Atypical behaviors of a tagged particle in ASEP

Sunder Sethuraman (U. Arizona)
SRS Varadhan (NYU)

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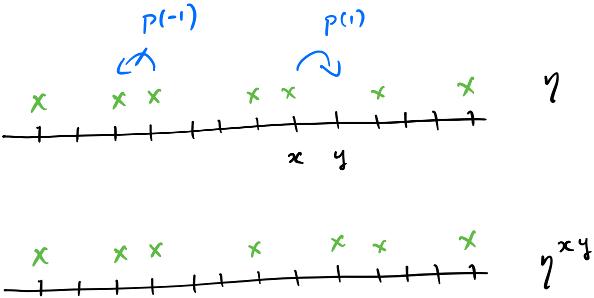
Random growth models and KPZ universality

Simple exclusion in 1D

We consider the simple exclusion process on $\mathbb{Z}$ consisting of a collection of continuous time RW's, with *nearest-neighbor* jump rates $p(\pm 1)$ going from x to $x \pm 1$, where jumps to occupied locations are suppressed.

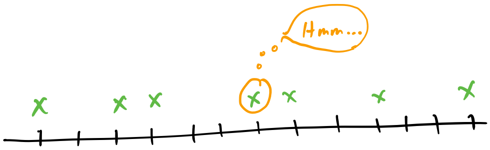


Although the particles are not labeled, it is a natural problem to 'tag' say one of them, and to follow its motion $\{X_t\}_{t\geq 0}$.

–This motion is undoubtedly influenced by the other particles, including 'bulk' mass notions,

but it does have a mind of its own,

especially in empty space.



Notation

Let

$$\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$$

be the configuration of the process at time t where

$$\eta(x) = \begin{cases} 1 & \text{if } x \text{ occupied} \\ 0 & \text{otherwise.} \end{cases}$$

The process is Markovian with generator

$$Lf(\eta) = \sum_{x,\pm} p(\pm 1)\eta(x)(1 - \eta(x \pm 1))\big\{f(\eta^{x,x\pm 1}) - f(\eta)\big\}.$$

TASEP $p(1) = 1$, $p(-1) = 0$.

ASEP $p = p(1) > p(-1) = 1 - p$.

SSEP $p(1) = p(-1) = 1/2$.

Stationary states

$$\nu_\rho = \prod_x \text{Bern}(\rho)$$

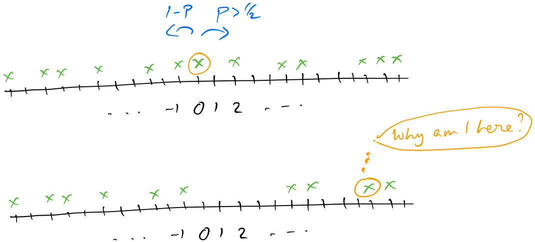
are invariant for $0 \leq \rho \leq 1$.

Question

What is the *typical* behavior of a tagged particle in ASEP,
starting from a stationary state,

when conditioned to deviate to an *atypical* position?

–That is, what is the likely structure of the process that allows a deviation?



Typical LLN behavior

Starting from $\nu_\rho(\cdot|\eta(0) = 1)$, with respect to ASEP, introducing a scaling factor N,

$$\frac{1}{N} X_{Nt} \to \gamma[1 - \rho]t,$$

as $N \uparrow \infty$, where $\gamma = p(1) - p(-1)$.

–implicit in the more general works,
Saada '87, Rezakhanlou '94.

Exact and approximate formulas

For TASEP, from queuing notions, e.g. Burke's theorem, starting from $\nu_\rho(\cdot|\eta(0) = 1)$,
X_t is *exactly* a Poisson process with rate $1 - \rho$ (!)

–However, in ASEP, Ferrari-Fontes showed an approximation:

$$X_t = Poiss(t) + \xi_t$$

where $Poiss(\cdot)$ is a Poisson process with rate $\gamma(1 - \rho)$
and ξ_t has a uniform in time bounded exponential moment,
$E\left[\exp\{\kappa(p(1), \rho)\xi_t\}\right] < \infty$.

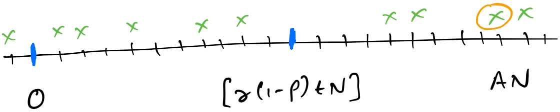
Question rephrased

Starting from $\nu_\rho(\cdot|\eta(0) = 1)$,

in ASEP, how does the system organize to achieve a deviation

$$\frac{1}{N} X_{Nt} \sim A \neq \gamma(1-\rho)t,$$

as $N \uparrow \infty$ for a fixed $t > 0$?



Rate function in TASEP

From the exact relation,
in TASEP starting from $\nu_\rho(\cdot|\eta(0) = 1)$, we get

$$P\left(\frac{1}{N}X_{Nt} \sim A\right) \sim \exp\left\{ -N\, I(A)\right\}$$

as $N \uparrow \infty$,
where

$$I(A) = A \log \frac{A}{(1-\rho)t} - A + (1-\rho)t.$$

–On the other hand, in ASEP, the Ferrari-Fontes relation gives a non-sharp large deviation upper estimate.

Main goals

We discuss strategies for lowerbounds, and also upperbounds in ASEP, which when they match identify the large deviation rate function in certain regions.

In particular, an ‘upper tail’ LDP is established.

–We will focus on TASEP to be concrete, but most of the arguments hold for ASEP as well. In TASEP, we have the advantage of knowing what cost to achieve, guiding some of the work.

Note: We often fix $t = 1$ to simplify notation. In this case, typically

$$\frac{1}{N} X_N \sim 1 - \rho.$$

Comment in SSEP

Before going further, we remark in SSEP, starting from stationary initial conditions, that large deviations are known:

$$P\left(X_N \sim \sqrt{N}A\right) \sim \exp\left\{-\sqrt{N}I(A)\right\}.$$

Here, the rate function I can be understood as a certain contraction of the rate function for the large deviations of the empirical measure (Diffusive scale).

A more explicit formula is also known.

–Derrida-Gerschenfeld 2009, SS-Varadhan 2013, Imamura-Mallick-Sasamoto 2017

References for the current

As the tagged particle has connection to local ‘currents’ (across a site), we mention a sample of the work in this vein.

—Upper and lower large deviations bounds for the current in TASEP (CGM) from deterministic initial condition.

Seppäläinen ’98, Olla-Tsai ’17

–Upper large deviations for the current in ASEP from wedge (step) initial condition.

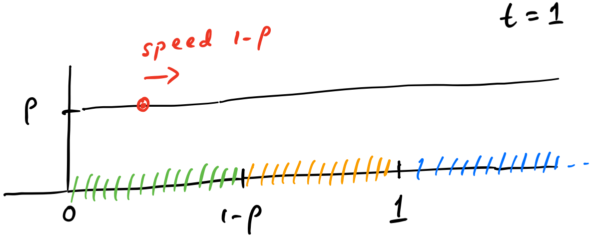
Damron-Petrov-Sivakoff ’18, Das-Zhu ’21

TASEP regimes

In TASEP, particles to the left of the origin do not interact with the tagged particle.

–There are three regimes:

- $A \geq 1$
- $1 - \rho \leq A \leq 1$
- $0 \leq A \leq 1 - \rho$



Sketch: $A \geq 1$

In this setting, the tagged particle *must* change its jump rate from the apriori rate 1 to something else to have a chance of reaching $A \geq 1$ in macro time $t = 1$.

–But, for the tagged particle to move, it must not be obstructed.

–Moving a macro amount of particles is quite costly.
An alternative is to *remove* some particles from the system.

–Such a cost would be $O(N)$, much less than $O(N^2)$ cost to speed-up $O(N)$ particles.

Connection with the 'bulk' mass flow

We first recall 'hydrodynamics' for (T)ASEP as it will be useful.

–Let $\rho_0 : \mathbb{R} \to [0,1]$ be an initial condition.

Starting from

$$\nu_{\rho_0(\cdot)} = \prod_x \mathrm{Bern}\Big(\rho_0\big(\tfrac{x}{N}\big)\Big),$$

we have the hydrodynamic limit:

$$\pi^N = \frac{1}{N}\sum_x \eta_{Nt}(x)\delta_{x/N} \Rightarrow \delta_{\rho(t,x)dx}.$$

Here, $\rho = \rho(t, x)$ satisfies:

$$\partial_t \rho + \gamma \partial_x \{\rho(1-\rho)\} = 0$$

where ρ is the unique ‘entropy’ solution.

–Rost ’81, Rezakhanlou ’91, Seppalainen ’98

Returning to the tagged particle, since jumps are nearest-neighbor in TASEP,

$$\{X_{tN} \geq AN\} = \left\{ \sum_{z=0}^{AN} \eta_{tN}(z) = 0 \right\}.$$

–In the continuum, when starting from $\nu_{\rho_0(\cdot)}$, let

$$h_t(x) = \int_0^x \rho(t,u)du.$$

Then,

$$h_t(A) = \int_0^A \rho(t,u)du = 0.$$

“All initial mass between 0 and A
has flowed beyond A at time t.”

Note $h_t(x)$ satisfies

$$\partial_t h_t + (\partial_x h_t)(1 - \partial_x h_t) = 0.$$

–After some manipulation, the Hopf-Lax formula solves for the entropy solution:

$$h_t^{ent}(x) = \sup_z \left\{ h_0(y) - tg\Big(\frac{x-z}{t}\Big) \right\}$$

where

$$g(u) = \begin{cases} \frac{1}{4}(1-u)^2 & \text{for } |u| \leq 1 \\ 0 & \text{for } u < -1 \\ u & \text{for } u > 1. \end{cases}$$

A superexponential estimate

It can be shown that

$$h_1^{ent}(A) \text{ cannot be larger than } \frac{1}{N}\sum_{z=1}^{AN}\eta_N(z),$$

without incurring super-exponential cost:

$$\limsup_{N\uparrow\infty}\frac{1}{N}\log P\Big(h_1^{ent}(A) > \frac{1}{N}\sum_{z=1}^{AN}\eta_N(z)+\epsilon\Big) = -\infty.$$

–Equivalently, $\sum_{z>AN}\eta_N(z)$ cannot be more than $N\int_{AN}^{\infty}\rho(t,u)du$; that is to speed up $O(N)$ particles has super-exponential cost.

Where to remove particles?

Hence, to move $\frac{1}{N}X_N \sim A$ (recall $t = 1$),
we have

$$h_1^{ent}(A) \leq h_1(A) = 0.$$

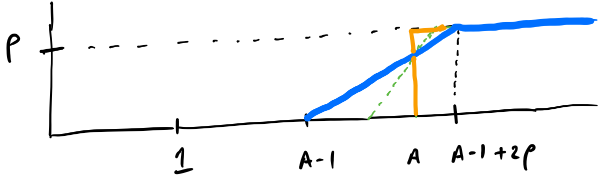
–In other words, for $z \geq A - 1$,

$$\begin{aligned} h_0(y) &\leq g(A - z) \\ &= \frac{1}{4}\big(1 - (A - z)\big)^2. \end{aligned}$$

–This suggests an initial density where

$$\partial_x h_0(z) = \rho(0, x) \ \stackrel{?}{\leq} \ \frac{1}{2}\big(1 - (A - z)\big)_+.$$

Plotting this is the ‘blue’ initial profile, which flows, according to the hydrodynamic equation, to the ‘yellow’ profile at time $t = 1$.



The cost of changing the initial profile is

$$
\begin{aligned}
&-(A-1)\log(1-\rho) \\
&+\int_{A-1}^{A-1+2\rho} u_0(z)\log\frac{u_0(z)}{\rho}+(1-u_0(z))\log\frac{1-u_0(z)}{1-\rho}dz \\
&=-A\log(1-\rho)-\rho.
\end{aligned}
$$

–Now, there is room for the tagged particle,
a Poisson rate 1 (unobstructed) process,
to get to AN in time N.

–It's optimal to change its rate to $A > 1$, with cost

$$A \log A - A + 1.$$

–Then, both tagged particle and the particle at $(A-1)N$ reach AN at the same macro time 1.

–Adding the costs, we get

$$A \log \frac{A}{1-\rho} - A + 1 - \rho = I(A).$$

–In ASEP, we will have to change a birth-death process instead of a Poisson process. This will give a more general formula for $I(A)$ involving p and q.

Sketch: $0 \leq A \leq 1 - \rho$

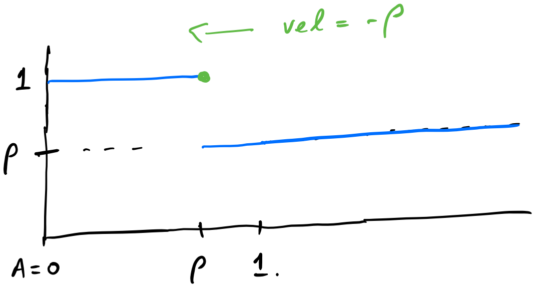
What if $A = 0$?
That is, the tagged particle doesn't move, up to time 1?

–One way: The tagged particle's clock does not ring up to time 1, with cost $-\log e^{-1} = 1$.

But, this is too large,
compared to what we should achieve,

$$I(A = 0) = 1 - \rho....$$

–Another way: Try to block with particles.



The shock, from Rankine-Huguniot with flux $u(1-u)$, moves at speed

$$1 - R - L = 1 - 1 - \rho = -\rho.$$

–If we maintain the shock,
via a nonentropic solution,
then the tagged particle would be blocked up to time 1.

To compute the cost, we invoke Jensen-Varadhan large deviations for the empirical measure π^N:

$$P(\pi^N \text{ follows } \zeta) \sim \exp\{-N I_{JV}(\zeta)\}$$

where

$$I_{JV}(\zeta) = \text{ positive charge of } '(\partial_t h(\zeta) + \partial_x g(\zeta)) dxdt'$$

and $h(u) = u \log(u) + (1-u) \log(1-u)$ and $g(u) = u(1-u) \log \frac{u}{1-u}$.

–Jensen-Varadhan 2000; Vilensky 2008; Quastel-Tsai 2022

The I_{JV} cost here of the nonentropy solution is

$$\begin{aligned} & L - R + LR \log \frac{R}{L} \\ & + (1-L)\log(1-L) - (1-R)\log(1-R) \\ & + L(1-R)\log(1-R) - (1-L)R\log(1-L) \end{aligned}$$

which reduces to $(1-\rho) + \rho \log(\rho)$.

–Adding, to the profile cost, which is $-\rho \log(\rho)$, we get

$$1 - \rho = I(A = 0).$$

–At the moment, we do not a similar formula in ASEP, as the I_{JV} cost is known only in TASEP.

Upper bounds

Are the costs of these strategies found by minimizing over all strategies?

–When $\gamma(1-\rho) \leq A$, we can prove this, and so derive an ‘upper tail’ LDP in ASEP.

Hence, this identifies in ASEP the formula for the rate $I(A)$.

Sketch: $A \geq 1$

Staying with TASEP to reduce notation, things boil down to understanding

$$
\begin{aligned}
&\frac{1}{N}\log P\Big(\frac{1}{N}\sum_{z=0}^{AN}\eta_N(z)=0\Big)\\
&\sim \frac{1}{N}\log P\Big(h_1^{ent}(A)\leq \frac{1}{N}\sum_{z=0}^{AN}\eta_N(z)=0\Big)\\
&\leq -\inf_{w_0:h_1^{ent,w_0}(A)=0}\int_0^\infty w_0(z)\log\frac{w_0(z)}{\rho}(1-w_0(z))\log\frac{1-w_0(z)}{1-\rho}dz.
\end{aligned}
$$

–This calculus of variations problem can be solved to yield the previous initial profiles, when $A>1-\rho$. Notably, the optimal evolutions in the 'upper tail' regimes are *entropic*.

Comments in ASEP

Many estimates carry over, but there is a new category.

–We have mentioned that the LDP holds in the 'upper tail' : $A > \gamma(1-\rho)$.

In the other categories, we can derive candidate lower bounds.

- $A \geq \gamma$
- $\gamma(1-\rho) \leq A \leq \gamma$
- $0 \leq A \leq \gamma(1-\rho)$
- $A \leq 0$

Form of the rate in the ASEP ‘upper tail’

We can identify, when $\gamma(1-\rho) \leq A \leq \gamma$, that the rate is in form

$$I(A) = A \log \frac{A}{\gamma(1-\rho)} - A + \gamma(1-\rho).$$

–This corresponds to the Ferrari-Fontes approximation by a Poisson process with rate $\gamma(1-\rho)$.

However, as alluded to before, when $A \geq \gamma$,
we derive that

$$I(A) = A \log c - pc - \frac{q}{c} + 1 \\ - \gamma\{ \log(1 - \rho) + \rho\},$$

where

$$c = \frac{A + \sqrt{A^2 + 4pq}}{2p}.$$

–This is a smaller cost than found from the Ferrari-Fontes approximation.

Happy birthday to Timo
and best wishes on this occasion!