

Tail bounds for KPZ models: a case study for ASEP

joint work with Benjamin Landon

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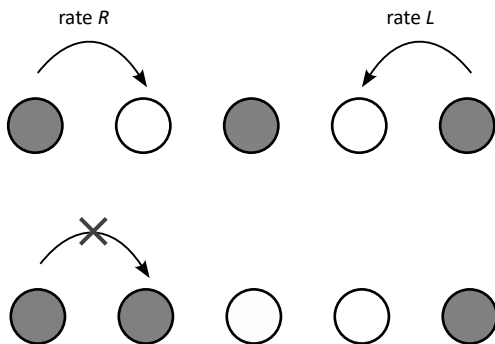
Goal of the talk

Goal 1: Report on what we learned from Timo's papers.

Goal 2: Complement one of Timo's breakthrough results, $T^{\frac{1}{3}}$ current fluctuations for ASEP (Balasz-Seppalainen, *Order of current variance and diffusivity in the asymmetric simple exclusion process*, Annals 2010).

The model: ASEP

Configuration $\eta(t) \in \{0, 1\}^{\mathbb{Z}}$ of particles performing biased continuous time walk with exclusion.

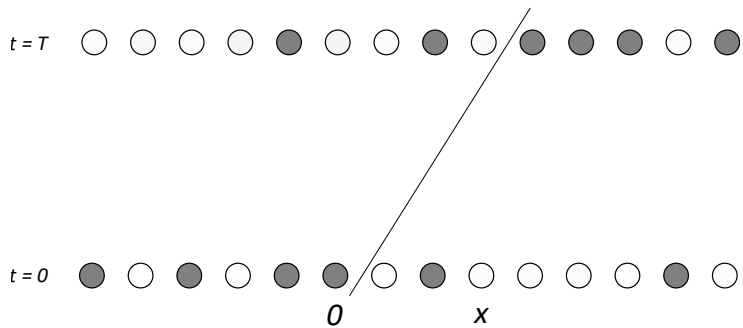


Current

Current: net number of particles that cross $[\frac{1}{2}, x + \frac{1}{2}]$ in time T .

$$J_T(x) = \underbrace{J_T(0)}_{\text{net flux across 0}} - \sum_{i=1}^x \eta_i(t).$$

Current



Current: Results

Balász-Seppäläinen, for the current in a characteristic direction $x_0 := (L - R)(2b - 1)T$:

$$\text{Var}(J_T(x_0)) \asymp T^{\frac{2}{3}}.$$

We have:

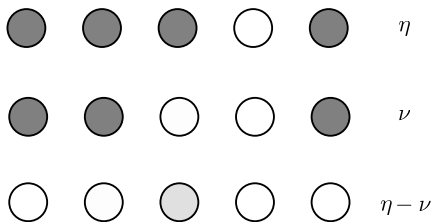
$$ce^{-Cu^{3/2}T^{-1/2}} \leq \mathbb{P}[J_T(x_0) - \mathbb{E}[J_T(x_0)] > u] \leq Ce^{-cu^{3/2}T^{-1/2}}$$

Tail behavior consistent with Baik-Rains limit (Aggarwal, 2016).

2nd class particle

2nd class particle: ASEP dynamics, but 1st class particles take precedence (switch positions instead of exclusion).

Arise from considering discrepancy between coupled, ordered configurations $\nu \leq \eta$:



Second class particle: Results

Balazs-Seppäläinen: position $Q(T)$ of second class particle started at 0 in a stationary (Bernoulli b) environment for $k < 3$

$$E[|Q(T) - x_0(T)|^k] \asymp T^{\frac{2}{3}k},$$

where

$$x_0(T) = (L - R)(2b - 1)T.$$

We show:

$$P[|Q(T) - x_0| > u] \leq Ce^{-cu^3T^{-2}}.$$

Ingredients

Two technical inputs come from some of my favorite papers of Timo's:

1. Microscopic concavity coupling (Bálasz-Seppäläinen)
2. Exponential formula (Elnur-Jianjigian-Seppäläinen)

This is combined with

3. Degeneration from stationary stochastic six vertex model (Borodin-Corwin-Gorin, Aggarwal)

Stochastic Six Vertex Model






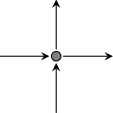
Configurations of arrows entering/exiting vertices of domain in \mathbb{Z}^2 .

At each vertex:

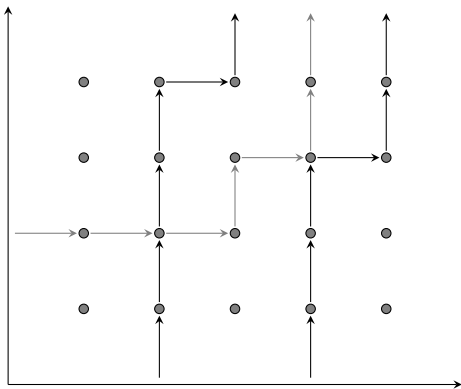
Number of incoming arrows = Number outgoing arrows

Stochastic Six Vertex Model

Six possible configurations, with weights:

					
1	δ_1	$1 - \delta_1$	δ_2	$1 - \delta_2$	1

Up-right paths



Weight of configuration:

$$1^{\# \text{type 1}} \delta_1^{\# \text{type 2}} (1 - \delta_1)^{\# \text{type 3}} \delta_2^{\# \text{type 4}} (1 - \delta_2)^{\# \text{type 5}} 1^{\# \text{type 6}}$$

Stationary Model

There is a stationary version of the model, considered by Aggarwal.

Choose arrow configurations along the boundaries $(x, 0)$ and $(0, y)$ to be Bernoulli with parameters b_1 and b_2 such that

$$\frac{b_1}{1 - b_1} = \kappa \frac{b_2}{1 - b_2}.$$

Then the probabilities of incoming and outgoing arrows along down-right paths are *invariant* for S6V.

Degeneration to ASEP

Consider S6V with

$$\delta_1 = \epsilon L, \delta_2 = \epsilon R$$

with initial data being (b_1, b_2) such that

$$\frac{b_1}{1 - b_1} = \frac{1 - \delta_1}{1 - \delta_2} \frac{b_2}{1 - b_2}.$$

$p_i(t)$: particles in S6V at height t

$X_i(t)$ particles in ASEP, Bernoulli b_2 data

$$q_i(t) = p_i(t) - t.$$

Degeneration to ASEP

For finite $S \subseteq \mathbb{Z}^n$:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} P [q_{i_1}(\lfloor \epsilon^{-1} t_1 \rfloor), \dots, q_{i_n}(\lfloor \epsilon^{-1} t_n \rfloor) \in S] \\ &= P [X_{i_1}(t_1), \dots, X_{i_n}(t_n) \in S]. \end{aligned}$$

As a consequence,

$$\lim_{\epsilon \rightarrow 0} P \left[H^{(b_1, b_2)}(x + \lfloor \epsilon^{-1} t \rfloor, \lfloor \epsilon^{-1} t \rfloor) > r \right] = P [J_t(x) \geq r].$$

Observed in Borodin-Corwin-Gorin, proved by Aggarwal, used extensively in Aggarwal and Borodin-Aggarwal.

Height function

For a S6V configuration in the rectangle

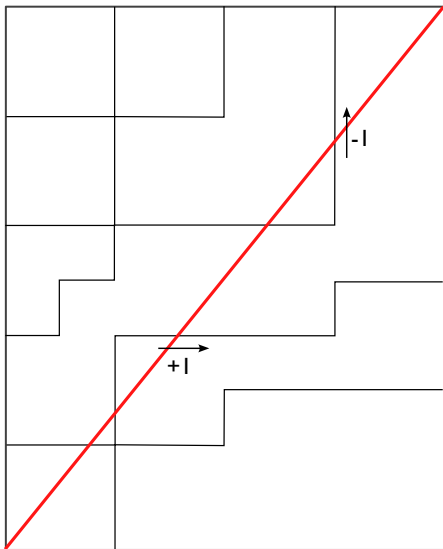
$$\{x \leq X, y \leq Y\}$$

the **height function** is defined by

$$H(X, Y) = \text{net flux of paths across line from origin to } (X, Y).$$

+1 if the line is traversed left-right, -1 if right-left.

Height function



Asymptotics of the height function

Borodin-Corwin-Gorin identify scaling limit:

$$\lim_{\epsilon \rightarrow 0} \epsilon H(\epsilon^{-1}x, \epsilon^{-1}y) = \mathcal{H}(x, y),$$

for explicit \mathcal{H} and Riemann boundary conditions (fixed densities along the axes). Also show Tracy-Widom limit after centering and rescaling by $\epsilon^{-1/3}$

Hydrodynamic limit for general boundary conditions by Aggarwal.

Tail estimates

Subject to a **characteristic direction** condition, for

$$(y(1 - \kappa))^{1/3} \leq u \leq cy(1 - \kappa),$$

we have

$$\mathbb{E} \left[H^{(b_1, b_2)}(x, y) - \mathbb{E}[H^{(b_1, b_2)}(x, y)] > u \right] \geq ce^{-Cu^{3/2}(y(1-\kappa))^{-1/2}},$$

and

$$\mathbb{E} \left[H^{(b_1, b_2)}(x, y) - \mathbb{E}[H^{(b_1, b_2)}(x, y)] > u \right] \leq Ce^{-cu^{3/2}(y(1-\kappa))^{-1/2}}.$$

Ideas in the proofs.

Coupling method (a.k.a. *Seppäläinen machine*)

General strategy, originating in Balázs-Cator-Seppäläinen.

Appears in Timo's works on O'Connell-Yor, ASEP, zero-range process, log-Gamma polymer, LPP, etc.

Based on simultaneously estimating two quantities:

Height function (passage time, log partition function, current...)

$$H(\theta, \eta)$$

Derivative (time spent on the boundary, first jump, second class particle...)

$$Q = \partial_{\theta} H.$$

The coupling method heuristic

Stationarity:

$$H = B(\theta) + R(\theta, \eta),$$

B : increments along $[(1, 1), (N, 1)]$

R : increments along $[(N, 1), (N, M)]$.

Rearrange:

$$\bar{R} = \bar{H} - \bar{B}$$

$$\text{Var}(R) = \text{Var}(H) + \text{Var}(B) - 2E[B\bar{H}].$$

The coupling method heuristic

$$\begin{aligned} E[B\bar{H}] &= \frac{d}{d\delta} E[e^{\delta\bar{B}}\bar{H}] \Big|_{\delta=0} \\ &= \frac{d}{d\delta} (E[\bar{H}(B_{\theta,\delta})]) \Big|_{\delta=0} \\ &:= E[Q] \end{aligned}$$

Assume $\text{Var}(R) = \text{Var}(B)$ (characteristic direction):

$$\text{Var}(H) = 2E[Q].$$

Exact version of KPZ relation

$$2\chi = \xi.$$

Convexity

$$Q = \frac{d}{d\delta} H(B_{\theta, \delta}, \eta) =: \partial_{\theta} H.$$

In his papers, Timo (**master of couplings**), bounds $P(Q > u)$ in terms of H by ingenious couplings.

Generally: if $\theta \mapsto H(\theta, \eta)$ is convex, then for $\lambda > \theta$:

$$Q \leq \frac{H(\lambda, \theta) - H(\theta, \theta)}{\lambda - \theta}.$$

Upper bound for χ

For example, for known integrable models, can typically show:

$$Q \lesssim \frac{1}{\lambda - \theta} (|\overline{H}(\theta, \theta)| + |\overline{H}(\lambda, \lambda)| + |E[H(\theta, \theta)] - E[H(\lambda, \lambda)]|).$$

We get

$$\text{Var}(H) := V \leq \frac{C}{\lambda - \theta} (V^{1/2} + N^{1/2}(\lambda - \theta) + N(\lambda - \theta)^2).$$

Optimize:

$$\lambda - \theta \sim N^{-1/3} \Rightarrow \chi = \frac{1}{3}.$$

A remarkable formula

Want to run the previous argument on an exponential scale.

Elnur, Jianjigian and Seppäläinen (EJS): general methodology to get concentration in stationary models (in their case, exponential LPP) using a formula due to Rains (2001), with a simple proof.

EJS formula

EJS's observation: for “any” model with product invariant measure on a quadrant, in the characteristic direction,

$$\mathbb{E} \left[\exp \left((\theta - \eta) (H(\theta, \eta) - \mathbb{E}[H(\theta, \theta)]) \right) \right] = \exp \left(c(\eta) N (\theta - \eta)^3 \right).$$

Cubic behavior $\rightarrow N^{1/3}$ fluctuations.

Consequence

For $\eta < \theta$:

$$\begin{aligned} H(\theta, \theta) &= H(\theta, \nu) + \int_{\eta}^{\theta} Q(\theta, u) du \\ &\leq H(\theta, \eta) + (\theta - \eta)Q(\theta, \theta) \end{aligned}$$

Subtract $E[H(\theta, \theta)]$, multiply by $\theta - \eta$ and exponentiate. Done provided we can control

$$E[e^{\epsilon^2 Q(\theta, \theta)}].$$

This argument gives moderate deviations on $N^{1/3}$ scale with $u^{3/2}$ exponent for *all* known integrable polymer models, and works for some non-integrable interacting diffusion models (Landon-S., 2022)

Back to S6V

The height function in the stationary six vertex model has the form:

$$\begin{aligned} H(x, y) &= \text{horizontal arrows through right side} \\ &\quad - \text{vertical arrows through bottom} \\ &:= R - B. \end{aligned}$$

EJS for S6V

Choose parameters:

$$e^\varepsilon \frac{a_1}{1 - a_1} = \frac{1 - \delta_1}{1 - \delta_2} \frac{a_2}{1 - a_2}$$

Then:

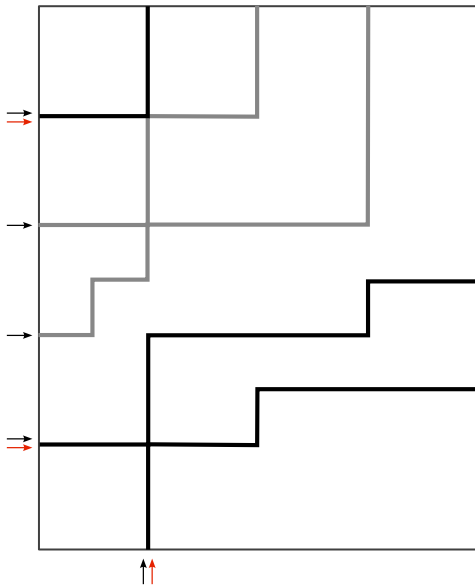
$$\mathbb{E} \left[\exp \left(\varepsilon H^{(a_1, a_2)}(x, y) \right) \right] = (e^\varepsilon a_1 + (1 - a_1))^y (e^{-\varepsilon} a_2 + (1 - a_2))^x.$$

Couplings and 2nd class paths

The role of Q is played by second class paths. As in ASEP, we can couple different boundary conditions for S_6V , and look at the discrepancies between them (“grey paths”).

Balázs and Seppäläinen’s famous second class particle arguments extend to that setting.

Second class paths



Exit point estimate

Let $a_1 < b_1$ and $a_2 < b_2$. Start a second class path at the bottom left corner.

$$\begin{aligned} & \mathbb{P}[\text{second class path exits through north}] \\ & \lesssim e^{-ck} + e^k \mathbb{E} \left[e^{\epsilon H^{(a_1, a_2)}(x, y)} \right]^{1/2} \mathbb{E} \left[e^{-\epsilon H^{(b_1, a_2)}(x, y)} \right]^{1/2}. \end{aligned}$$

Future directions

- ▶ Lower tail bounds: for some models, geometric argument is available.
- ▶ Non product form invariant measures: half-space models.

Thanks a lot for listening & Happy Birthday to Timo!