RSK construction of the KPZ fixed point

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based on joint work with

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Order of current variance and diffusivity in the asymmetric simple exclusion process

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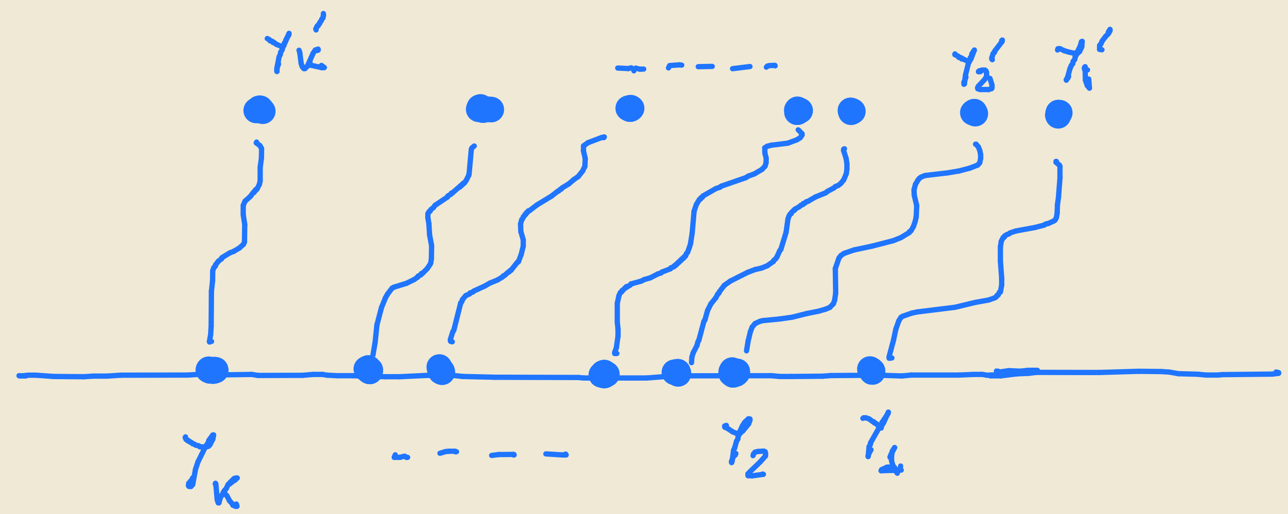
The core formula of KPZ fixed point

(Matetski-Quastel-Remenik)



$$\mathbb{P}\left(\{ y_{k_i}(t) \geq s_i \}_{i=1,\dots} \right) = \det\left(I - \chi_s K \chi_s \right)_{\ell^2(\{1,\dots,n\} \times \mathbb{Z})}$$

with $\chi_{s_i}(k_i, x) := \mathbb{1}_{x < s_i}$

$$K(m,x;\, n,x') := -Q_{(m,n]}\, \mathbb{1}_{m>n} + S_{[1,m],(0,t]} \circ S^{\mathrm{epi}(y)}_{[1,n],(0,t]}(x,x')$$

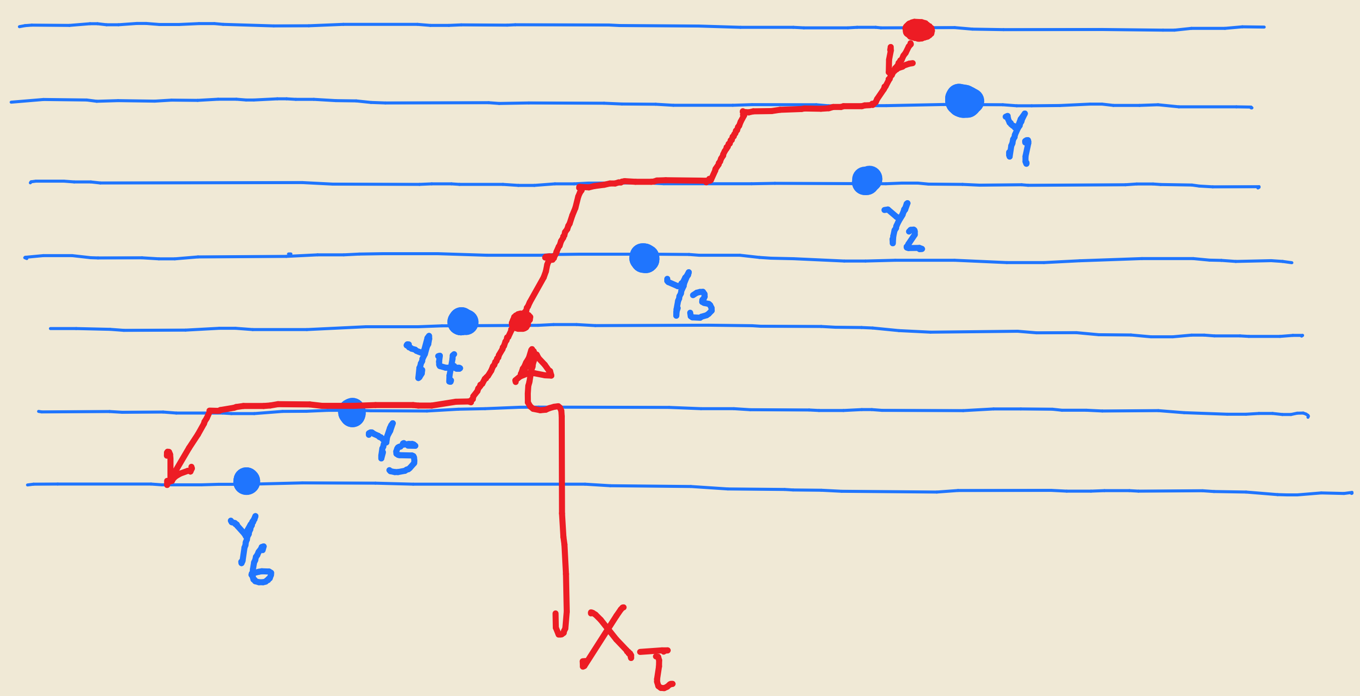
$$S^{\mathrm{epi}(y)}_{[1,n],(0,t]}(x,y) := \mathbb{E}_x\left[\bar{S}_{[\tau+1,n],(0,t]}(X_\tau, y)\, \mathbb{1}_{\tau < n} \right]$$

& $(X_n)_{n\geq 1}$ a geometric random walk with

$$\mathbb{P}\left(X_{i+1} = y \mid X_i = x \right) \propto Q_i(x,y) := q_i^{y-x} \, \mathbb{1}_{y < x}$$

$$Q_{(m,n]} := Q_{m+1} \circ \cdots \circ Q_n$$

$$\tau := \min \{ n : X_n > y_{n+1} \}$$



The details of TASEP

$$\mathbb{P}_t\left(\underset{k}{\circ}\,\curvearrowright \right) = \frac{p_t\, q_k}{1 + p_t\, q_k}$$

with $\quad q_k > 1 \quad \& \quad p_t\, q_k < 1$

Update rule : sequential from first to last particle

TASEP with inhomogeneous rates :

Hydrodynamics : Krug - Seppäläinen '99

Emrah 16'

Emrah - Janjigian - Seppäläinen '21

Integrable : Johansson '00

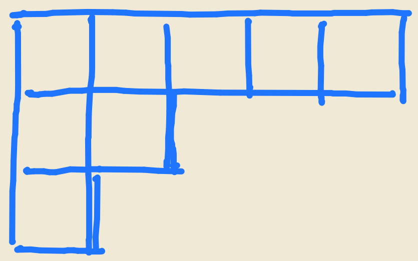
Borodin - Pesche '08

& many more

Robinson - Schensted - Knuth

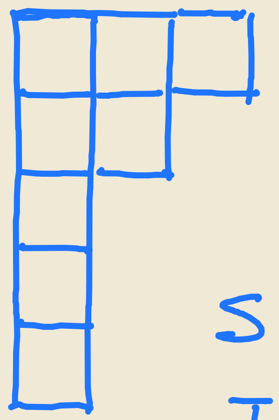
row insertion, column insertion, dual-row, dual-column

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ & & \vdots & & & \end{bmatrix} \xleftrightarrow{\text{dual-column}}$$



shape

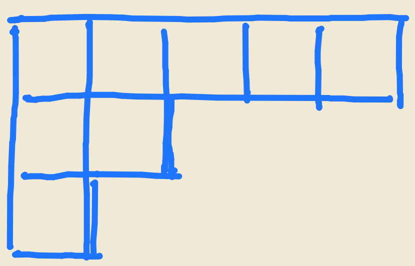
$\lambda = (5, 2, 1)$



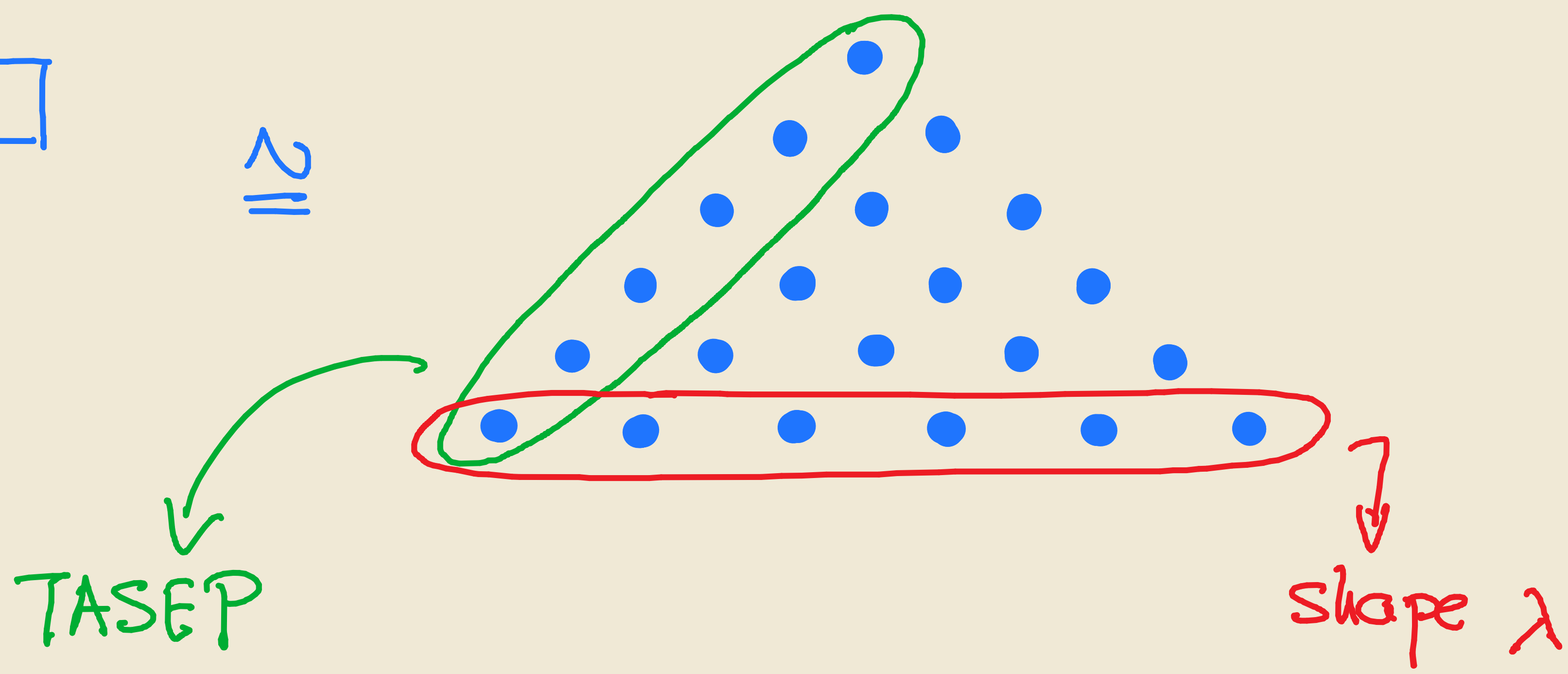
Shape

$\lambda^T = (3, 2, 1, 1, 1)$

Gelfand - Tsetlin



$\cong$



Probabilities

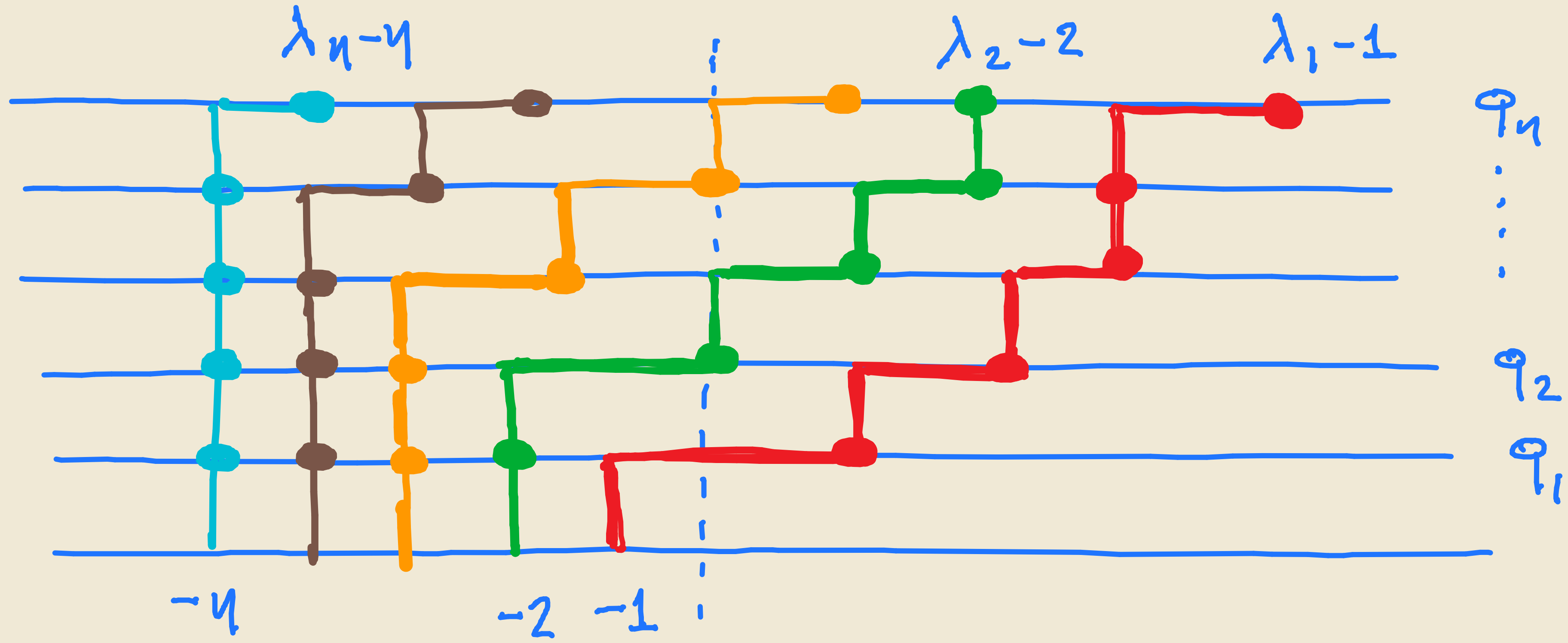
If $(W_{ij})_{i\geq 1,\, j=1,\dots,k}$ independent $\in \{0,1\}$

$$\mathbb{P}(W_{ij}=1) = \frac{p_i q_j}{1+p_i q_j}$$

Then $$\mathbb{P}(\operatorname{sh} P = \operatorname{sh} Q^T = \lambda) = \frac{1}{\prod_{i,j}(1+p_i q_j)} s_\lambda(q)\, s_{\lambda^T}(p)$$

Schur functions & paths

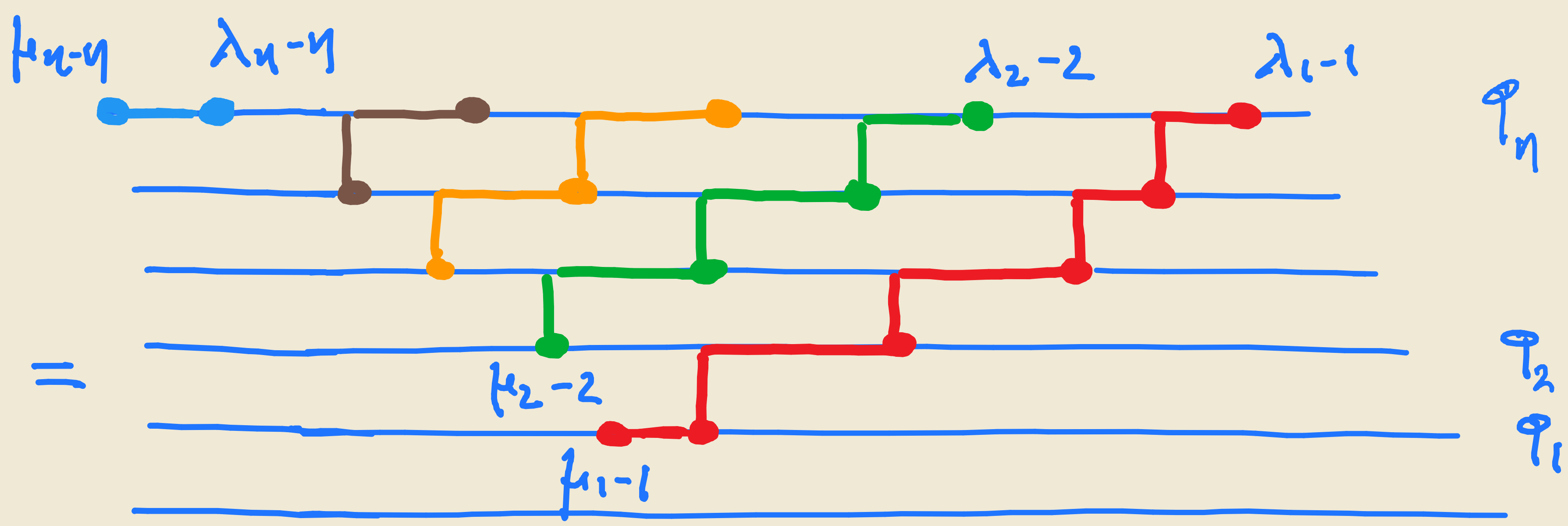
$s_\lambda(q_1,\dots,q_n) =$



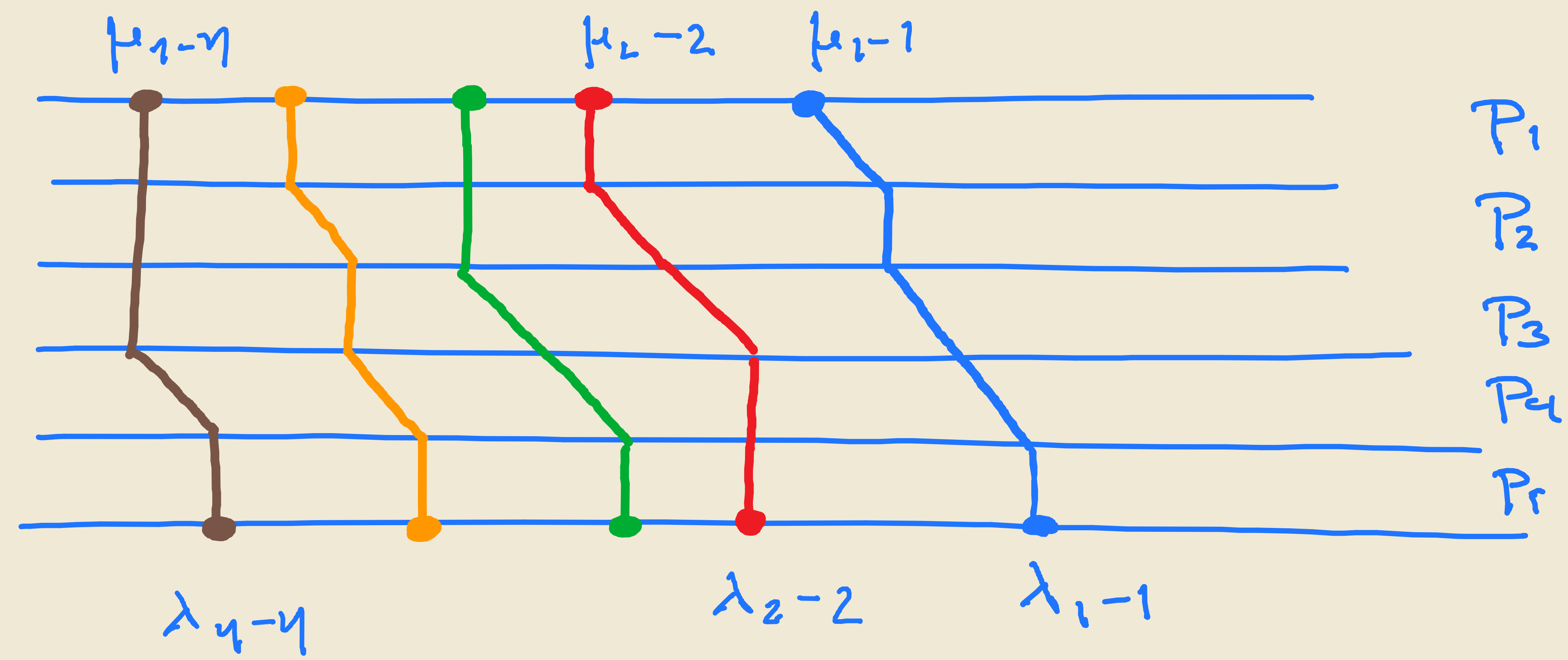
$$= \det\left(\hat{Q}_1 \circ \hat{Q}_2 \circ \cdots \circ \hat{Q}_n(-i, \lambda_j - j)\right)_{i,j\le n}$$

with $\hat{Q}_i(x,y) = q_i^{\,y-x}\, \mathbb{1}_{y \geq x}$

$s_{\lambda/\mu}(q_1,\dots,q_n) =$

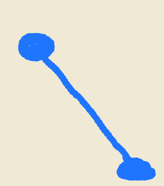


$$s_{\lambda^T/\mu^T}(p_1,\dots,p_n) =$$

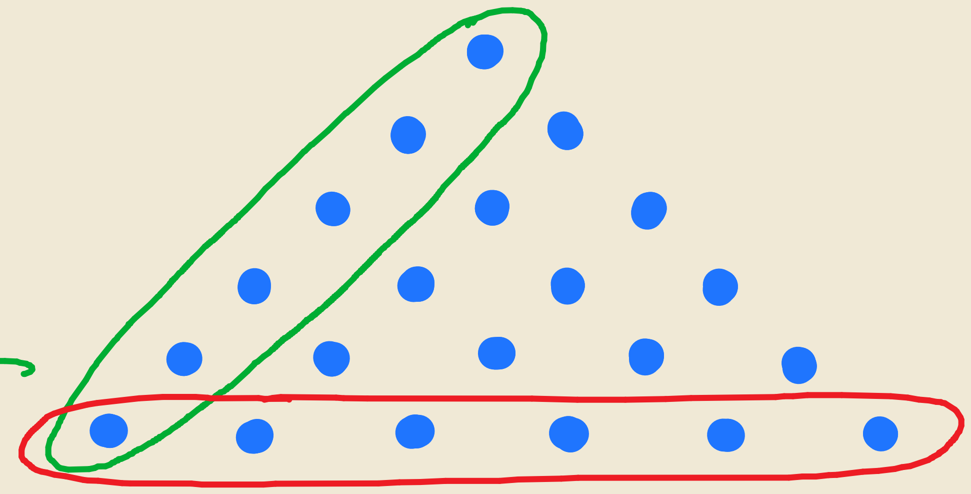


Weights

$R_i($  $) = 1$

$R_i($  $) = p_i$

Intertwining & TASEP transition



TASEP transition kernel: $P(y, y')$

shape λ transition kernel $S(\mu, \lambda)$

intertwiner $\Lambda(\lambda, y)$

$$\Lambda P = S \Lambda$$

cf. Dieker–Warren '08

$$\Rightarrow \quad P = \Lambda^{-1} S \Lambda$$

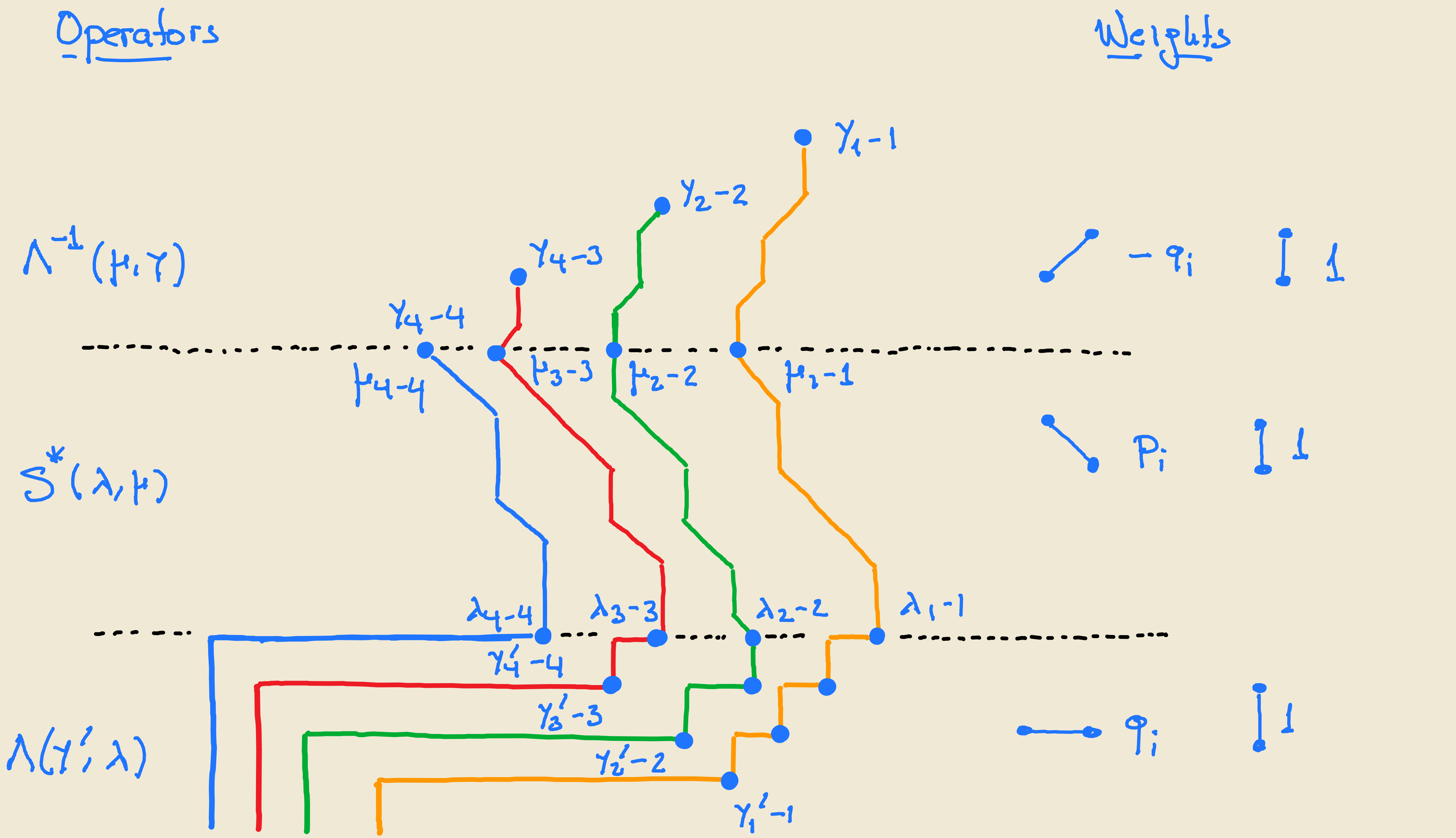
Λ : kernel of Schur functions fixing shape & left edge

$$\Lambda^{-1}(y, \mu) \overset{\text{DW08}}{=\!=} \det\left(e_{y_j - \mu_i - j + i}(0, \dots, 0, q_{j+1}, \dots, q_N) \right)_{i,j}$$

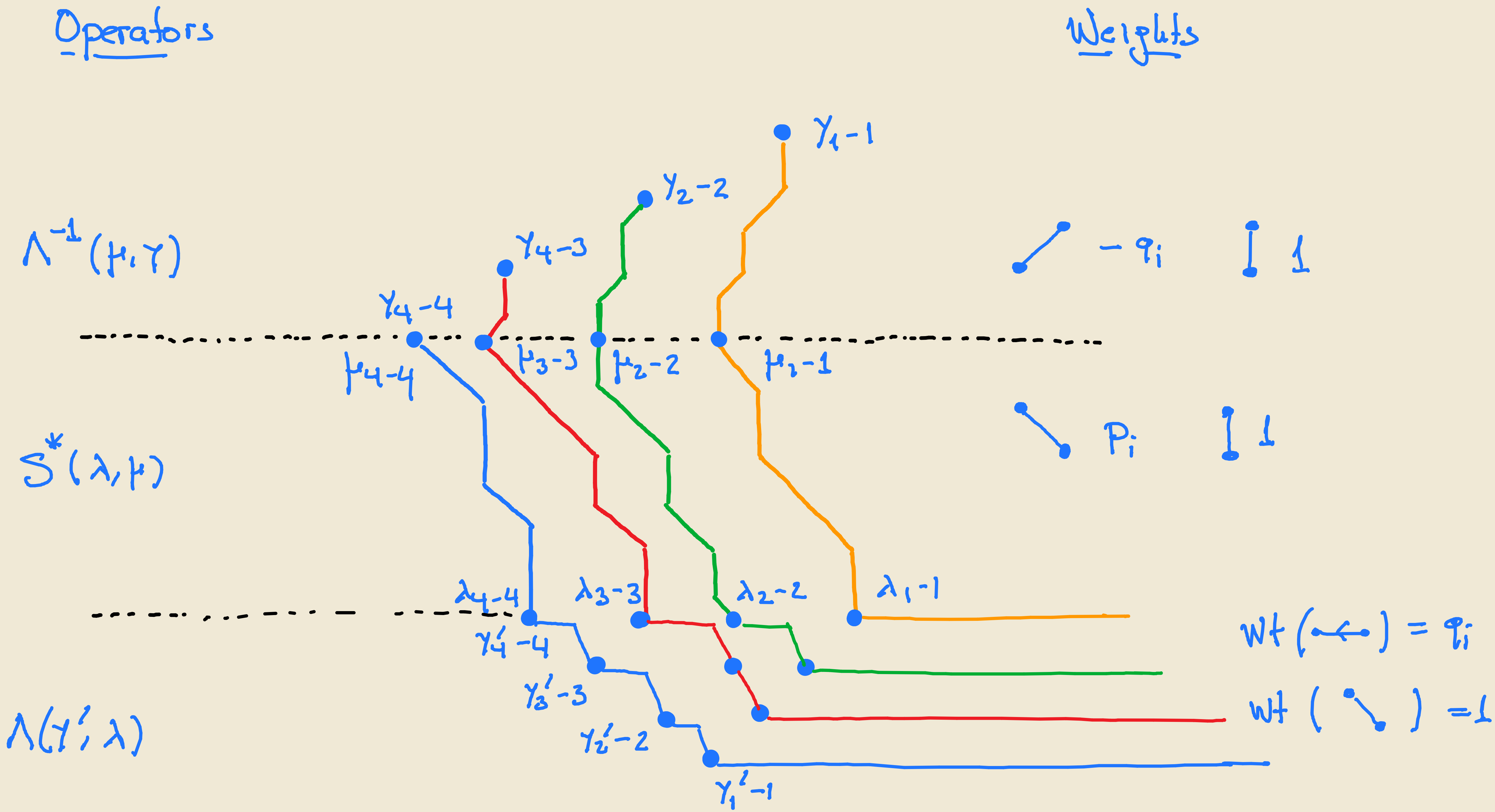
$$= \det\left((-1)^{N-j} Q^{-1}_{(j,N]}(\mu_i - i, y_j - j) \right)_{i,j}$$

with $Q_i^{-1}(x, y) = -1_{\substack{y \\ x}} + q_i 1_{\substack{y=x+1 \\ x}}$

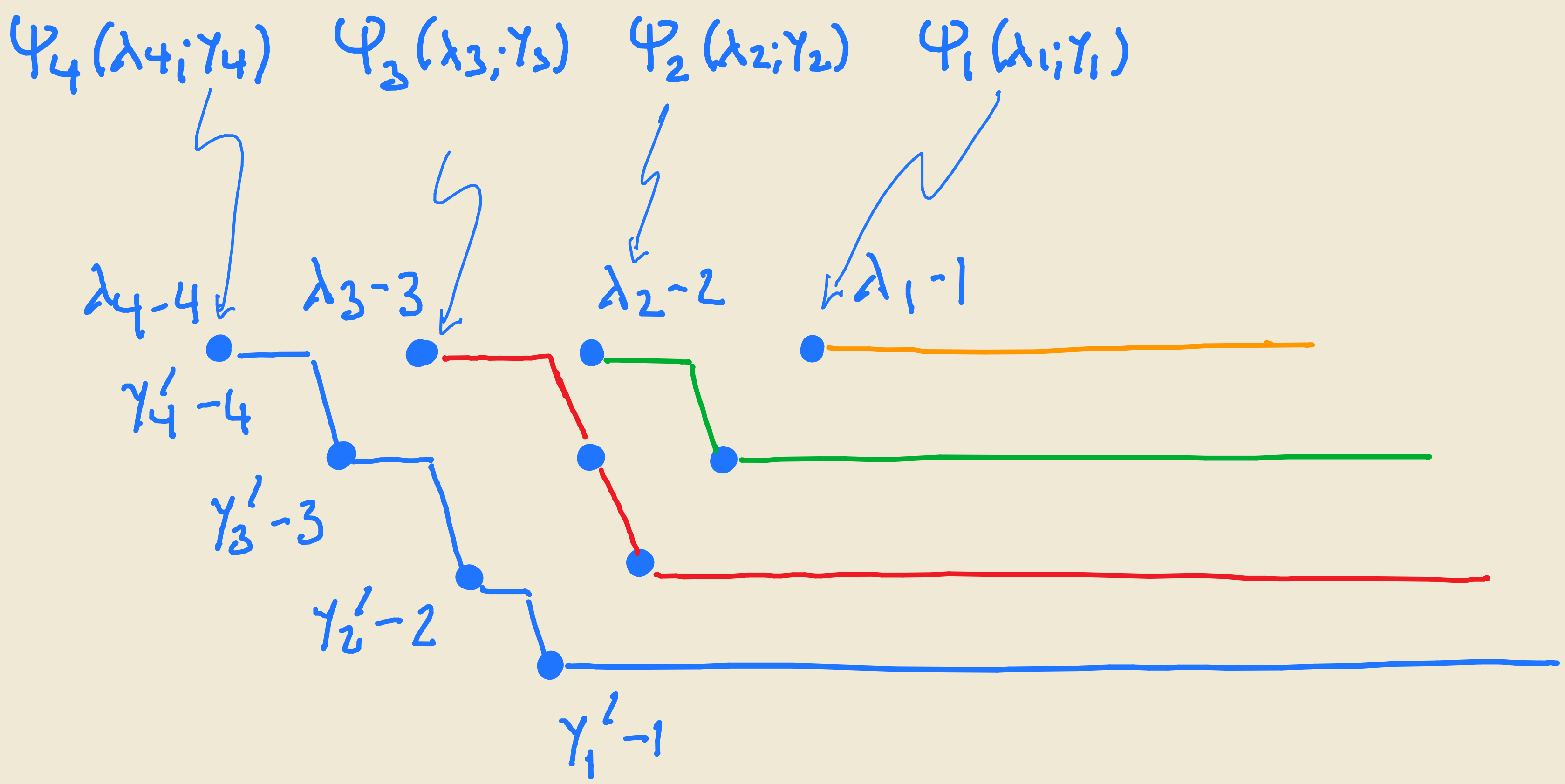
The path picture of intertwining



More appropriate path representation

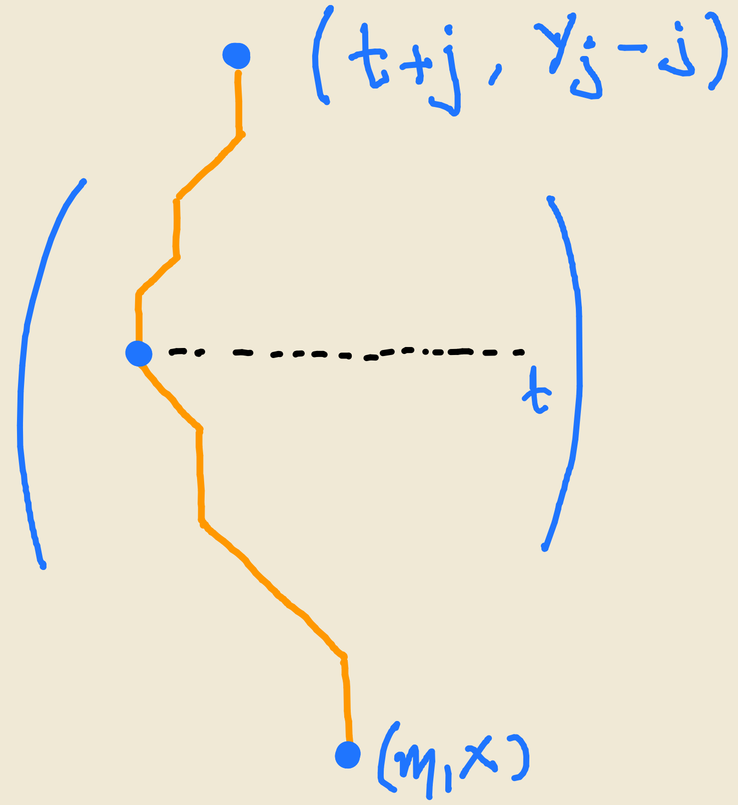


Restricted determinantal process



with

$$\Psi_i^m(x) = \mathrm{wt}\left(\vphantom{\Big|} \right.$$



$$\text{Total weight} \propto \prod_{k=1}^{N} \det\left(Q_k(x_{i-1}^{k-1}, x_j^k)\right)_{i,j \le k} \cdot \det\left(\Psi_i^N(x_j^N)\right)_{i,j \le N}$$

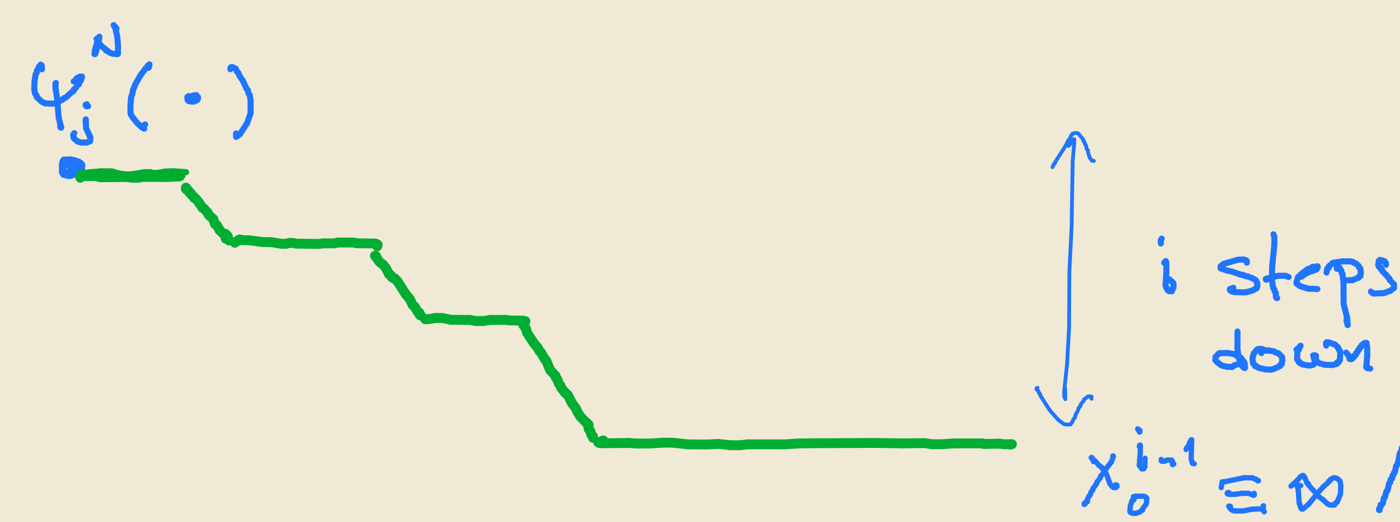
TASEP transition = restriction left marginal

$$= \det(I + K)$$

$$K(m, x; n, y) = -Q_{(m,n]}(x,y)\,\mathbb{1}_{n>m} + \sum_{i=1}^{N} \Psi_i^{m}(x) \cdot \underbrace{\sum_{j=1}^{N} M^{-1}_{i,j}\, Q_{[j,n]}(x_0^{j-1}, y)}_{\Phi_j^{n}(y)}$$

$\Phi_j^n(y)$ biorthogonal ensemble

with

$$M_{ij} = \mathrm{wt}\Big($$ 

$$\Big)$$

$$= \sum_{y} Q_i \circ Q_{i+1} \circ \cdots \circ Q_N(\infty, z)\, \varphi_j^N(z)$$

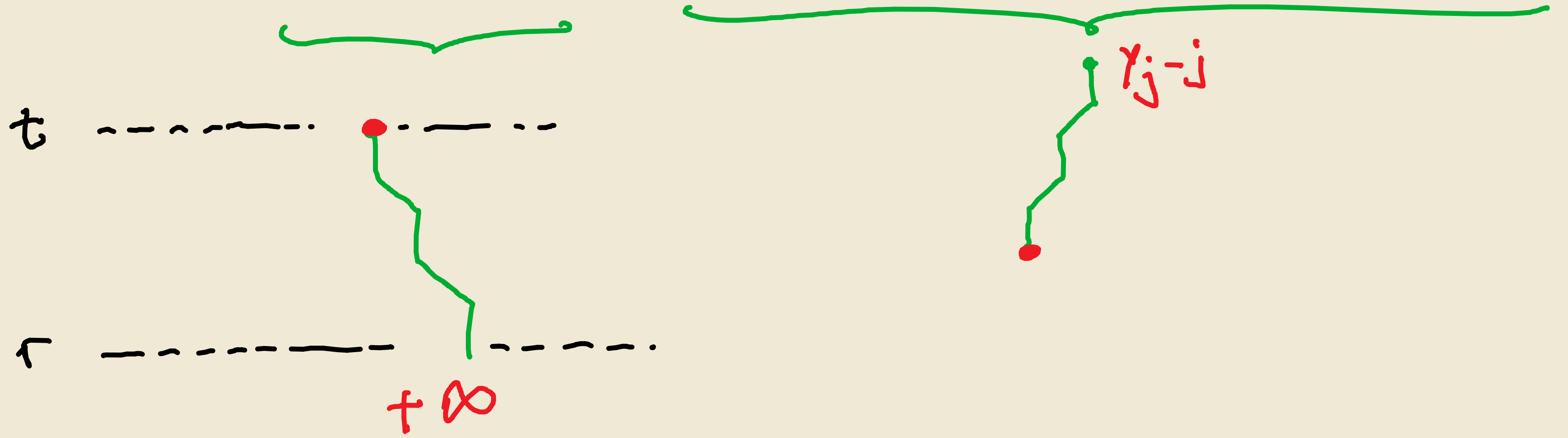
Key Observation : M_{ij} is upper-triangular

$$M_{ij} = \sum_{z} Q_i \circ \cdots \circ Q_N (\infty, z) \cdot \varphi_j^N(z)$$

$$= \sum_{z} Q_i \circ \cdots \circ Q_N (\infty, z) \cdot R_{(r,t]} \circ Q_N^{-1} \circ \cdots \circ Q_{j+1}^{-1} (z, y_j - j)$$

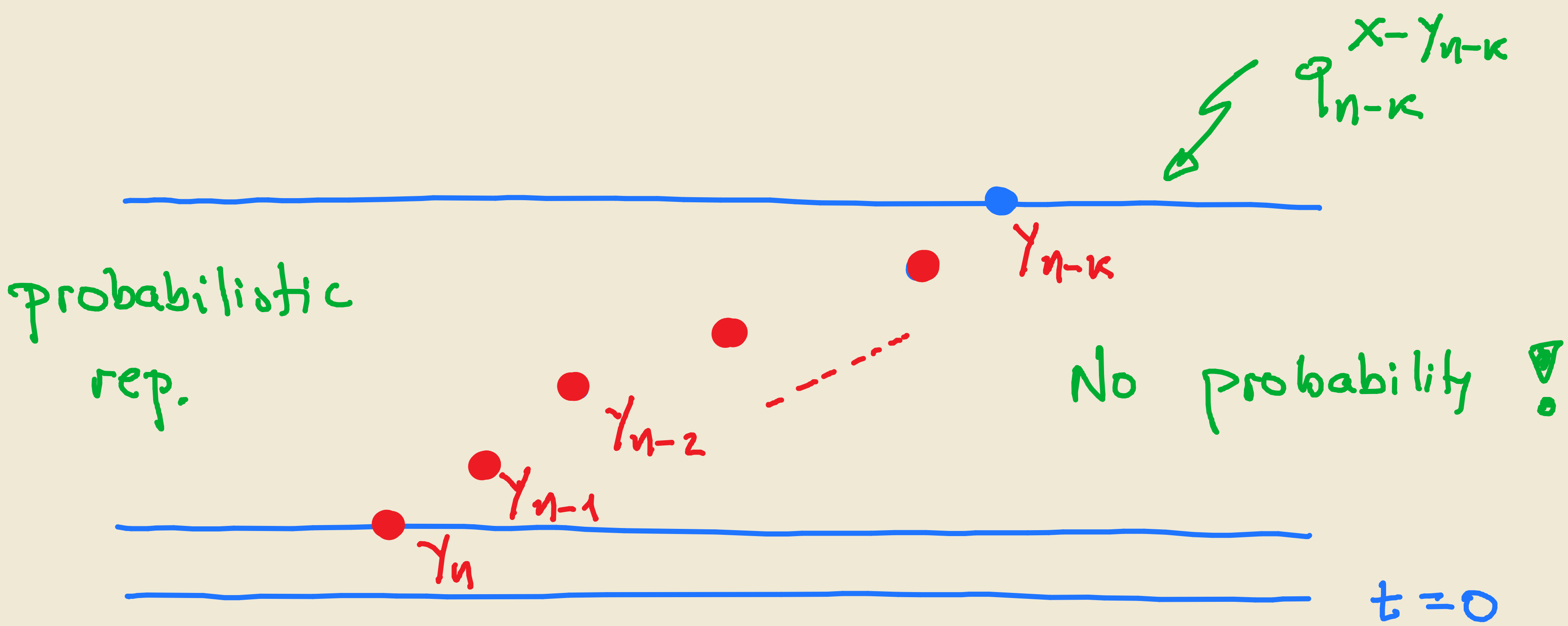
$$= R_{(r,t]} \circ Q_i \circ \cdots \circ Q_N \circ Q_N^{-1} \circ \cdots Q_{j+1}^{-1} (\infty, y_j - j)$$

$$\overset{\text{if } i>j}{=} R_{(r,t]} \circ Q_{j+1}^{-1} \circ \cdots \circ Q_{i-1}^{-1} (\infty, y_j - j)$$



$$= 0$$

The boundary value problem



If

$$h^n_k(\ell+1, x) = h^n_k(\ell, \cdot) \circ Q^{-1}_{n-\ell}(x) \qquad x \in \mathbb{Z},\ \ell < k$$

$$h^n_k(\ell, y_{n-\ell}) = 0 \qquad \ell < k$$

$$h^n_k(k, x) = \varphi_{n-k}^{x-y_{n-k}} \qquad x \in \mathbb{Z}$$

then

$$\Phi^n_{n-k}(x) = h^n_k(0, \cdot) \circ \left(R_{(r,t]}\right)^{-1}(x)$$

Thanks