C*-envelopes and Nuclearity related properties of Operator Systems

PREETI

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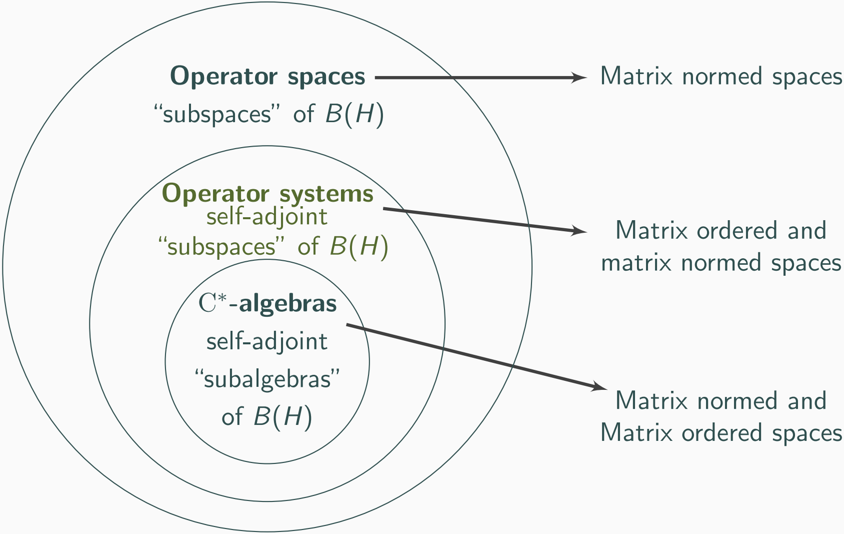
For a Hilbert Space H,



Abstract operator space

A normed space V with a sequence of norm $\|\cdot\|_n : M_n(V) \to [0,\infty) : n \in \mathbb{N}$ (known as *matrix norm*) is said to be an *abstract operator space*; if the following conditions are satisfied:

1. $\|v \oplus w\|_{n+m} \leq \max\{\|v\|_n, \|w\|_m\}, v \in M_n(V)$ and $w \in M_m(V)$;
2. $\|\alpha v \beta\|_n \leqslant \|\alpha\|\|v\|_n\|\beta\| \ \forall \ \alpha \in M_{m,n}, \ \beta \in M_{m,n}, \ v \in M_n(V)$.

- $\phi : V \to W$ between operator spaces V and W is said to be *completely bounded* (abbreviated as c.b.) if

$$\|\phi\|_{cb} := \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty,$$

  where $\phi_n : M_n(V) \to M_n(W)$ is defined by

$$\phi_n((x_{ij})) = (\phi(x_{ij})) \text{ for all } (x_{ij}) \in M_n(V).$$

- ϕ is a complete isometry if each ϕ_n is so

Ruan(1988)

If V is an abstract operator space , then V is completely isometrically isomorphic to a linear subspace $B(\mathcal{H})$ for some $\mathcal{H}$.

As in the case of operator spaces, one can consider an abstract definition independent of associated Hilbert space

Definition

An Abstract operator system is a triple $\{V, \{\mathcal{C}_n\}_{n=1}^{\infty}, e)\}$, where V is a complex $*$-vector space, $\{\mathcal{C}_n\}_{n=1}^{\infty}$ is a matrix ordering on V, and $e \in V_h$ is an Archimedean matrix order unit.

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- $\mathcal{C}_n \subset M_n(V)_h$ is a cone
- $\mathcal{C}_n \cap (-\mathcal{C}_n) = \{0\}$
- $A \in M_{m,n} \implies A\mathcal{C}_n A^* \subset \mathcal{C}_m$.
- $e_n := diag(e, e, \ldots, e) \in M_n(V)_h$ satisfies:

$$x \in M_n(V)_h \implies re_n - x \in \mathcal{C}_n \quad \text{for some} \quad r > 0, \quad \text{and}$$

$$te_n + x \in \mathcal{C}_n \text{ for all } t > 0 \implies x \in \mathcal{C}_n$$

- Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A linear map $\phi : \mathcal{S} \to \mathcal{T}$ is said to be **completely positive** provided

$$(\phi(s_{ij})) \in M_n(\mathcal{T})^+ \text{ for all } (s_{ij}) \in M_n(\mathcal{S})^+$$

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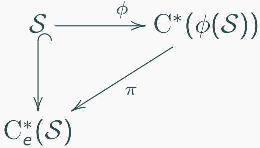
Theorem (Choi-Effros(1977))

If $(V, \{\mathcal{C}_n\}_{n=1}^{\infty}, e)$ is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$, a concrete operator system $\mathcal{S} \subseteq B(\mathcal{H})$, and a complete order isomorphism $\phi : V \to \mathcal{S}$ with $\phi(e) = I$.

Thus one can identify abstract and concrete operator systems and refer to them simply as operator systems.

$\mathrm{C}^*_e(\mathcal{S})$: C^*-envelope of $\mathcal{S}$ (Arveson 1969, Hamana 1979)

The C^*-**envelope** $\mathrm{C}^*_e(\mathcal{S})$ is the C^*-algebra generated by an isomorphic copy of $\mathcal{S}$ that enjoys the following universal "minimality" property: For every isomorphic copy $\phi(\mathcal{S})$ of $\mathcal{S}$, there is a unique surjective unital $*$-homomorphism $\pi : \mathrm{C}^*(\phi(\mathcal{S})) \to \mathrm{C}^*_e(\mathcal{S})$ such that $\pi(\phi(s)) = s$ for every s in $\mathcal{S}$, i.e. the following diagram commutes:



C^*-covers

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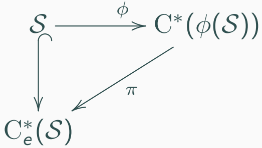

$\mathrm{C}_u^*(\mathcal{S})$: Universal C^*-cover of $\mathcal{S}$ (Kirchberg-Wasserman 1998)

The maximal C^*-cover generated by $\mathcal{S}$.

Definition

Given operator system $(\mathcal{S}, \{\mathcal{P}_n\}_{n=1}^{\infty}, e_1)$ and $(\mathcal{T}, \{\mathcal{Q}_n\}_{n=1}^{\infty}, e_2)$, an **operator system structure** on $\mathcal{S} \otimes \mathcal{T}$ is a family $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of cones, where $\mathcal{C}_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying :

- **(T1)** $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n\}_{n=1}^{\infty}, e_1 \otimes e_2)$ is an operator system,
- **(T2)** $\mathcal{P}_n \otimes \mathcal{Q}_m \subseteq \mathcal{C}_{nm}$ for all $n, m \in \mathbb{N}$, and
- **(T3)** If $\phi : \mathcal{S} \to \mathbb{M}_n$ and $\psi : \mathcal{T} \to \mathbb{M}_m$ are unital completely positive maps, then $\phi \otimes \psi : \mathcal{S} \otimes \mathcal{T} \to \mathbb{M}_{mn}$ is a unital completely positive map.

By **an operator system tensor product** we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, denoted $\mathcal{S} \otimes_\tau \mathcal{T}$.

Motivated by the Choi-Effros' characterization:

KPTT 2011

A lattice of functorial partially ordered tensor products were introduced:

$$\mathrm{min} \leq \mathrm{e} \leq \mathrm{er}, \mathrm{el} \leq \mathrm{c} \leq \mathrm{max}.$$

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- *minimal* (**min**): $\mathcal{S} \otimes_{\text{min}} \mathcal{T} \subset B(\mathcal{H}) \otimes_{\mathrm{C}^*\text{-min}} B(\mathcal{K})$
- *maximal* (**max**): $\mathcal{S} \otimes_{\text{max}} \mathcal{T} \subset A \otimes_{\mathrm{C}^*\text{-max}} B$
- *maximal commuting* (**c**): $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T} \subset \mathrm{C}^*_u(\mathcal{S}) \otimes_{\text{max}} \mathrm{C}^*_u(\mathcal{T})$
- *enveloping* (**e**): $\mathcal{S} \otimes_{\mathrm{e}} \mathcal{T} \subset I(\mathcal{S}) \otimes_{\text{max}} I(\mathcal{T})$
- *left enveloping* (**el**): $\mathcal{S} \otimes_{\mathrm{e}} \mathcal{T} \subset I(\mathcal{S}) \otimes_{\text{max}} \mathcal{T}$
- *right enveloping* (**er**): $\mathcal{S} \otimes_{\mathrm{e}} \mathcal{T} \subset \mathcal{S} \otimes_{\text{max}} I(\mathcal{T})$.

ess-tensor product (FKPT 2014)

$$\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{T} \subset \mathrm{C}_e^*(\mathcal{S}) \otimes_{\mathsf{max}} \mathrm{C}_e^*(\mathcal{T})$$

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Proposition (GL 16)

For any two unital C^*-algebras A and B, we have

$$A \otimes_{\mathrm{ess}} B = A \otimes_{\mathrm{c}} B = A \otimes_{\max} B.$$

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Proposition (GL 16)

ess is symmetric non-functorial, and hence is different from all known tensor products.

(α, β)-nuclear (KPTT 2013)

The identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$.

Unlike C^*-algebra there are several notions of nuclearity.

Properties	Operator systems	Equivalent Nuclearity
Exactness	K-P-T-T 2013	(min,el)-nuclear
osLLP	K-P-T-T 2013	(min,er)-nuclear
DCEP	K-P-T-T 2013	(el,c)-nuclear
WEP	K-P-T-T 2013 - Han 2011	(el,max)-nuclear
C^*-nuclearity	Kavruk 2014	(min,c)-nuclearity
CPFP	Han-Paulsen 2011	(min,max)-nuclear

Table: Structural properties and equivalent nuclearities for operator systems

Properties	**C^*-algebra**	**Operator spaces**	**Operator systems**
Exactness	Kirchberg 1978	Pisier 1995	KPTT 2013
Local Lifitng Property	Kirchberg 1993	Kye-Ruan 1999	KPTT 2013
WEP	Lance 1982	Pisier 2003	KPTT 2013-Han 2011
DCEP	Arveson 1969	Paulsen 2011	KPTT 2013
CPFP	Choi-Effros 1975-Kirchberg 1977	Kirchberg 1995	Han-Paulsen 2011
Nuclearity	Takesaki 1964-Lance 1973	Does not ex-end	Kavruk 2014

Operator system from discrete group (FKPT 2014)

Given a countable discrete group G and generating set $\mathfrak{u}$ of G,

$$\mathcal{S}(\mathfrak{u}) := \mathrm{span}\{1, u, u^* : \ u \in \mathfrak{u}\} \subset \mathrm{C}^*(G),$$

where $\mathrm{C}^*(G)$ is the full group C^*-algebra of the group G and $u \in G$ is identified with $\delta_u \in \mathrm{C}^*(G)$.

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For $\mathbb{F}_n$, the free group with n-generators
$\rightsquigarrow \mathcal{S}(\mathfrak{u})$ is a $(2n+1)$-dimensional operator system independent of the generating set $\mathfrak{u}$
$\rightsquigarrow \mathcal{S}_n \subset \mathrm{C}^*(\mathbb{F}_n)$.
In general, such independence is not expected.

Lance 1973

For a discrete group G, $\mathrm{C}^*(G)$ is nuclear $\iff G$ is amenable.

Lance 1973

For a discrete group G, $\mathrm{C}^*(G)$ is nuclear $\iff$ G is amenable.

Question: : What kind of nuclearity is observed by $\mathcal{S}(\mathfrak{u})$ if G is amenable?
$\rightsquigarrow$ $\mathrm{C}^*_e(\mathcal{S}(\mathfrak{u})) = \mathrm{C}^*(G)$! (Kavruk, 2014)

More general question

Question: : What kind of nuclearity is observed by an operator system $\mathcal{S}$ if its C^*-envelope $\mathrm{C}^*_e(\mathcal{S})$ is nuclear?

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- Kirchberg and Wassermann 1998 gave an example of a $(\min, \max)$-nuclear operator system whose C^*-envelope is non-nuclear.

More general question

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- Kirchberg and Wassermann 1998 gave an example of a $(\min, \max)$-nuclear operator system whose C^*-envelope is non-nuclear.
- The other direction is equally mysterious.

Theorem (GL 16)

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The exhaustive list of nuclear group operator systems.

Theorem (GL 16)

$\mathfrak{u}$ be a minimal generating set of a finitely generated group G.
$\mathcal{S}(\mathfrak{u})$ is $(\min, \max)$-nuclear if and only if $|G| \leq 3$.

Graph operator systems (KPTT 2011)

Given a finite graph G with n-vertices,

$$\mathcal{S}_G = \mathrm{span}\{\{E_{i,j} : (i,j) \in G\} \cup \{E_{i,i} : 1 \leq i \leq n\}\} \subseteq M_n(\mathbb{C}),$$

where $\{E_{i,j}\}$ is the standard system of matrix units in $M_n(\mathbb{C})$ and (i,j) denotes (an unordered) edge in G.

Nuclearity of graph operator systems

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Theorem (GL 16)

$\mathcal{S}_G$ is $(\min, \max)$-nuclear if and only if each component of G is complete.

Kirchberg 2000

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Question: Are separable exact operator system embeddable in $\mathcal{O}_2$?

Embeddings of exact operator systems

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Answer NO!

Kirchberg-Wasserman (1998) and Lupini (2014) constructed examples that are exact but non-embeddable in $\mathcal{O}_2$.

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Question: Under what conditions operator systems are embeddable in $\mathcal{O}_2$?

PZ 2016

For the generators $s_1, s_2, \cdots, s_n$ $(n \geq 2)$ of the Cuntz algebra $\mathcal{O}_n$ and identity I, **the Cuntz operator system**

$$\mathcal{S}_{\mathcal{O}_n} := span\{I, s_1, s_2, \cdots, s_n, s_1^*, s_2^*, \cdots, s_n^*\} \subset \mathcal{O}_n.$$

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- For $s_1, s_2, \cdots$ in $\mathcal{O}_\infty \rightsquigarrow$
  $\mathcal{S}_{\mathcal{O}_\infty} = span\{I, s_1, s_2, \cdots, s_1^*, s_2^*, \cdots\} \subset \mathcal{O}_\infty$.

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- For $s_1, s_2, \cdots$ in $\mathcal{O}_\infty \rightsquigarrow$ $\mathcal{S}_{\mathcal{O}_\infty} = span\{I, s_1, s_2, \cdots, s_1^*, s_2^*, \cdots\} \subset \mathcal{O}_\infty$.
- If an operator system $\mathcal{S}$ has a simple C^*-cover (A, i) then $A \cong \mathrm{C}_e^*(\mathcal{S})$.

Theorem (LK 17)

Let $\mathcal{S}$ be a separable operator system with C^*-envelope. Then there exists a unital complete order embedding of $\mathcal{S}$ into $\mathcal{O}_2$ if and only if $\mathrm{C}^*_e(\mathcal{S})$ is exact.

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Corollary (LK 17)

For any simple, unital, separable and nuclear C^*-algebra A, we have $\mathrm{C}^*_e(A \otimes_{\text{min}=\max} \mathcal{S}_{\mathcal{O}_2}) \cong \mathcal{O}_2$.

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(LK 17)

For any unital, simple, nuclear, separable and purely infinite C^*-algebra A, $\mathrm{C}^*_e(A \otimes_{\mathsf{min}=\max} \mathcal{S}_\infty) \cong A$.

Theorem (KL 18)

Let $\{\mathcal{S}_i\}_{i=1}^{\infty}$ be an increasing collection of operator systems such that that the complete order embedding $\mathcal{S}_i \subset \mathcal{S}_{i+1}$ extends to a $*$-homomorphism, then we have

$$\mathrm{C}_e^*(\varinjlim \mathcal{S}_i) = \varinjlim \mathrm{C}_e^*(\mathcal{S}_i).$$

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$$C_e^*(\varinjlim \mathcal{S}_i) = \varinjlim C_e^*(\mathcal{S}_i).$$

And, moreover if each $\mathcal{S}_i$ is separable, exact and contains enough unitaries of $C_e^*(\mathcal{S}_i)$, then $\varinjlim \mathcal{S}_i$ embeds into $\mathcal{O}_2$.

Theorem (LKR 18)

Let $(\mathcal{S}, \{M_n(\mathcal{S})^+\}_{n=1}^{\infty}, e_{\mathcal{S}})$ *and* $(\mathcal{T}, \{M_n(\mathcal{T})^+\}_{n=1}^{\infty}, e_{\mathcal{T}})$ *be operator systems. The family* $\{\mathcal{C}_n\}_{n=1}^{\infty}$ *defined as*

$$\mathcal{C}_n = \{\alpha \otimes_{\lambda_j} (v_1, v_2, \cdots, v_m)\alpha^* \mid v_t \in M_j(V_t)^+, \alpha \in M_{n,\tau(j)}, j \in \mathbb{N}, t = 1, 2, \cdots,$$

satisfying $(\mathrm{O1}) - (\mathrm{O3})$, *is a matrix ordering on* $\mathcal{S} \otimes \mathcal{T}$ *with order unit* $e_{\mathcal{S}} \otimes e_{\mathcal{T}}$.

Definition

Let $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ fulfills conditions $(\mathrm{O1}) - (\mathrm{O3})$, and let

$$\mathcal{C}_n^{\lambda} := \{P \in M_n(\mathcal{S} \otimes \mathcal{T}) \;|\; r(e_{\mathcal{S}} \otimes e_{\mathcal{T}})_n + P \in \mathcal{C}_n, \quad \text{for all } r > 0\}$$

be the Archimedeanization of the matrix ordering $\mathcal{C}_n$ for all $n \geq 1$. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^{\lambda}\}_{n=1}^{\infty}, e_{\mathcal{S}} \otimes e_{\mathcal{T}})$ the λ- *operator system tensor product* of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\lambda} \mathcal{T}$.

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Theorem (LKR 18)

The mapping $\lambda : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *sending* $(\mathcal{S}, \mathcal{T})$ *to* $\mathcal{S} \otimes_{\lambda} \mathcal{T}$ *is a functorial operator system tensor product.*

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Theorem (AKL 18)

$$\varinjlim(\mathcal{S}_k \otimes_{\lambda} \mathcal{T}) \overset{c.o.i.}{\cong} (\varinjlim \mathcal{S}_k) \otimes_{\lambda} \mathcal{T}.$$

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Thank you for your kind attention!

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