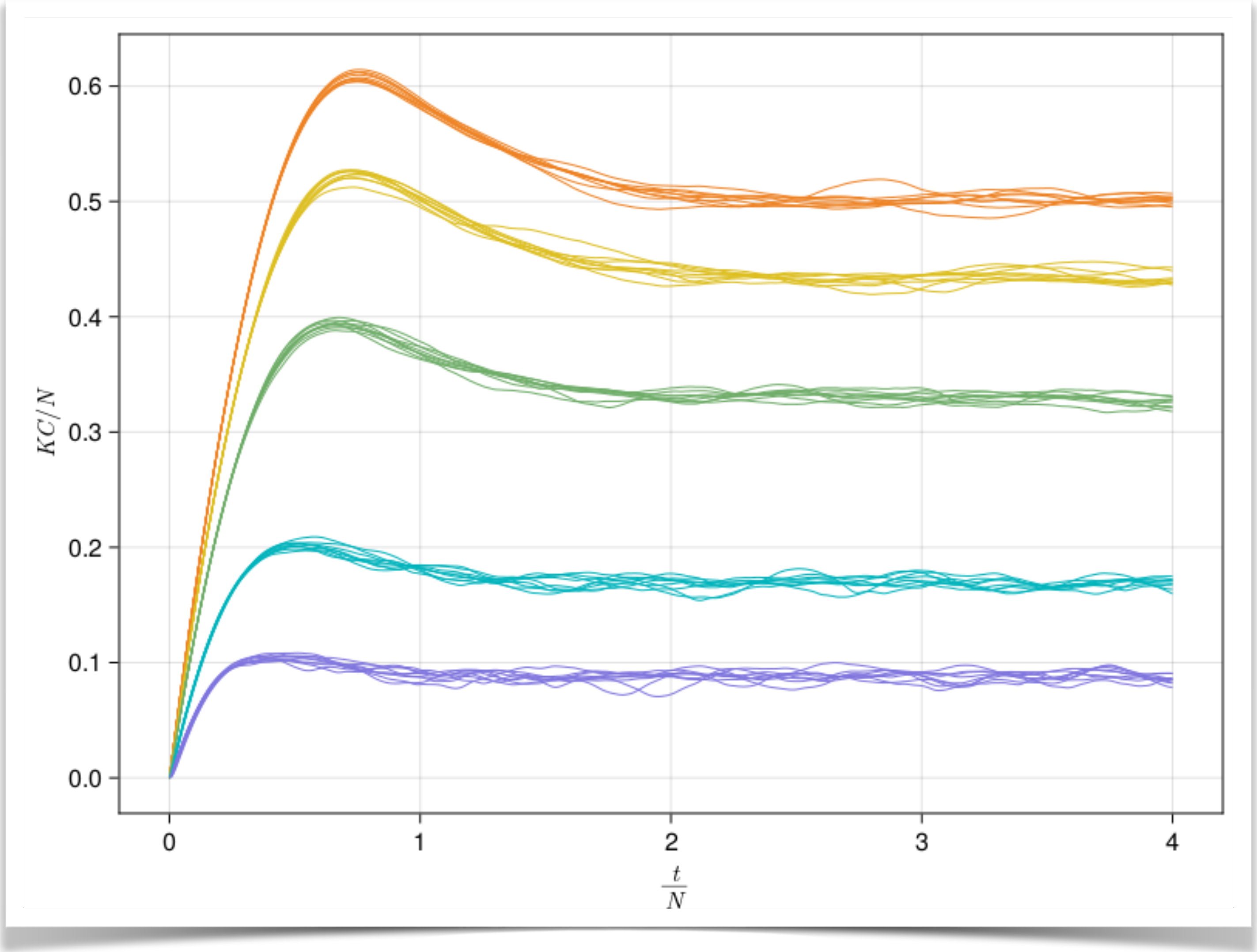
Long times, chaos, and spread complexity



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Based on: Balasubramanian, Caputa, J.M.M, and Q. Wu arXiv[2208.08452]
Balasubramanian, J.M.M and Q. Wu arXiv[2202.06957,2306.xxxxx]

Motivations

- Complexity and Quantum Chaos
- Complexity and long times
- Maldacena's AdS/CFT version of the information paradox
- Complexity and the volume of the black hole interior

From Krylov complexity to spread complexity

Krylov complexity was introduced in

[Parker, Cao, Avdoshkin, Scaffidi, Altman, 2018]

As a measure of operator complexity

We will describe directly the generalization to quantum states

The idea is very simple and starts by noticing that

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \sum_n \frac{(it)^n}{n!} H^n |\psi(0)\rangle \equiv \frac{(it)^n}{n!} |\tilde{\psi}_n\rangle$$

From Krylov complexity to spread complexity

From such vectors we can derive the Krylov basis

$|\psi_n\rangle = H^n|\psi(0)\rangle$ ⟶ Gram-Schmidt procedure ⟶ $|K_n\rangle$

This is the solution of the Lanczos algorithm **[Viswanath, Muller, 1994]**

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \quad |K_n\rangle = b_n^{-1}|A_n\rangle$$

where the Lanczos coefficients are defined by

$$a_n = \langle K_n|H|K_n\rangle, \qquad b_n = \langle A_n|A_n\rangle^{1/2}$$

with the initial conditions

$$b_0 \equiv 0 \qquad |K_0\rangle = |\psi(0)\rangle$$

From Krylov complexity to spread complexity

This implies that the Hamiltonian in the Krylov basis is a tridiagonal matrix

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$\begin{pmatrix} a_0 & b_1 & & & & \\ b_1 & a_1 & b_2 & & & \\ & b_2 & a_2 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{N-2} & a_{N-2} & b_{N-1} \\ & & & & b_{N-1} & a_{N-1} \end{pmatrix}$$

Available numerically stable algorithms for computing the 'Hessenberg form'

Another starting point is the 'survival amplitude'

$$S(t) = \langle\psi(t)|\psi(0)\rangle = \langle\psi(0)|e^{iHt}|\psi(0)\rangle$$

From Krylov complexity to spread complexity

Finally, to compute complexity, we expand the time evolving state in the Krylov basis

$$|\psi(t)\rangle = \sum_n \psi_n(t)\,|K_n\rangle$$

↓

$$C(t) = C_{\mathscr{K}}(t) = \sum_n n\,p_n(t) = \sum_n n\,|\psi_n(t)|^2$$

From Krylov complexity to spread complexity

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This quantity also follows from minimizing the spread of the wave function over all choices of basis

[Balasubramanian, Caputa, Magan, Wu, 2022]

'Spread complexity'

From Krylov complexity to spread complexity

It is illuminating to apply this framework to the TFD state

$$|\psi_\beta\rangle \equiv \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta E_n}{2}} |n, n\rangle$$

Under time evolution one obtains

$$|\psi_\beta(t)\rangle = e^{-iHt} |\psi_\beta\rangle = |\psi_{\beta+2it}\rangle$$

In AdS/CFT, TFD are dual to eternal BH and this evolution might describe aspects of the BH interior

[Maldacena, 2001] [Hartman, Maldacena, 2013] [Susskind, 2016]

What is important for us is that the survival amplitude is

$$S(t) = \langle \psi_{\beta+2it} | \psi_\beta \rangle = \frac{Z_{\beta-it}}{Z_\beta}$$

the analytically continued partition function
and the survival probability is the Spectral Form Factor (SFF)

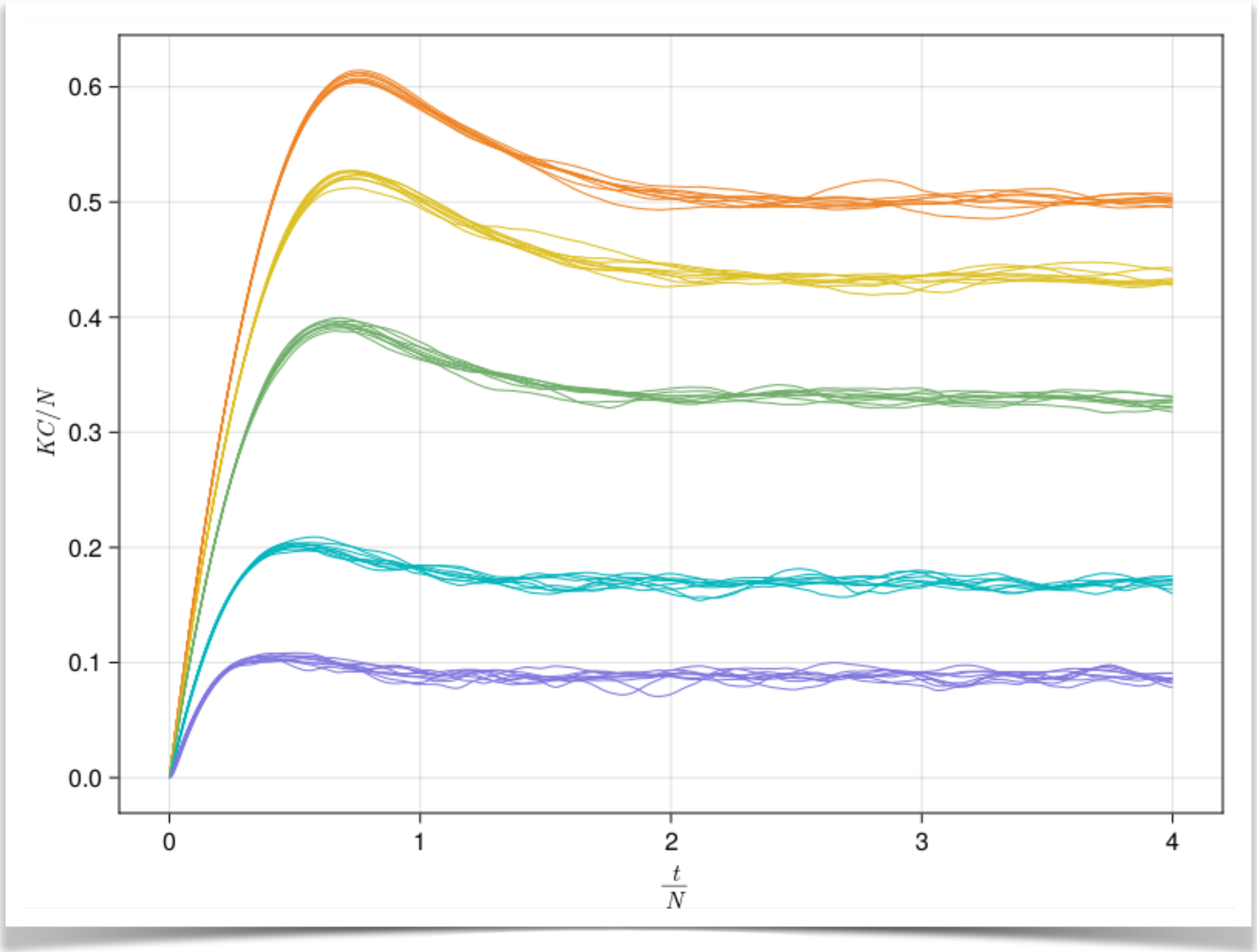
The SFF plays a leading role in quantum chaos and in recent research in quantum BH

[Guhr, Groeling, Weidenmuller, 1997] [Cotler, Gur-Ari, Hanada, Polchinski, Saad,Shenker, Stanford, Streicher, Tezuka, 2017]

Random Matrices and long times

Using Lanczos coefficients we can compute probabilities and quantum complexity

Different hues correspond to the GUE ensemble for different temperatures

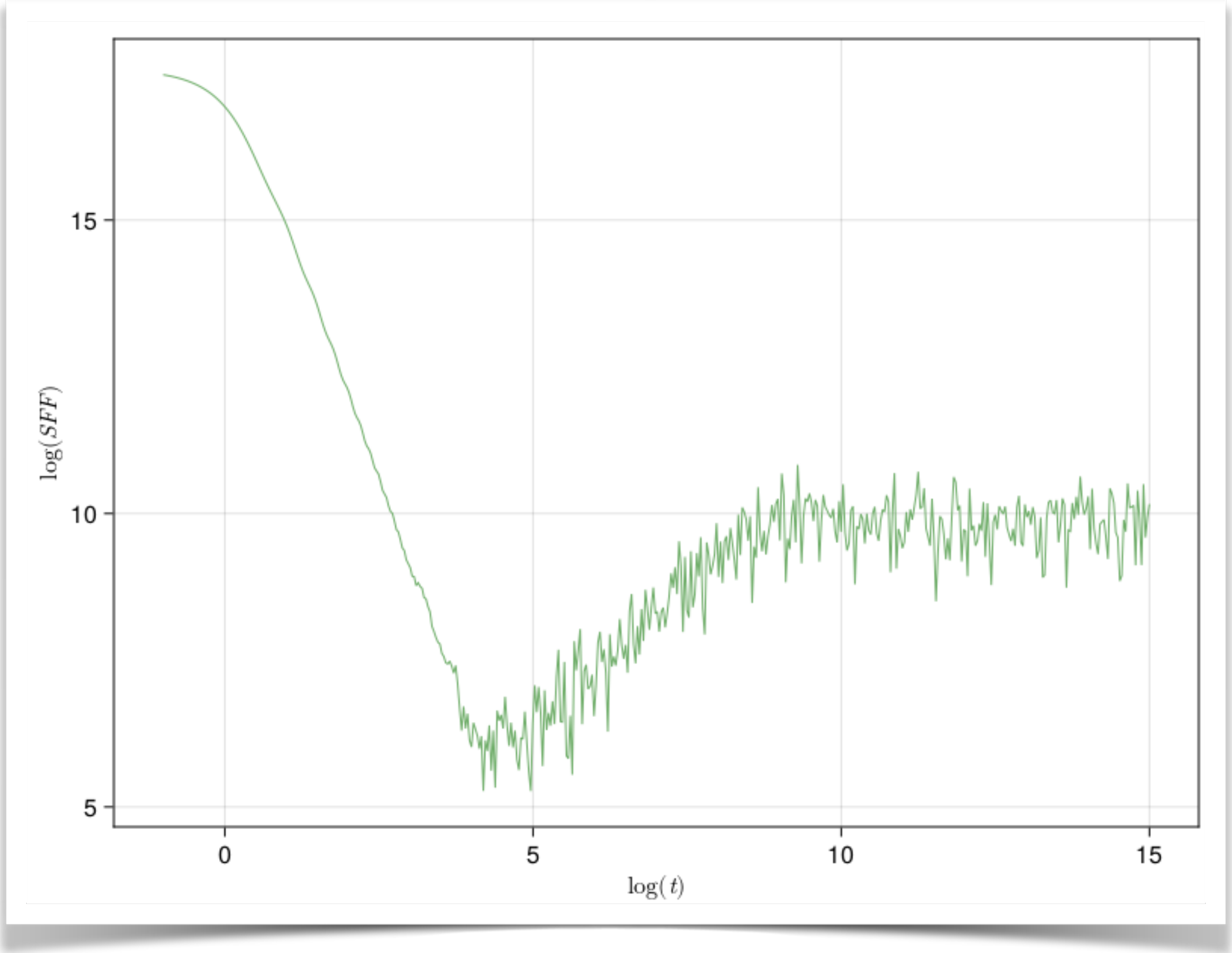


The GUE complexity displays four regimes: A ***ramp***, a ***peak***, a ***slope*** and a ***plateau***

Random Matrices and long times

We can contrast the behavior with that of the SFF

$$SFF = |S(t)|^2 = |\langle \psi_{\beta+2it} | \psi_\beta \rangle|^2 = |\frac{Z_{\beta-it}}{Z_\beta}|^2$$



The SFF shows a slope, dip, ramp, plateau. Dip and slope originated in spectral rigidity

[Guhr, Groeling, Weidenmuller, 1998] [Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka, 2017]

Tridiagonalizing Random Matrices

Starting from a generic random state and running the Lanczos algorithm we arrived at

$$\begin{pmatrix} a_0 & b_1 & & & & \\ b_1 & a_1 & b_2 & & & \\ & b_2 & a_2 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{N-2} & a_{N-2} & b_{N-1} \\ & & & & b_{N-1} & a_{N-1} \end{pmatrix}$$

Where the Lanczos coefficients are random.

The problem is then to find the statistics of the Lanczos coefficients, namely the joint probability distribution

$$p(a_0, \cdots, a_{N-1}, b_1, \cdots, b_{N-1})$$

And generic correlation functions

$$\overline{a_m \cdots a_n \, b_r \cdots b_s}$$

Tridiagonalizing Random Matrices

The joint probability distribution turns out to be

$$p(a_0, \cdots, a_{N-1}, b_1, \cdots, b_{N-1}) \propto \prod_{n=1}^{N-1} b_n^{(N-n)\beta-1} e^{-\frac{\beta N}{4} Tr(V(H))}$$

By taking the logarithm we obtain an effective action

$$S_{eff} \equiv \ln p(a_0, \ldots, a_{n-1}, b_1, \ldots, b_{N-1}) = \sum_n ((N-n)\beta - 1)\ \ln b_n - \frac{\beta N}{4} Tr(V(H))$$

Doing this and taking the thermodynamic limit is somewhat technical. The end result is

$$S_{eff} = \sum_n ((N-n)\beta - 1)\ \ln b_n - \frac{\beta N}{4} \sum_n \int dE \frac{V(E)}{\pi\sqrt{4b_n^2 - (E-a_n)^2}}$$

In the thermodynamic (large-N) limit it is better to write everything in terms of $x = n/N$

$$\frac{S_{eff}}{\beta N^2} = \int dx\, (1-x) \ln b(x) - \frac{1}{4} \int dx \int dE \frac{V(E)}{\pi\sqrt{4b(x)^2 - (E-a(x))^2}}$$

Tridiagonalizing Random Matrices

The saddle point equations for the Lanczos coefficients are

$$4(1-x) = b(x)\frac{\partial}{\partial b(x)}\int dE\,\frac{V(E)}{\pi\sqrt{4b(x)^2-(E-a(x))^2}}$$

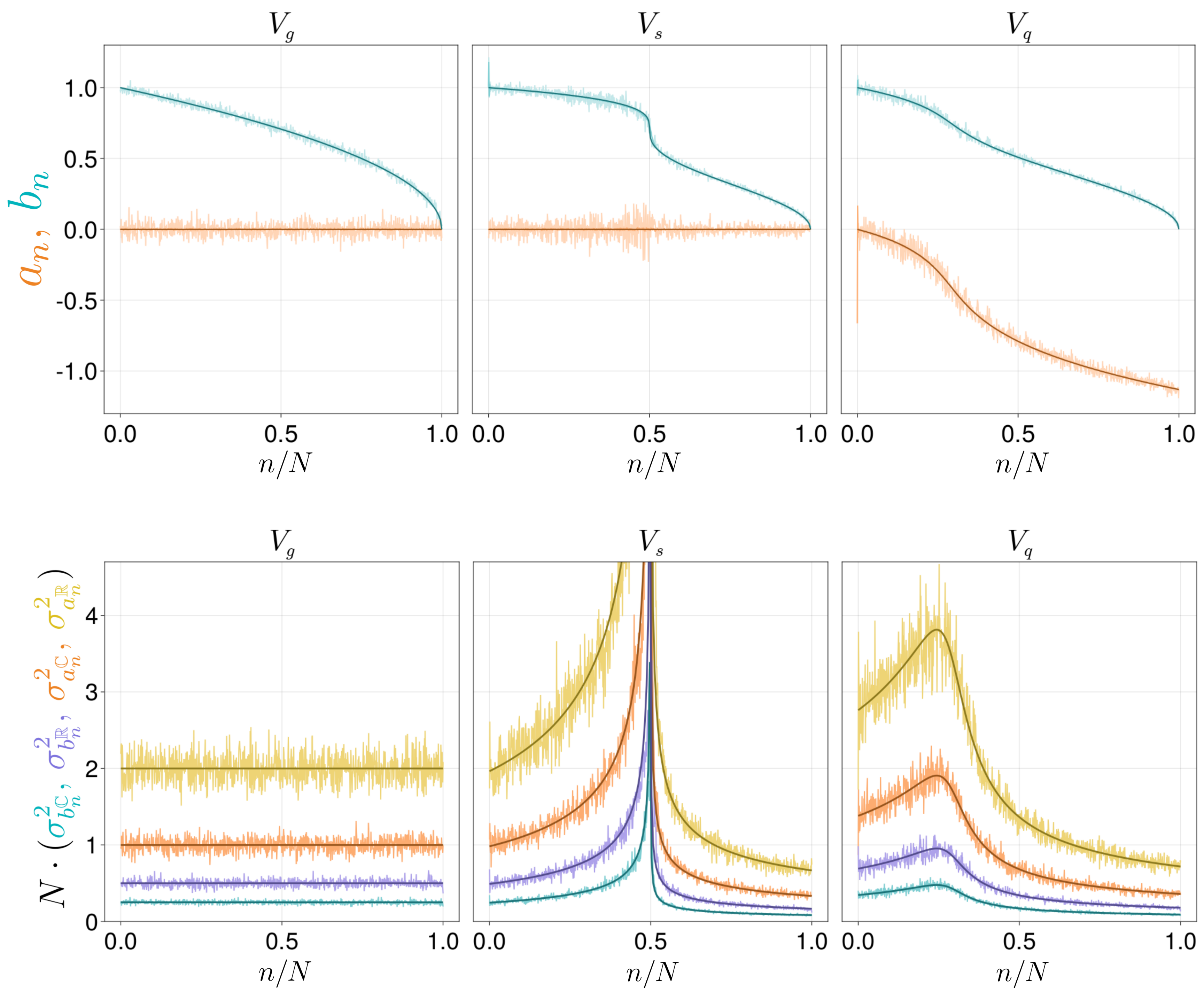
$$0 = \frac{\partial}{\partial a(x)}\int dE\,\frac{V(E)}{\pi\sqrt{4b(x)^2-(E-a(x))^2}}$$

By expanding the saddle point equations around the average we can compute the two point functions (covariances) of the Lanczos coefficients

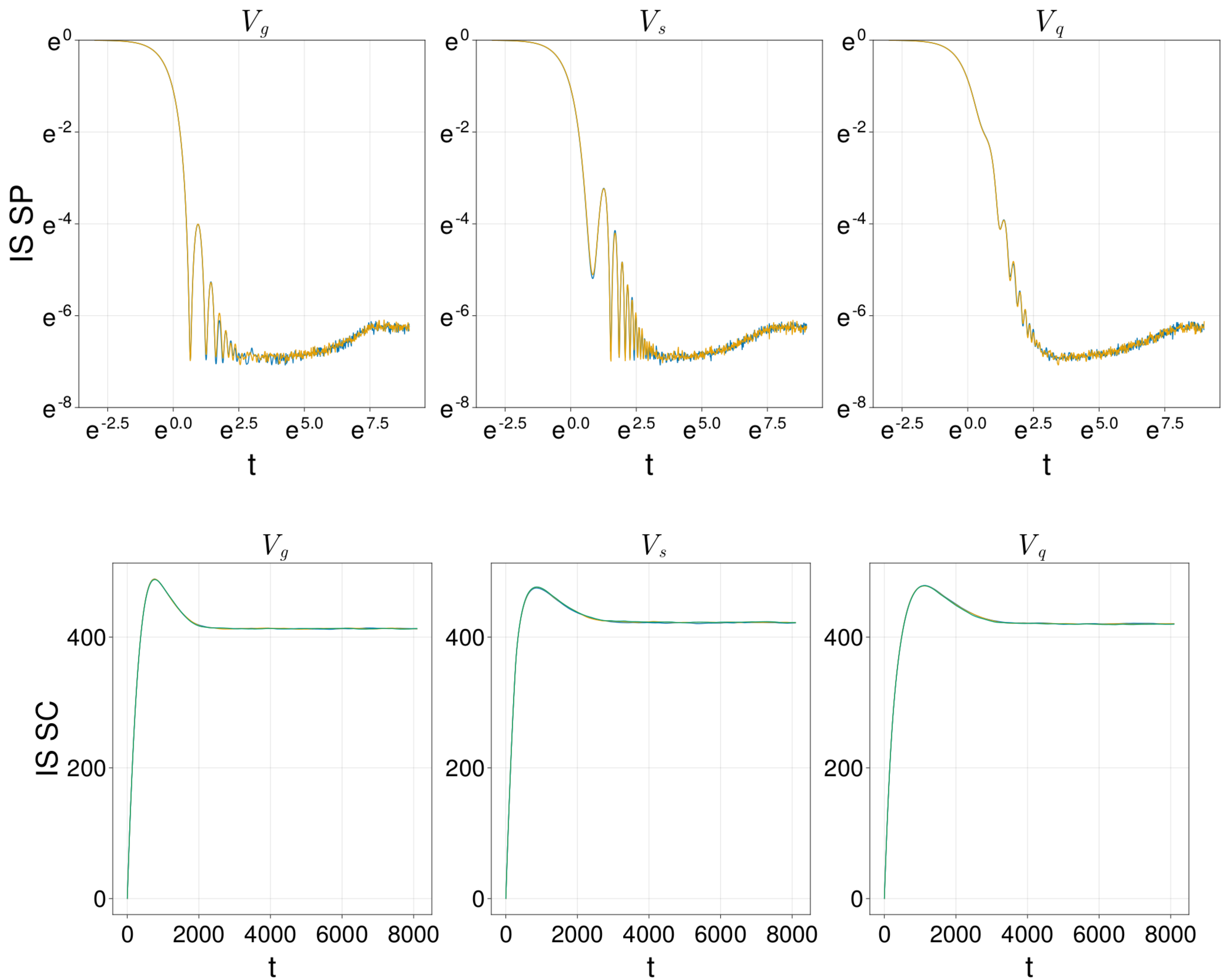
$$\Delta S_{eff} \equiv -\frac{1}{2}\left(\,\delta a_i\, M_{ij}^{aa}\,\delta a_i + 2\delta a_i\, M_{ij}^{ab}\,\delta b_i + \delta b_i\, M_{ij}^{bb}\,\delta b_i\,\right)$$

Tridiagonalizing Random Matrices

Numerical verification for three different potentials



Tridiagonalizing Random Matrices



Discussion

- Complexity and Quantum Chaos

Thermofield Double Spread complexity is a functional of the spectrum. It is a functional of the spectral form factor.

- Complexity and long times

Spread complexity displays four regimes. Complexity slope controlled by spectral rigidity. It codifies the universality classes of chaotic behavior.

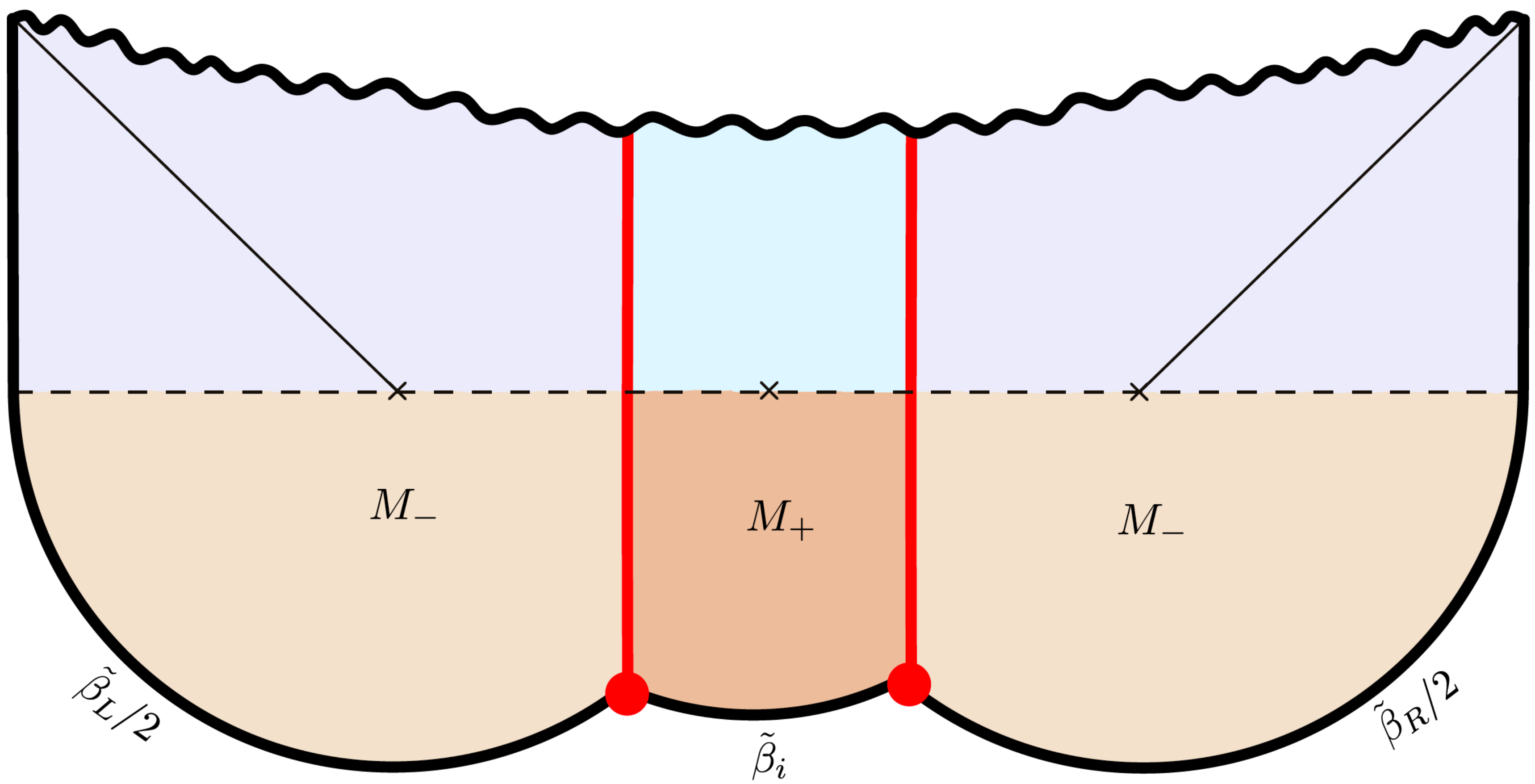
- Maldacena's AdS/CFT version of the information paradox

Saturation of survival probability (spectral form factor) equivalent to saturation of spread complexity. Unification of two problems. These are further equivalent to the vanishing of the Lanczos coefficients. We were able to provide a proof for RMT with a compact spectrum. In the way a new tridiagonal approach to RMT is developed

- Spread complexity and the volume of the black hole interior

[Balasubramanian, Lawrence, Magan, Sasieta, 2022] [Erdmenger, Jian, Xian 2023] [Rabinovici, Sanchez-Garrido, Conner, 2023]

Discussion



[Balasubramanian, Lawrence, Magan, Sasieta, 2022]

The proper mass of the shell, related to number of operator insertions, is unconstrained from above

The volume of the Einstein-Rosen bridge roughly proportional to the mass of the shell $V_{ren} = ml/(d-1)$

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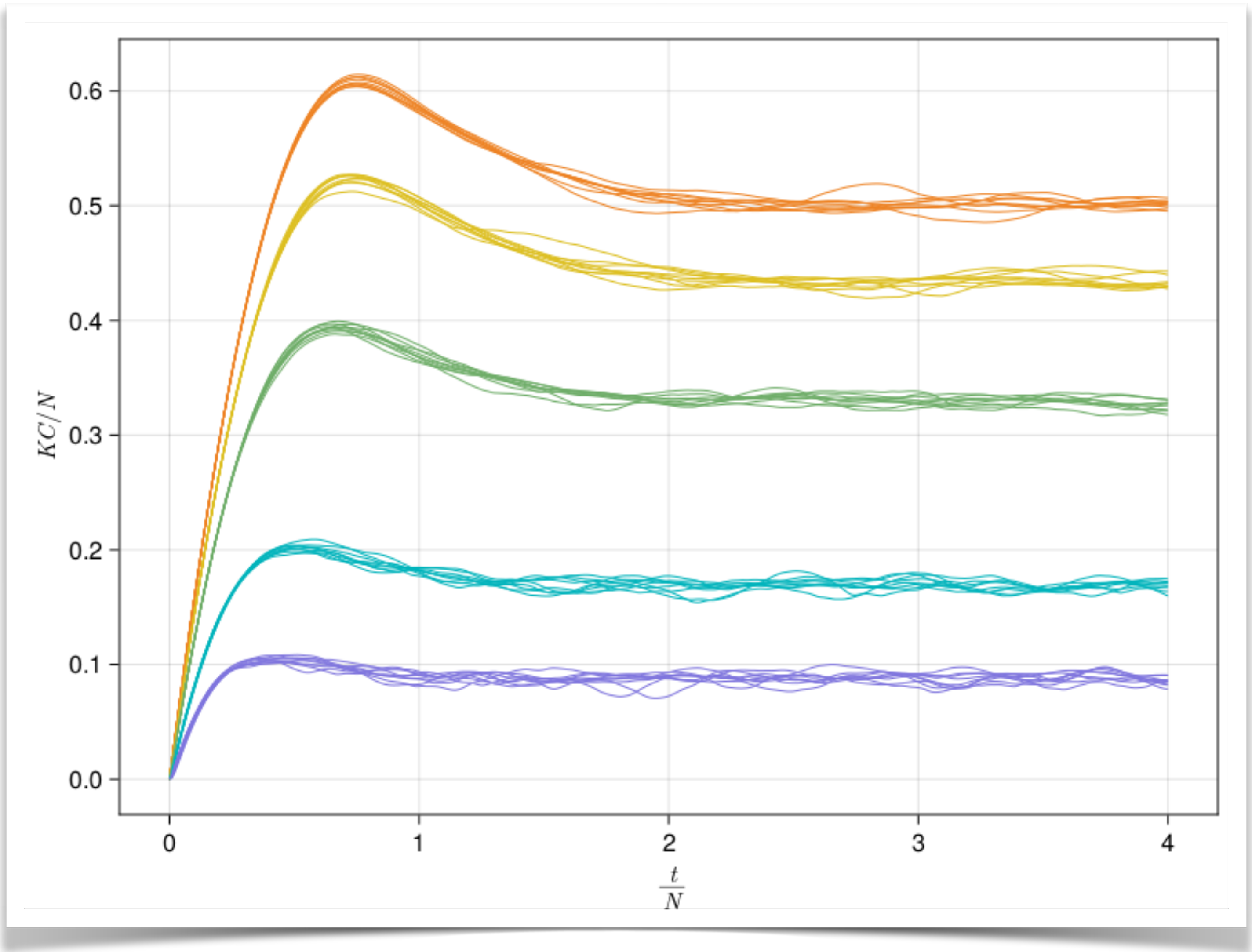

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