Differentiability of shape functions for directed polymers in continuous space

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joint work with Douglas Dow

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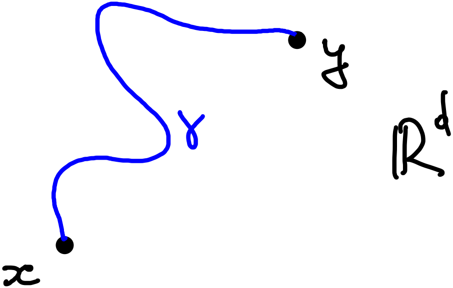
Optimal paths in random environments (zero temperature)



$$A_\omega(x, y) = \inf \left\{ A_\omega(\gamma) \;\middle|\; \text{admissible } \gamma : \; x \rightsquigarrow y \right\}$$

Shape function

$$\Lambda(v) = \lim_{T \to \infty} \frac{1}{T} A_\omega(0, Tv)$$

or

$$\Lambda(v) = \lim_{n \to \infty} \frac{1}{n} A_\omega(0, x_n), \qquad \frac{x_n}{n} \to v.$$

Exact shape functions (including the temperature > 0 case)

- Corner growth, i.i.d. exponential weights: Rost (1981)
- (generalized) Hammersley process: Hammersley (1972), Aldous, Diaconis (1995), Cator, Pimentel (2011)
- Euclidean FPP: Howard, Newman (1997)
- O'Connell–Yor polymers: Baryshnikov (2001), Gravner, Tracy, Widom (2001), Hambly, Martin, O'Connell (2002), Moriarty, O'Connell (2007)
- Log-gamma polymers: Seppäläinen (2012)
- Burgers equation, quadratic L: Bakhtin, Cator, Khanin (2014), Bakhtin (2016), Bakhtin, Li (2019)
- KPZ equation: Janjigian, Rassoul-Agha, Seppäläinen (recent)

Shape function

- Always convex
- In all explicit examples, differentiable and strictly convex
- Strict convexity would imply existence-uniqueness of one-sided geodesics and infinite volume polymer measures (thermodynamic limits) with a given slope, and even just differentiability allows to make pretty strong claims in that direction [Janjigian, Rassoul-Agha, Seppäläinen 2020,2022]
- Differentiability at the edge of the percolation cone for lattice FPP, LPP [Auffinger, Damron 2013].

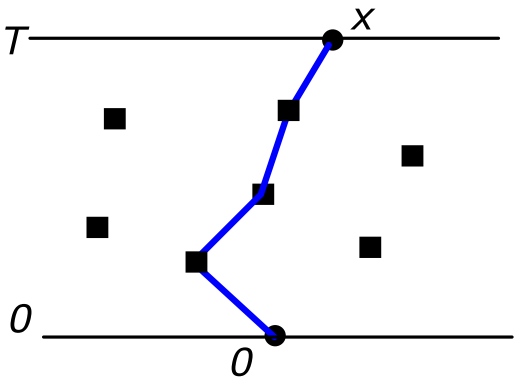
This talk:

Several classes of models where the shape function is not known precisely but differentiability holds:

- continuous space directed polymers
  - zero temperature
  - positive temperature
- homogenization in HJB equations with dynamic environments.

Shear invariant case (Burgers/KPZ type models)

Poissonian points in space-time $\mathbb{R} \times \mathbb{R}$:



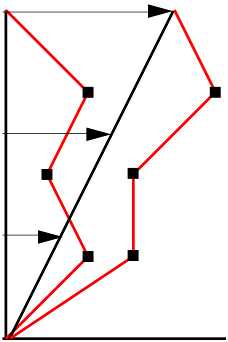
$$A^{0,T}(\gamma) = \int_0^T \dot{\gamma}(t)^2 dt - \#\{\text{Poisson points on } \gamma\}$$

[Bakhtin, Cator, Khanin 2014]

Shear invariant case (Burgers/KPZ type models)

$$A^{0,T}(\gamma) = \int_0^T \dot\gamma(t)^2 dt - \#\{\text{Poisson points on } \gamma\}$$

If $\gamma(0) = \gamma(T) = 0$ and $\Xi_v\gamma(t) = \gamma(t) + vt$, then



$$\int_0^T \left(\frac{d}{dt}\Xi_v\gamma(t)\right)^2 dt$$

$$= \int_0^T (\dot\gamma(t) + v)^2 dt$$

$$= \int_0^T \dot\gamma(t)^2 dt + 2v \int_0^T \dot\gamma(t) dt + \int_0^T v^2 dt$$

$$= \int_0^T \dot\gamma(t)^2 dt + Tv^2.$$

PPP is distributionally invariant under Ξ_v, so $\Lambda(v) = \Lambda(0) + v^2$.

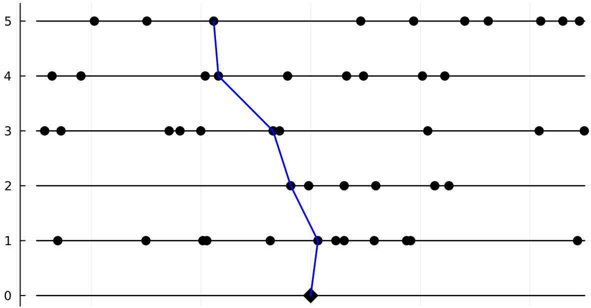
Several other shear invariant models

$$A(\gamma) = \sum_k (\gamma_{k+1} - \gamma_k)^2 + \sum_k F_k(\gamma_k),$$

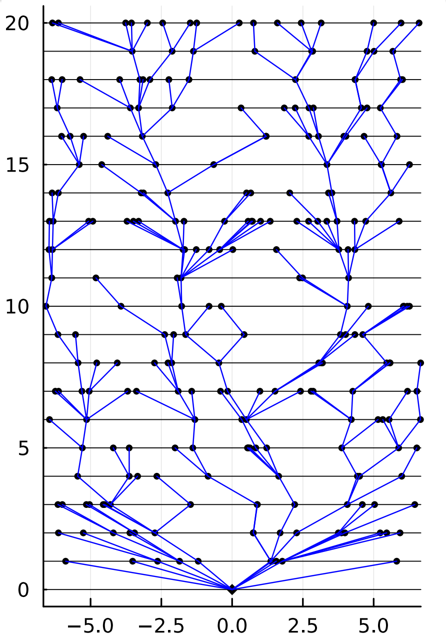
where F_k are i.i.d. in time and stationary in space [Bakhtin, 2016], or simply

$$A(\gamma) = \sum_k (\gamma_{k+1} - \gamma_k)^2$$

but require γ_k to coincide with one of the Poisson points on $\{k\} \times \mathbb{R}$.



Trees of minimizers

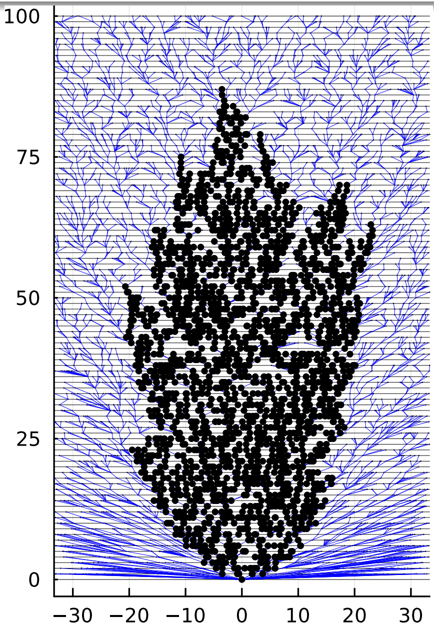


paths minimizing

$$A = \sum (\gamma_{k+1} - \gamma_k)^2$$

for various endpoints.

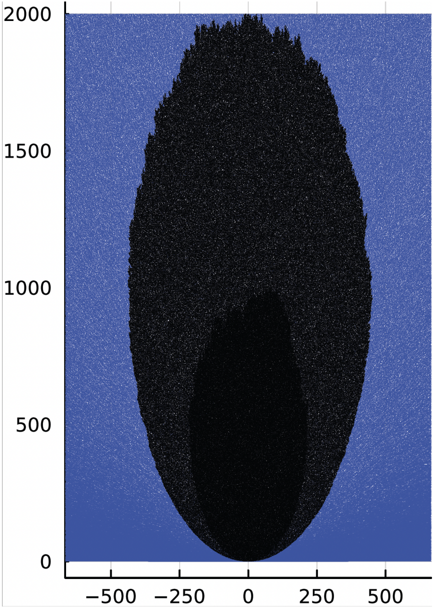
Limit shape



A point is shown in black if there is a path to that point with

$$A = \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})^2 < \mathbf{20}$$

Limit shape



A point is shown in black if there is a path to that point with

$$A = \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})^2 < \mathbf{400}$$

The boundary of the limit shape is an ellipse

$$x^2 + C\left(t - \frac{1}{2C}\right)^2 = \frac{1}{4C}$$

with $C = \Lambda(0)$.

General (nonquadratic) action

$$A(\gamma) = \sum_k L(\Delta_k \gamma) + \sum_k F_k(\gamma_k) \qquad (\Delta_k \gamma = \gamma_{k+1} - \gamma_k)$$

Theorem [with Douglas Dow] Assume that

- F is i.i.d. in time, stationary in space, continuous, bounded from below, $\mathbb{E}\, F(0) < \infty$;
- $L \in C^2$, $\lim_{|v|\to\infty} L(v) = +\infty$, $\limsup_{|v|\to\infty} \frac{L''(v)}{L(v)} < \infty$.

  (Doesn't have to be convex, e.g. $L(v) = v^{2p} + \sum_{k=0}^{2p-1} a_k v^k$)

Then there is a deterministic, convex, and **differentiable** shape function Λ: for each $v \in \mathbb{R}$, with probability 1,

$$\Lambda(v) = \lim_{n\to\infty} \frac{1}{n} A^n(0, nv),$$

$$\Lambda'(v) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} L'(\Delta_k \gamma^n(v)),$$

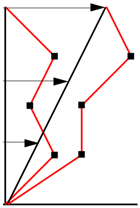
where $\gamma^n(v)$ realizes $A^n(0, nv) = \inf\{A(\gamma) |\ \gamma_0 = 0,\ \gamma_n = nv\}$.

Using shear $\Xi_v(t,x) = (t, x+tv)$

For a path γ with $\gamma_0 = \gamma_n = 0$

$$B(v,\gamma) = \sum_k L(\Delta_k \gamma + v) + \sum_k F_k(\gamma_k)$$

$$B^n(v) = \inf\{B(v,\gamma) \mid \gamma_0 = \gamma_n = 0\}$$



Since $\Xi_v F = \Xi_{-v} F = F$ in distribution,

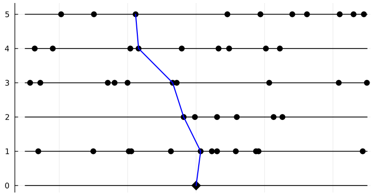
$$(B^n(v))_{n\in\mathbb{N}} \stackrel{d}{=} (A^n(0,nv))_{n\in\mathbb{N}},$$

so

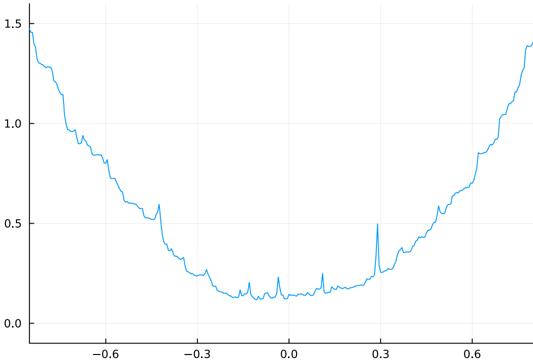
$$\Lambda(v) = \lim_{n\to\infty} \frac{1}{n} A^n(0,nv) = \lim_{n\to\infty} \frac{1}{n} B^n(v).$$

In addition, $(B^n(v))_{v\in\mathbb{R}}$ is “nicer” than $(A^n(0,nv))_{v\in\mathbb{R}}$

Poisson points model with $A(\gamma) = \sum_k (\Delta_k \gamma)^4$



$$\frac{1}{n} A^n(0, nv) \text{ for } n = 200$$



Poisson points model with $A(\gamma) = \sum_k (\Delta_k \gamma)^4$

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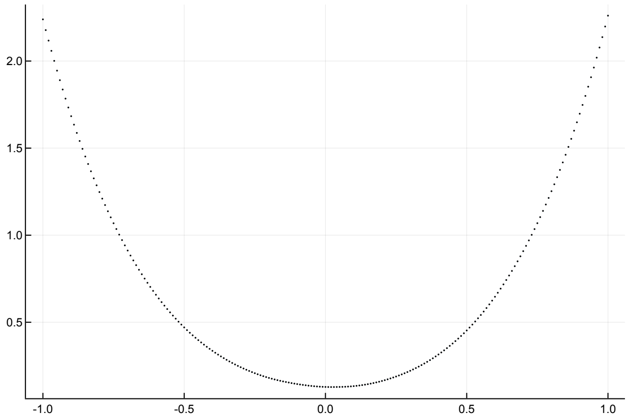
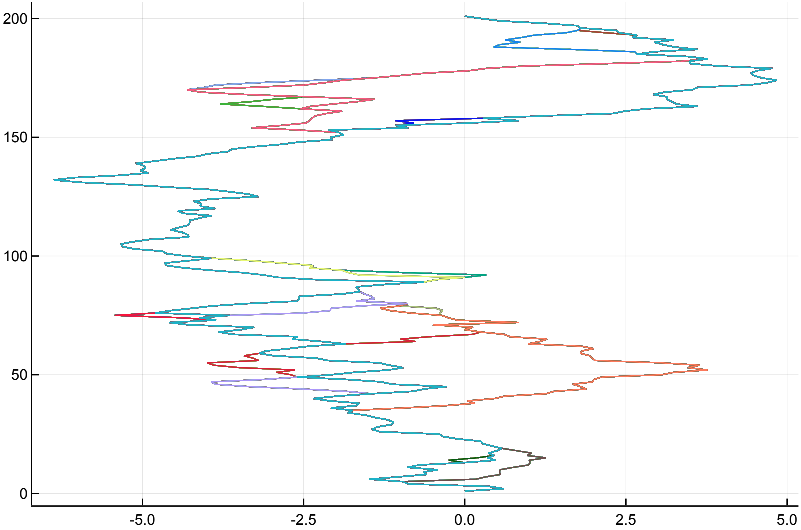


Poisson points model with $A(\gamma) = \sum_k (\Delta_k \gamma)^4$

Optimal paths realizing $B^n(v)$, $v \in [-1, 1]$.



Each path serves a range of values of v.

Proof of differentiability

Use $\gamma^n(v)$, the optimal path for $B^n(v)$ to estimate $B^n(w)$ for $w \approx v$.

$$B^n(w) \le B^n(w, \gamma^n(v))$$

$$\le B^n(v) + (w - v) \sum_{k=0}^{n-1} L'(\Delta_k \gamma(v) + v)$$

$$+ \frac{(w-v)^2}{2} \sum_{k=0}^{n-1} L''\big(\Delta_k \gamma(v) + v + s(w - v)\big)$$

For $w - v > 0$,

$$\frac{B^n(w) - B^n(v)}{n(w-v)} \le \frac{1}{n} \sum_{k=0}^{n-1} L'(\Delta_k \gamma(v) + v) + C(w - v)$$

$$\frac{\Lambda(w) - \Lambda(v)}{w - v} \le \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} L'(\Delta_k \gamma(v) + v) + C(w - v)$$

$$\partial^+ \Lambda(v) \le \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} L'(\Delta_k \gamma(v) + v).$$

Proof of differentiability

Use $\gamma^n(v)$, the optimal path for $B^n(v)$ to estimate $B^n(w)$ for $w \approx v$.

$$
\begin{aligned}
B^n(w) &\le B^n(w, \gamma^n(v)) \\
&\le B^n(v) + (w-v)\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v) \\
&\qquad + \frac{(w-v)^2}{2}\sum_{k=0}^{n-1} L''\big(\Delta_k\gamma(v)+v+s(w-v)\big)
\end{aligned}
$$

For $w - v < 0$,

$$\frac{B^n(w)-B^n(v)}{n(w-v)} \ge \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v) + C|w-v|$$

$$\frac{\Lambda(w)-\Lambda(v)}{w-v} \ge \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v) + C|w-v|$$

$$\partial^-\Lambda(v) \ge \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v).$$

Proof of differentiability

So

$$\partial^+\Lambda(v) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v)$$

$$\le \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v) \le \partial^-\Lambda(v).$$

But Λ is convex, so

$$\partial^-\Lambda(v) \le \partial^+\Lambda(v).$$

Therefore,

$$\partial^-\Lambda(v) = \partial^+\Lambda(v) = \Lambda'(v) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v)+v)$$

In terms of the optimal path from $(0,0)$ to (n, nv):

$$\Lambda'(v) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma(v))$$

Doesn't quite imply strict convexity

$$\Lambda'(v) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} L'(\Delta_k \gamma(v)) = \langle L'(\Delta\gamma(v))\rangle$$

For example, if $L(v) = v^4$, then $L'(v) = 4v^3$,

$$\Lambda'(v) = 4\langle(\Delta\gamma(v))^3\rangle.$$

It is natural to expect this to strictly grow in v (strict convexity) because

$$\langle\Delta\gamma(v)\rangle = v.$$

But the first moment does not control the third moment, so this is not obvious.

Positive temperature case. Average free energy.

$$Z^n(y) = \int \exp\Big[-A(\gamma)\Big]\, \delta_0(d\gamma_0)d\gamma_1 \ldots d\gamma_{n-1}\delta_y(d\gamma_n)$$

$$= \int \exp\Big[-\sum_{k=0}^{n-1} L(\Delta_k\gamma) - \sum_{k=0}^{n-1} F_k(\gamma_k)\Big]\delta_0(d\gamma_0)d\gamma_1 \ldots d\gamma_{n-1}\delta_y(d\gamma_n)$$

Need more requirements: $\mathbb{E}\sup_{|x|\le 1/2} F_k(x) < \infty,$

$$\liminf_{|v|\to\infty} |L'(v)| > 0, \quad \limsup_{|v|\to\infty} \frac{|L'(v)|}{|L(v)|^\theta} < \infty \quad \text{for some } \theta \in (0,1).$$

Theorem [with Douglas Dow] There is a deterministic, convex, **differentiable** $\Lambda : \mathbb{R} \to \mathbb{R}$ s.t. for every $v \in \mathbb{R}$, with prob. 1,

$$\Lambda(v) = -\lim_{n\to\infty} \frac{1}{n} \log Z^n(nv),$$

$$\Lambda'(v) = \lim_{n\to\infty} \mu^n_{nv}\Big(\frac{1}{n}\sum_{k=0}^{n-1} L'(\Delta_k\gamma)\Big)$$

(μ^n_{nv} is the polymer measure on paths connecting $(0,0)$ to (n, nv))

The main estimate (after applying the shear)

All paths connect 0 to 0 (not 0 to Tv)

$$\log \tilde{Z}^n(w) = \log \int \exp\Big(-\sum_k \Big[F_k(\gamma_k) + L(\Delta_k\gamma + w)\Big]\Big) \mathrm{d}\gamma$$

$$\geq \log \int e^{-\sum_k \left[F_k(\gamma_k)+L(\Delta_k\gamma+v)+(w-v)L'(\Delta_k\gamma+v)+\frac{1}{2}(w-v)^2L''(\dots)\right]} \mathrm{d}\gamma$$

$$= \log \tilde{Z}^n(v)$$

$$+ \log \frac{1}{\tilde{Z}^n(v)} \int e^{-\sum_k \left[F_k(\gamma_k)+L(\Delta_k\gamma+v)+(w-v)L'(\Delta_k\gamma+v)+\frac{1}{2}(w-v)^2L''(\dots)\right]} \mathrm{d}\gamma$$

$$= \log \tilde{Z}^n(v) + \log \tilde{\mu}^n_v\Big(e^{-\sum_k \left[(w-v)L'(\Delta_k\gamma+v)+\frac{1}{2}(w-v)^2L''(\dots)\right]}\Big)$$

$$\geq \log \tilde{Z}^n(v) - (w-v)\tilde{\mu}^n_v\Big(\sum_{k=0}^{n-1} L'(\Delta_k\gamma + v)\Big) - \frac{1}{2}(w-v)^2\tilde{\mu}^n_v\Big(\dots\Big)$$

Continuous time, HJB eqns, non-white noise, in $\mathbb{R}^d$, $d \geq 1$

$$A^t(\gamma) = \int_0^t L(\dot{\gamma}_s)ds + \int_0^t F(s, \gamma_s)ds$$

- We no longer assume F is white it time:

$$F(t,x) = \sum_i \varphi_i(t-t_i, x-x_i) = \int_{\mathbb{R}\times\mathbb{R}^d\times\mathcal{C}} \varphi(t-s, x-y)\mathbf{N}(ds, dy, d\varphi)$$

  (t_i, x_i) are Poisson points in $\mathbb{R} \times \mathbb{R}^d$, each convolved with its own random φ_i (i.i.d., C^2, uniformly bounded support, an exp-moment)
- $L : \mathbb{R}^d \to \mathbb{R}$ convex, twice differentiable,

$$\lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty$$

$$\limsup_{|v|\to\infty} \sup_{|r|\leq\delta} \frac{\|\nabla^2 L(v+r)\|}{L(v)} < \infty \quad \text{for some } \delta > 0.$$

Continuous time, HJB eqns, non-white noise, in $\mathbb{R}^d$, $d \geq 1$

$$A^t(\gamma) = \int_0^t L(\dot{\gamma}_s)ds + \int_0^t F(s, \gamma_s)ds$$

$$A(t,x) = \inf \left\{A^t(\gamma) \,:\, \gamma(0) = 0,\ \gamma(t) = x\right\}$$

$$\partial_t A(t,x) + H(\nabla A(t,x)) = F(t,x), \quad t \in (0,\infty),\ x \in \mathbb{R}^d$$

The Hamiltonian H is the Legendre–Fenchel transform of L

$$H(p) = \sup_{v \in \mathbb{R}^d} \{\langle p, v\rangle - L(v)\}, \quad p \in \mathbb{R}^d$$

$$\lim_{t \searrow 0} A(t,x) = \begin{cases} 0, & x = 0, \\ +\infty, & x \neq 0. \end{cases}$$

Theorem [with Douglas Dow]

Under these conditions, there is a convex deterministic and **differentiable** function $\Lambda : \mathbb{R}^d \to \mathbb{R}$ such that for every $v \in \mathbb{R}^d$, with probability 1,

$$\Lambda(v) = \lim_{t\to\infty} \frac{1}{T} A(T, Tv)$$

$$\nabla\Lambda(v) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Big[\nabla L(\dot{\gamma}_t^T(v)) + \Theta(t, \gamma_t^T(v))\Big] dt,$$

where

$$\begin{aligned}\Theta(t,x) &= \int (t-s)\nabla\varphi(t-s, x-y)\mathbf{N}(ds, dy, d\varphi) \\ &= \partial_v \int \varphi(t-s, x-y+v(t-s))\mathbf{N}(ds, dy, d\varphi)\Big|_{v=0}\end{aligned}$$

Shape Theorem. HJB Homogenization Version.

Corollary

For every $t \in (0, \infty)$ and $x \in \mathbb{R}^d$, with probability 1,

$$\lim_{\epsilon \searrow 0} \epsilon A(t/\epsilon, x/\epsilon) = t\Lambda(x/t)$$

The nonrandom function $\overline{U}(t,x) = t\Lambda(x/t)$ is the fundamental viscosity solution of the deterministic HJB equation

$$\partial_t \overline{U}(t,x) + \overline{H}(\nabla \overline{U}(t,x)) = 0,$$

where $\overline{H}$ is the Legendre–Fenchel transform of Λ:

$$\overline{H}(p) = \sup_{v \in \mathbb{R}^d} \{\langle v, p \rangle - \Lambda(v)\}, \quad p \in \mathbb{R}^d.$$

Moreover, $\overline{H}$ is **strictly convex** (no flat edges), and $\overline{U}(t,x)$ is a **classical** solution which is C^1 for all $t > 0$, $x \in \mathbb{R}^d$.

Existing homogenization results for dynamic environments

Zero viscosity HJB:

- Schwab 2009
- Bakhtin, Cator, Khanin 2014, Bakhtin 2016 (for quadratic L in the context of Burgers equation),
- Seeger 2021

Positive viscosity HJB:

- Kosygina, Varadhan 2008
- Jing, Souganidis, Tran 2017
- Bakhtin, Li 2019 (Burgers equation, quadratic L)
- Janjigian, Rassoul-Agha, Seppäläinen (KPZ eqn, quadratic L)

Further questions

- continuous FPP with asymmetries
- (parabolic) HJB with positive viscosity/diffusion
- More general potentials (we still need shear invariance for the background process)?
- Does this say anything about lattice models?
- Strict convexity? Use our formulas for $\nabla\Lambda$?