Hidden temperature in the KMP model

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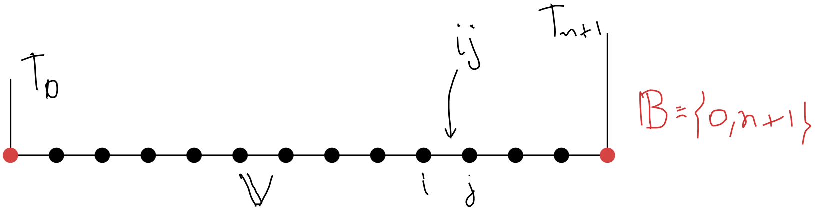
Setup

graph $(\mathbb{V} \cup \mathbb{B}, E)$. $\mathbb{V}$ bulk, $\mathbb{B}$ boundary $\qquad \overline{\mathbb{V}} := \mathbb{V} \cup \mathbb{B}$

Oriented edges $E \subset \{ij : i, j \in \mathbb{V} \cup \mathbb{B}\}$

(Orientation only used to define dynamics.)

Boundary conditions Fixed real positive "temperatures" T_b, $b \in \mathbb{B}$.

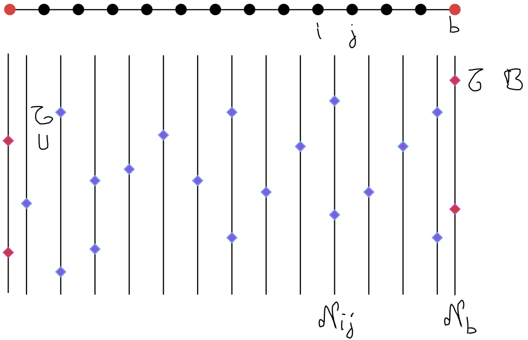


Marked Poisson process $(\mathcal{N}, \mathcal{U}, \mathcal{B})$

$\mathcal{N}_{ij}$, $\mathcal{N}_b$ are independent rate 1 Poisson process on $\mathbb{R}$

$$\mathcal{N} = \bigcup_{ij \in E} \mathcal{N}_{ij} \cup \bigcup_{b \in \mathbb{B}} \mathcal{N}_b$$

Marks: $\tau \in \mathcal{N}_{ij}$: $U(\tau)$ Uniform$[0,1]$; $\tau \in \mathcal{N}_b$: $B(\tau)$ Exponential(1).

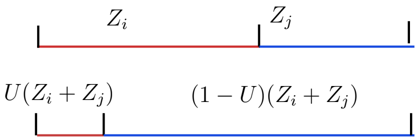


KMP Dynamics

$\underline{Z}(t) = (Z_i(t))_{i \in \overline{\mathbb{V}}}$ governed by $(\mathcal{N}, \mathcal{U}, \mathcal{B})$ and boundary conditions $T_{\mathbb{B}}$.

If $\tau \in \mathcal{N}_{ij}$, $U = U(\tau)$, then

$$Z_i, Z_j \quad \longrightarrow \quad U(Z_i + Z_j)\ ,\ (1 - U)(Z_i + Z_j)$$



If $\tau \in \mathcal{N}_b$, and $B = B(\tau)$, then

$$Z_b \quad \longrightarrow \quad T_b B$$

When $T_i \equiv 1$ the product of exponential(1) is reversible.

Goal: Describe invariant measures for general $T_{\mathbb{B}}$.

Opinion model with extremists

$O_i(t)$ in $\mathbb{R}_+$ is the opinion of individual i at time t.

Extremist at boundary vertex b does not change opinion: $O_b(t) \equiv T_b$.

$\underline{O}(t) = (O_i(t))_{i \in \overline{\mathbb{V}}}$ governed by $(\cup_{ij}\mathcal{N}_{ij}, \mathcal{U})$:

When $\tau \in \mathcal{N}_{ij}$, $U = U(\tau)$:

$$\begin{array}{lcll} O_i & \longrightarrow & UO_i + (1-U)O_j, & i \in \mathbb{V} \\ O_j & \longrightarrow & UO_i + (1-U)O_j, & j \in \mathbb{V} \\ O_b & \text{stays at } T_b & & b \in \mathbb{B} \end{array}$$

Bulk i and j adopt a convex combination of their previous opinions.

Extremist b commits but does not comply.

Call $\underline{O}^{\text{stat}}$ a realization of the unique invariant measure.

Invariant measure for KMP

Theorem (DFG, to be ArXived). *Consider*

$\underline{O}^{\text{stat}} \stackrel{\text{law}}{=}$ *invariant measure for opinion model with boundary* $T_{\mathbb{B}}$*.*

$\underline{X}$ *iid Exponential(1);* $\underline{X}$ *and* $\underline{O}^{\text{stat}}$ *independent*

$\underline{Z}$ *defined by* $Z_i := X_i \, O_i^{\text{stat}}$

Then, the law μ^Z *of* $\underline{Z}$ *is invariant for KMP with boundary* $T_{\mathbb{B}}$*.*

In other words, the invariant measure μ^Z satisfies

$$\int \varphi(z)\, \mu^Z(dz) = \int \nu^O(do) \int d\underline{y}\, \varphi(\underline{y}) \prod_i \frac{1}{o_i}\, e^{-y_i/o_i}$$

where ν^O is the law of $\underline{O}^{\text{stat}}$.

Equilibrium KMP (zero current)

Denote $(\underline{X}(t))_{t\in\mathbb{R}}$ a stationary realization of KMP governed by $(\mathcal{N}, \mathcal{U}, \mathcal{B})$ and boundary $T_b \equiv 1$.

$X(t) \overset{\text{law}}{=}$ "iid exp(1)", also reversible.

Reverse process: $\underline{X}^*(t) := \underline{X}(-t-)$. Define

$$V(\tau) := \frac{X_i(\tau-)}{X_i(\tau-) + X_j(\tau-)}, \quad \tau \in \mathcal{N}_{ij};$$

$$A(\tau) := X_b(\tau-), \quad \tau \in \mathcal{N}_b.$$

Dual marks: Marked points $(-\tau, V(\tau))$ and $(-\tau, A(\tau))$ govern $X^*(t)$,

$(\mathcal{N}, \mathcal{V}, \mathcal{A}) \overset{\text{law}}{=} (\mathcal{N}, \mathcal{U}, \mathcal{B})$,

$X(t) = X^*(-t)$ is independent of $(\tau, V(\tau))$, $(\tau, A(\tau))$, $\tau < t$.

Coupling equilibrium KMP and the opinion model

Proposition 1. *Let*

$$\underline{X}(t) := \textit{Equilibrium KMP governed by } (\mathcal{N}, \mathcal{U}, \mathcal{B})$$
$$\underline{O}(t) := \textit{Opinion model governed by } (\mathcal{N}, \mathcal{V})$$

If $\underline{X}(0)$ *and* $\underline{O}(0)$ *are independent, then*

$\underline{X}(t)$ *and* $\underline{O}(t)$ *are independent for each* t*.*

Proof. $X(t)$ independent of dual marks of the past of t

$O(t)$ is a function of the dual marks of the past of t □

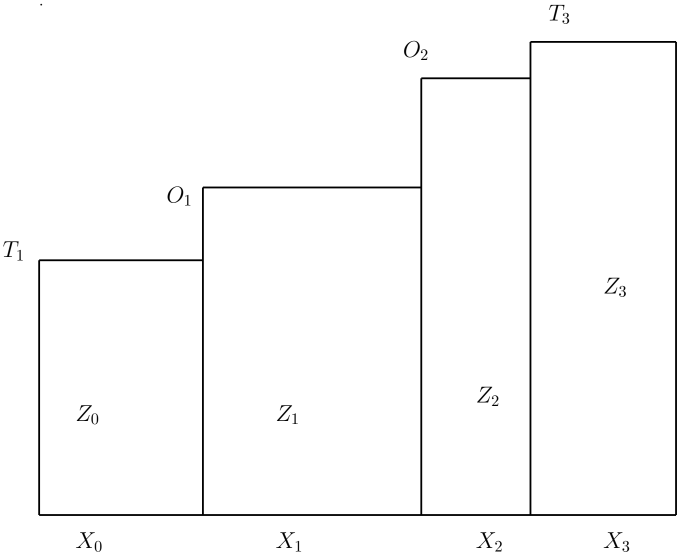
Equilibrium and Nonequilibrium KMP via the opinion model

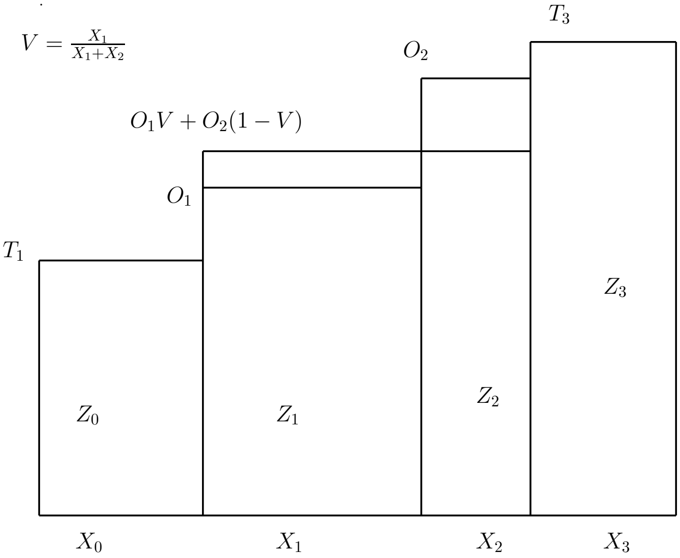
Theorem (DFG). *Let*

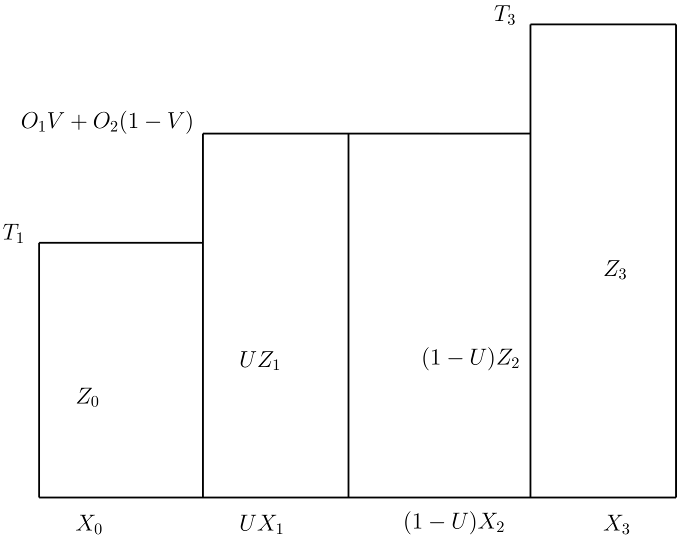
$$\underline{X}(t) := \text{Equilibrium KMP governed by } (\mathcal{N}, \mathcal{U}, \mathcal{B})$$
$$\underline{O}(t) := \text{Opinion model governed by } (\mathcal{N}, \mathcal{V}), \text{ boundary } T_{\mathbb{B}}$$
$$\text{and arbitrary initial } \underline{O}(0).$$
$$\underline{Z}(t) := \big(X_i(t)\, O_i(t)\big)_{i \in \overline{\mathbb{V}}}.$$

Then

$(\underline{Z}(t))_{t \geq 0}$ *is KMP with boundary* $T_{\mathbb{B}}$*, governed by* $(\mathcal{N}, \mathcal{U}, \mathcal{B})$*, and initial* $\underline{Z}(0) = \underline{X}(0)\, \underline{O}(0)$*.*


T_3
O_2
O_1
T_1
Z_3
Z_2
Z_0
Z_1
X_0
X_1
X_2
X_3


$V = \frac{X_1}{X_1+X_2}$
T_3
O_2
$O_1V + O_2(1-V)$
O_1
T_1
Z_3
Z_2
Z_0
Z_1
X_0
X_1
X_2
X_3


T_3
$O_1V + O_2(1-V)$
T_1
Z_3
UZ_1
$(1-U)Z_2$
Z_0
X_0
UX_1
$(1-U)X_2$
X_3

Stationary opinion distribution

No explicit expression.

One dimensional one individual case:

$\mathbb{V} = \{1\}$, $\mathbb{B} = \{0, 2\}$, $E = \{01, 12\}$, boundary values $T_0 < T_2$.

Markov chain $O_1(t)$ in $[T_0, T_2]$.

Invariant measure:

$$\frac{do}{\pi\sqrt{(o - T_0)(T_2 - o)}}\ \mathbf{1}\{o \in [T_0, T_2]\},$$

by an explicit computation.

For arbitrary $n \geq 1$ we have duality.

Coalescing random walks in random environment

Family of backward walks $(R_{k,s}(t))_{t\in(-\infty,s]}$, indexed by $k \in \overline{\mathbb{V}}$, $s \in \mathbb{R}$.

Governed by $(\cup_{ij}\mathcal{N}_{ij}, \mathcal{V}, \mathcal{U}')$, with initial $R_{k,s}(s) = k$.

Let $\tau \in \mathcal{N}_{ij}$, $V = V(\tau)$, $U' = U'(\tau)$ Uniform$[0,1]$,

If $R_{k,s}(\tau+) = \ell \in \{i, j\}$, define

$$R_{k,s}(\tau) := \begin{cases} i & \ell \in \mathbb{V} \text{ and } U' < V \\ j & \ell \in \mathbb{V} \text{ and } U' > V \\ j & \ell = j \text{ and } j \in \mathbb{B} \quad \text{(absorption)} \end{cases}$$

Coalescing walks absorbed at the boundary.

$R_{k,s}(\cdot)$ is just a backwards nearest neighbor symmetric random walk.

But, fixing $(\mathcal{N}, \mathcal{V})$, "random walk in space-time random environment".

(Classical) Duality for the opinion model

Proposition 2 (Duality)**.**

Let $O(t)$ be the opinion model governed by $(\mathcal{N}, \mathcal{V})$ with initial $O(0)$.

Let R be the backward random walks governed by $(\mathcal{N}, \mathcal{V}, \mathcal{U}')$. Then:

$$O_i(s) = \mathbb{E}\big(O_{R_{i,s}(0)}(0) \,\big|\, (\mathcal{N}, \mathcal{V})\big), \qquad \mathbb{P}\text{-a.s.}, \quad i \in \overline{\mathbb{V}}, s > 0.$$

Define

$$O_i^{\mathrm{stat}}(s) := \mathbb{E}\big(T_{R_{i,s}(\theta_{i,s})} \,|\, (\mathcal{N}, \mathcal{V})\big), \qquad i \in \overline{\mathbb{V}},\ s \in \mathbb{R}.$$

$$\theta_{i,s} := \sup\{t < s : R_{i,s}(t) \in \partial\mathbb{V}\} \qquad \text{hitting time of } \mathbb{B}.$$

Proposition 3 (Invariant measure)**.** *The law of O^{stat} is invariant for $O(t)$.*

In particular, when $\mathbb{V} = \{1, n\}$, $\mathbb{B} = \{0, n+1\}$,

$$\mathbb{E}O^{\mathrm{stat}}(i) = (T_{n+1} - T_0)\tfrac{i}{n}.$$

Spiking disagreement edges in $\{0,1\}^E$

Indicator function of edge disagreement

$$D_{ij}(t) := \mathbf{1}\{O_i(t) \neq O_j(t)\}, \qquad ij \in E.$$

Governed by $(\mathcal{N}, \mathcal{V})$: for $\tau \in \mathcal{N}_{ij}$,

$$D_{k\ell}(\tau) := \begin{cases} 0 & k\ell = ij \text{ and } j \in \mathbb{V} \\ 1 & |\{k,\ell\} \cap \{i,j\}| = 1 \\ D_{k\ell}(\tau-) & \text{else} \end{cases}$$

The process is Markov in the set of configuration with no neighboring zeroes.

Can compute density: In the one dimensional case:

$\mathbb{E}D^{\text{stat}}(ij) = \frac{2}{3}$ if $ij \in \mathbb{V}$.

Spiking process and Brownian web

D process $\mathbb{V} = \{1, \ldots, L\}$ is "coalescing random walks with births"

Analogous to the space-time discrete process of coalescing random walks, which are created at all times (Arratia [1]).

Conjecture: in a diffusive limit, the trajectories of the process D converge to the Brownian web in the space-time band $[0, 1] \times \mathbb{R}$, with absorbing boundary conditions, see Fontes, Isopi, Newman, Ravishankar [8] and Toth and Werner [17].

Discrete KMP

Same graph, same boundary. Process $K(t) \in \{0, 1, 2, \ldots\}$ with discrete uniforms to make the repartition of K_i and K_j at updating times.

Denote $\underline{k} = (k_i)_{i \in \overline{\mathbb{V}}}$, $\underline{s} = (s_i)_{i \in \overline{\mathbb{V}}}$, let φ be a test function, and define μ^K by

$$\int \mu^K(d\underline{k})\, \varphi(\underline{k}) := \int \mu^O(d\underline{s}) \sum_{\underline{k}} \Big(\varphi(\underline{k}) \prod_{i \in \overline{\mathbb{V}}} \big(\tfrac{1}{s_i+1}\big)^{k_i} \tfrac{s_i}{s_i+1} \Big),$$

A coupling between continuous KMP $\underline{Z}(t)$ and discrete KMP $\underline{K}(t)$ shows that μ^K is invariant for $\underline{K}(t)$.

Background KMP The Kipnis Marchioro Presutti model (KMP) [13]

In the case that the graph is a one dimensional chain then the model is one of the few examples for wich the large deviations rate functional for the invariant measure can be computed exactly [3, 4].

Background Opinion model

Our opinion model is a variation of a huge class of processes, we quote the survey by Castellano, Fortunato and Loreto [5]. Positive numbers representing opinions sit at the vertices of a graph. At interaction times of an edge, the involved individuals update their opinions to lower their opinion difference. In the Deffuant model, the updating occurs only when the modulus of their current opinion difference does not superate a theshold parameter. The Deffuant model was introduced in [7]. A vertex that agrees with the other vertex to update but does not do it is called extremist. Weisbuch, Deffuant and Amblart [19] include extremists, and show that the population splits in two, each piece taking the opinion of one of the expremists. Vazquez, Krapivsky and Redner [2, 18] consider a population with three opinions left/center/right, where center interacts with left and right, but they do not interact between them. The final state is either dominated by centrists, or split population in leftists and rightist or all population goes to one of the extremes. Lanchier, [14] and Lanchier and Li [15] studies consensus in Deffuant in infinite graphs. Gantert, Heydenreich, Hirscher [9] study local and global agreement in the Deffuant model. In the gossip model the interacting vertices take the mean value of their opinions. Picci and Taylor [16] show convergence of the classic gossip model (with no exetremists) to consensus. Hirscher [12] study Deffuant in infinite volume with extremists (called overlay determined individuals), individuals that do not update the new agreed value after the interaction. Haggstrom [11] phase transition in the Deffuant model in $\mathbb{Z}$. Gómez-Serrano, Graham and Le Boudec [10] prove convergence of the empirical measure of the Deffuant model to a limit satisfying a differential equation, whose solutions concentrate around several finite opinions, and at last converge to consensus (check). The opinion is a conserved quantity in the models of the above papers. Dagès and Bruckstein [6] consider a complete graph, and at each interaction time update the involved vertices with two independent random variables, uniformly distributed between the current opinions; this is a non conservative dynamics.

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