Classifying boundary fluctuations for uniformly random Gelfand-Tsetlin patterns

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(Based on a forthcoming joint work with Kurt Johansson)

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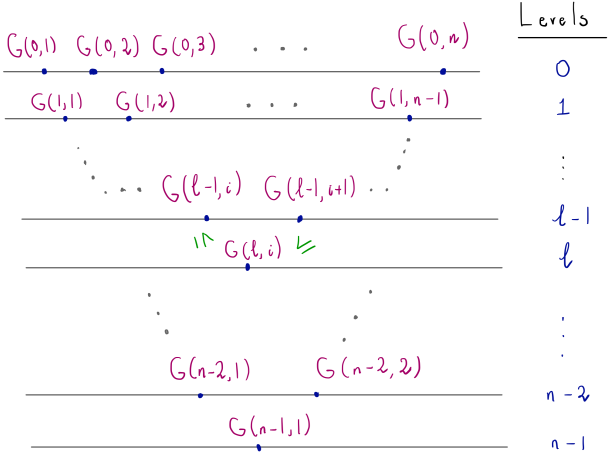
I. Uniformly random Gelfand-Tsetlin patterns

Gelfand-Tsetlin (GT) pattern

- A **GT pattern** G of **depth** n:



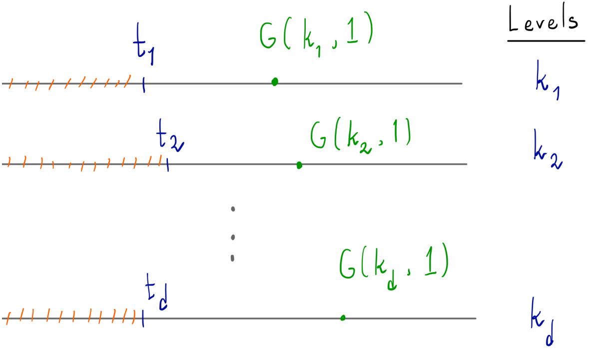
- $\mathrm{G}(\ell, i)$ denotes entry $i \in [n-\ell]$ on **level** $\ell \in [n-1] \cup \{0\}$.

Uniformly random GT patterns with fixed level zero

- Let $\mathsf{a} = (a_i)_{i \in [n]}$ be an **increasing** sequence of length $n \in \mathbb{Z}_{>1}$.
- Construct a random GT pattern $\mathrm{G} = \mathrm{G}^{\mathsf{a}}$ of depth n as follows.
- **Level zero:** $\mathrm{G}(0, i) = a_i$ for $i \in [n]$.
- The remaining entries $\{\mathrm{G}(\ell, i) : \ell \in [n-1], i \in [n-\ell]\}$ (particles) are **uniformly distributed**.

Multi-level distribution of first particles

- Let $\mathrm{k} = (k_i)_{i\in[d]} \in [n-1]^d$ be increasing and $\mathrm{t} = (t_i)_{i\in[d]} \in \mathbb{R}^d$ for some **dimension** $d \in [n-1]$.
- **CDF:** Define $\mathrm{F}_d(\mathrm{k},\mathrm{t}) = \mathrm{F}^{\mathrm{a}}_d(\mathrm{k},\mathrm{t}) = \mathbb{P}\{\mathrm{G}(k_i,1) \geqslant t_i : i \in [d]\}$.



Multi-level distribution of first particles

- From the determinantal structure **[Metcalfe '13]**,

$$F_d(\mathrm{k},\mathrm{t}) = \det[1 - \mathrm{K}]_{L^2(\bigcup_{i\in[d]}\{k_i\}\times\mathbb{R}_{<t_i})}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\mathrm{r}\in[d]^m} \int_{\mathbb{R}^m} \mathrm{dx} \prod_{i\in[m]} 1\{x_i < t_{r_i}\} \cdot \det_{i,j\in[m]} [\mathrm{K}(k_{r_i}, x_i; k_{r_j}, x_j)].$$

- **Metcalfe's kernel** $\mathrm{K} : ([n-1]\times\mathbb{R})^2 \to \mathbb{C}$ is of the form

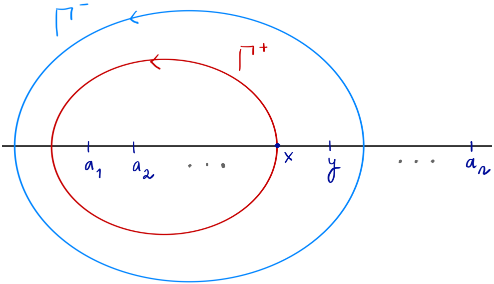
$$\mathrm{K}(k,x;\ell,y) = (\phi + \mathrm{I})(k,x;\ell,y)$$

for $k,\ell \in [n-1]$ and $x,y \in \mathbb{R}$.

Correlation kernel

- **Heat part:** $\phi(k,x;\ell,y) = 1_{\{x\geqslant y\}}1_{\{k>\ell\}}\dfrac{(y-x)^{k-\ell-1}}{(k-\ell-1)!}$.
- **Integral part:** $\mathrm{I}(k,x;\ell,y) = \mathrm{I}^{a}(k,x:\ell,y) =$

$$\frac{1}{4\pi^2}\frac{\ell!}{(k-1)!}\oint_{\Gamma^-}\mathrm{d}w\frac{\prod_{i\in[n]}(w-a_i)}{(w-y)^{\ell+1}}\oint_{\Gamma^+}\mathrm{d}z\frac{(z-x)^{k-1}}{\prod_{i\in[n]}(z-a_i)}\frac{1}{w-z}.$$



II: Five regimes of boundary fluctuations

Fluctuation regimes

- Five regimes of multi-level fluctuations of first particles.

1. **Bounded-level regime**
2. **(Generalized) Gaussian regime**
3. **Weierstrass regime**
4. **(Generalized) Baik-Ben Arous-Péché (BBP) regime**
5. **Airy regime**

Fluctuation regimes

- The regimes differ in **scaling**, and whether the terms of the level zero data (a_n) contribute **individually** and/or **collectively** to the limit process.

Regime	**Individual contribution**	**Aggregate contribution**	**Scaling exponent**
Bounded-level	Yes	Both possible	1
Gaussian	Yes	Yes	$1/2$
Weierstrass	Yes	No	$1/3$
BBP	Yes	Yes	$1/3$
Airy	No	Yes	$1/3$

- There can be **infinitely many outliers**.

Cauchy transform of level zero

- Let $\mathsf{a} = (a_i)_{i\in[n]}$ be an increasing sequence of length $n \in \mathbb{Z}_{>1}$.

- Define the (negative) **Cauchy transform** $\mathrm{A} = \mathrm{A}^{\mathsf{a}}$ of a by

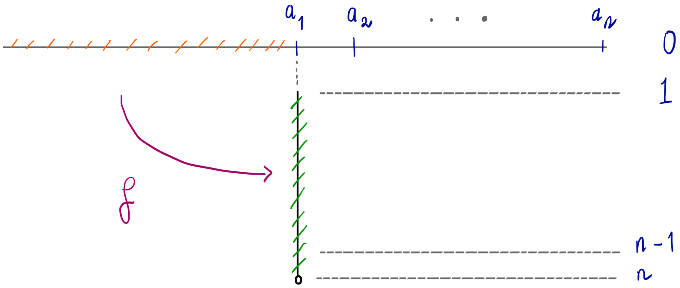
$$\mathrm{A}(z) = \sum_{i=1}^{n} \frac{1}{a_i - z} \quad \text{for } z \in \mathbb{C} \setminus \mathsf{a}.$$

- The fluctuation regimes can be described entirely in terms of the sequence $\mathrm{A}_n = \mathrm{A}^{\mathsf{a}_n}$.

Tracking levels

- **Level function:** Define $\rho = \rho^{\mathsf{a}} = \dfrac{\mathrm{A}^2}{\mathrm{A}'}$.

- $\rho : (-\infty, a_1) \to (1, n)$ is a decreasing bijection.



Tracking "curvature"

- **Curvature function:** Define $\kappa = \kappa^{\mathsf{a}} = \frac{1}{2}\mathrm{A}'' - \frac{(\mathrm{A}')^2}{\mathrm{A}}$.

- $\kappa > 0$ on $(-\infty, a_1)$.

- κ is connected to the **"curvature"** of the boundary.

Fluctuation regimes

- Consider a sequence of uniform GT patterns $\mathrm{G}_n = \mathrm{G}^{\mathsf{a}_n}$ of depth $\ell_n \in \mathbb{Z}_{>1}$ and level zero $\mathsf{a}_n = (\mathsf{a}_n(i))_{i \in [\ell_n]}$.

- Consider a sequence of levels $p_n \in [\ell_n - 1]$.

- **Critical (saddle) points:** Let $\zeta_n = \rho_n^{-1}(p_n)$ where $\rho_n = \rho^{\mathsf{a}_n}$.

Bounded-level regime

- **Scaling:** $(\mathsf{a}_n(i) - \zeta_n)\mathrm{A}_n(\zeta_n) \overset{n\to\infty}{\longrightarrow} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$.

- $b_1 < \infty$.

- The depth sequence $\ell_n \to \infty$.

Gaussian regime

- $(\mathrm{a}_n(1) - \zeta_n)\mathrm{A}_n(\zeta_n) \to \infty$.

- **Scaling:** $(\mathrm{a}_n(i) - \zeta_n)\mathrm{A}'_n(\zeta_n)^{1/2} \stackrel{n\to\infty}{\longrightarrow} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$.

- $b_1 < \infty$.

Weierstrass regime

- $(\mathsf{a}_n(1) - \zeta_n)\mathrm{A}_n'(\zeta_n)^{1/2} \to \infty$.

- **Scaling:** $(\mathsf{a}_n(i) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \overset{n\to\infty}{\longrightarrow} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$ and $b_1 < \infty$.

- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to \sum_{i=1}^{\infty} \dfrac{1}{b_i^3} \in (0, 1]$ where $\kappa_n = \kappa^{\mathsf{a}_n} = \dfrac{1}{2}\mathrm{A}'' - \dfrac{(\mathrm{A}')^2}{\mathrm{A}}$.

BBP regime

- $(\mathsf{a}_n(1) - \zeta_n)\mathrm{A}_n'(\zeta_n)^{1/2} \to \infty$.

- **Scaling:** $(\mathsf{a}_n(i) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \overset{n\to\infty}{\longrightarrow} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$

  and $b_1 < \infty$.

- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to \kappa_0 \in (\sum_{i=1}^{\infty} \dfrac{1}{b_i^3}, 1]$ where $\kappa_n = \kappa^{\mathsf{a}_n} = \dfrac{1}{2}\mathrm{A}'' - \dfrac{(\mathrm{A}')^2}{\mathrm{A}}$.

Airy regime

- **Scaling:** $(a_n(1) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \to \infty$.

- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to \kappa_0 \in (0,1]$ where $\kappa_n = \kappa^{a_n} = \dfrac{1}{2}\mathrm{A}'' - \dfrac{(\mathrm{A}')^2}{\mathrm{A}}$.

- The condition that $\kappa_0 > 0$ is technical.

~~(Hypothetical)~~ degenerate regime

- **Scaling:** $(\mathsf{a}_n(1) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \to \infty$.

- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to 0$ where $\kappa_n = \kappa^{\mathsf{a}_n} = \dfrac{1}{2}\mathrm{A}'' - \dfrac{(\mathrm{A}')^2}{\mathrm{A}}$.

- ~~Likely vacuous~~.

III: Limit kernels for large levels

Limiting kernel for the Gaussian regime

- **Limit parameters:** Let $\mathsf{b} = (b_i)_{i \in \mathbb{Z}_{>0}}$ be a nondecreasing sequence on $[1, \infty]$ such that $b_1 < \infty$ and $\sum_{i=1}^{\infty} \frac{1}{b_i^2} \leqslant 1$.

- The **limit kernel** $\widehat{\mathrm{K}}^{\mathsf{II},\mathsf{b}} : ((-\infty, b_1) \times \mathbb{R})^2 \to \mathbb{C}$ is of the form

$$\widehat{\mathrm{K}}^{\mathsf{II},\mathsf{b}}(u, s; v, t) = (\widehat{\phi}^{\mathsf{II},\mathsf{b}} + \widehat{\mathrm{I}}^{\mathsf{II},\mathsf{b}})(u, s; v, t)$$

for $u, v < b_1$ and $s, t \in \mathbb{R}$.

Limiting kernel for the Gaussian regime

- The **heat part** is given by $\widehat{\phi}^{\mathsf{II},\mathsf{b}}(u, s; v, t) =$

$$\frac{1_{\{u<v\}}}{\sqrt{2\pi}} \frac{1}{\sqrt{\mathrm{Q}^{\mathsf{b}}(v) - \mathrm{Q}^{\mathsf{b}}(u)}} \exp\left\{-\frac{1}{2} \cdot \frac{(\mathrm{R}^{\mathsf{b}}(u) - \mathrm{R}^{\mathsf{b}}(v) - s + t)^2}{\mathrm{Q}^{\mathsf{b}}(v) - \mathrm{Q}^{\mathsf{b}}(u)}\right\}.$$

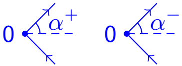
- $\mathrm{Q}^{\mathsf{b}}(u) = \sum_{i=1}^{\infty} \left\{ \frac{1}{(b_i - u)^2} - \frac{1}{b_i^2} \right\}.$

- $\mathrm{R}^{\mathsf{b}}(u) = u^2 \sum_{i=1}^{\infty} \frac{1}{b_i(b_i - u)^2}.$

Limiting kernel for the Gaussian regime

- The **integral part** is given by $\widehat{\mathrm{I}}^{\mathrm{II,b}}(u,s;v,t) =$

$$-\frac{1}{4\pi^2} \int \int \exp\{-\mathrm{f}^{\mathrm{II,b}}_{u,s}(z) + \mathrm{f}^{\mathrm{II,b}}_{v,t}(w)\} \frac{\mathrm{d}w\ \mathrm{d}z}{w-z}$$



where $\alpha^+ \in (0, \pi/4)$ and $\alpha^- \in (\pi/4, 3\pi/4)$.

- $\mathrm{f}^{\mathrm{II,b}}_{u,s}(z) = \frac{1}{2} \cdot z^2 \cdot \sum_{i=1}^{\infty} \frac{1}{(b_i - u)^2} + \mathrm{W}^{\mathrm{b}}_1(z) + z \cdot (s - \mathrm{R}^{\mathrm{b}}(u))$

Limiting kernel for the Gaussian regime

- $\mathrm{W}_1^{\mathrm{b}}$ denotes the **Weierstrass sum** of order 1 given by

$$\mathrm{W}_1^{\mathrm{b}}(z) = \sum_{i=1}^{\infty} \left\{ \log\left(1 - \frac{z}{b_i}\right) + \frac{z}{b_i} \right\} \quad \text{for } z \in \mathbb{C} \setminus [b_1, \infty)$$

  where the logarithms are the principal branch.

- The series above converges since $\displaystyle\sum_{i=1}^{\infty} \frac{1}{b_i^2} < \infty$.

Limiting kernel for the Gaussian regime

- If $b_2 = \infty$ then the limit process is a **Brownian motion**.

- The case $b_1 = b_N < b_{N+1} = \infty$ for some $N \in \mathbb{Z}_{>0}$ recovers the extended versions of the **generalized Gaussian kernels** of Baik-Ben Arous-Péché **[Baik-Ben Arous-Péché '06], [Knizel-Petrov-Saenz '18], [Imamura-Sasamoto '05,'07]**.

Limiting kernel for the Weierstrass regime

- **Limit parameters:** Let $\mathrm{b} = (b_i)_{i \in \mathbb{Z}_{>0}}$ be a nondecreasing sequence on $[1, \infty]$ such that $b_1 < \infty$ and $\sum_{i=1}^{\infty} \frac{1}{b_i^3} \leqslant 1$.

- The **limit kernel** $\widehat{\mathrm{K}}^{\mathsf{III},\mathrm{b}} : ((-\infty, b_1) \times \mathbb{R})^2 \to \mathbb{C}$ is of the form

$$\widehat{\mathrm{K}}^{\mathsf{III},\mathrm{b}}(u, s; v, t) = (\widehat{\phi}^{\mathsf{III},\mathrm{b}} + \widehat{\mathrm{I}}^{\mathsf{III},\mathrm{b}})(u, s; v, t)$$

for $u, v < b_1$ and $s, t \in \mathbb{R}$.

Limiting kernel for the Weierstrass regime

- The **heat part** is given by $\widehat{\phi}^{\mathrm{III,b}}(u,s;v,t) = \widehat{\phi}^{\mathrm{II,b}}(u,s;v,t) =$

$$\frac{1_{\{u<v\}}}{\sqrt{2\pi}} \frac{1}{\sqrt{\mathrm{Q}^{\mathrm{b}}(v) - \mathrm{Q}^{\mathrm{b}}(u)}} \exp\left\{ -\frac{1}{2} \cdot \frac{(\mathrm{R}^{\mathrm{b}}(u) - \mathrm{R}^{\mathrm{b}}(v) - s + t)^2}{\mathrm{Q}^{\mathrm{b}}(v) - \mathrm{Q}^{\mathrm{b}}(u)} \right\}.$$

- $\mathrm{Q}^{\mathrm{b}}(u) = \sum_{i=1}^{\infty} \left\{ \frac{1}{(b_i - u)^2} - \frac{1}{b_i^2} \right\}.$

- $\mathrm{R}^{\mathrm{b}}(u) = u^2 \sum_{i=1}^{\infty} \frac{1}{b_i(b_i - u)^2}.$

Limiting kernel for the Weierstrass regime

- The **integral part** is given by $\widehat{\mathrm{I}}^{\mathrm{III,b}}(u,s;v,t) =$

$$-\frac{1}{4\pi^2}\int_{0\,\angle\,\alpha^+}\int_{0\,\angle\,\alpha^-} \exp\{-\mathrm{f}^{\mathrm{III,b}}_{u,s}(z)+\mathrm{f}^{\mathrm{III,b}}_{v,t}(w)\}\frac{\mathrm{d}w\,\mathrm{d}z}{w-z}$$

  where $\alpha^+ \in (\pi/6,\pi/4)$ and $\alpha^- \in (\pi/2,3\pi/4)$.

- $\mathrm{f}^{\mathrm{III,b}}_{u,s}(z) = \mathrm{W}^{\mathrm{b}}_2(z) + \frac{1}{2}\cdot z^2\mathrm{Q}^{\mathrm{b}}(u) + z\cdot(s-\mathrm{R}^{\mathrm{b}}(u))$

Limiting kernel for the Weierstrass regime

- $\mathrm{W}_2^{\mathsf{b}}$ denotes the **Weierstrass sum** of order 2 given by

  $$\mathrm{W}_2^{\mathsf{b}}(z) = \sum_{i=1}^{\infty} \left\{ \log\left(1 - \frac{z}{b_i}\right) + \frac{z}{b_i} + \frac{z^2}{2b_i^2} \right\} \quad \text{for } z \in \mathbb{C} \setminus [b_1, \infty)$$

  where the logarithms are the principal branch.

- The series above converges since $\sum_{i=1}^{\infty} \frac{1}{b_i^3} < \infty$.

Limiting kernel for the Weierstrass regime

- If $\sum_{i=1}^{\infty} \frac{1}{b_i^2} < \infty$ then $\widehat{\mathrm{K}}^{\mathsf{III,b}} = \widehat{\mathrm{K}}^{\mathsf{II,b}}$ but the two regimes still differ in **scaling**.
- If $\sum_{i=1}^{\infty} \frac{1}{b_i^2} = \infty$ then the limit process is **novel**.
- A similar kernel appeared in **exponential LPP** with **growing inhomogeneous parameters** **[Johansson 07]**.

Limiting kernel for the BBP regime

- **Limit parameters:** Let $\mathrm{b} = (b_i)_{i\in\mathbb{Z}_{>0}}$ be a nondecreasing sequence on $[1,\infty]$ such that $b_1 < \infty$ and $\sum_{i=1}^{\infty} \frac{1}{b_i^3} < \kappa_0$ for some $\kappa_0 \in (0,1]$.

- The **limit kernel** $\widehat{\mathrm{K}}^{\mathsf{IV},\mathrm{b},\kappa_0} : ((-\infty, b_1) \times \mathbb{R})^2 \to \mathbb{C}$ is of the form

$$\widehat{\mathrm{K}}^{\mathsf{IV},\mathrm{b},\kappa_0}(u,s;v,t) = (\widehat{\phi}^{\mathsf{IV},\mathrm{b},\kappa_0} + \widehat{\mathrm{I}}^{\mathsf{IV},\mathrm{b},\kappa_0})(u,s;v,t)$$

for $u, v < b_1$ and $s, t \in \mathbb{R}$.

Limiting kernel for the BBP regime

- The **heat part** is given by $\widehat{\phi}^{\mathrm{IV},\mathrm{b},\kappa_0}(u,s;v,t) =$

$$\frac{1_{\{u<v\}}}{\sqrt{2\pi}} \frac{1}{\sqrt{2(v-u)K_0^{\mathrm{b},\kappa_0} + \mathrm{Q}^{\mathrm{b}}(v) - \mathrm{Q}^{\mathrm{b}}(u)}}$$

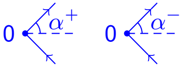
$$\cdot \exp\left\{ -\frac{1}{2} \cdot \frac{\left((u^2-v^2)K_0^{\mathrm{b},\kappa_0} + \mathrm{R}^{\mathrm{b}}(u) - \mathrm{R}^{\mathrm{b}}(v) - s + t\right)^2}{2(v-u)K_0^{\mathrm{b},\kappa_0} + \mathrm{Q}^{\mathrm{b}}(v) - \mathrm{Q}^{\mathrm{b}}(u)} \right\}.$$

- $K_0^{\mathrm{b},\kappa_0} = \kappa_0 - \sum_{i=1}^{\infty} \frac{1}{b_i^3} > 0.$

Limiting kernel for the BBP regime

- The **integral part** is given by $\widehat{\mathrm{I}}^{\mathsf{IV},\mathsf{b},\kappa_0}(u,s;v,t) =$

$$-\frac{1}{4\pi^2}\int\int \exp\{-\mathrm{f}_{u,s}^{\mathsf{IV},\mathsf{b},\kappa_0}(z)+\mathrm{f}_{v,t}^{\mathsf{IV},\mathsf{b},\kappa_0}(w)\}\frac{\mathrm{d}w\ \mathrm{d}z}{w-z}$$



  where $\alpha^+ \in (\pi/6,\pi/2)$ and $\alpha^- \in (\pi/2, 5\pi/6)$.

- $$\mathrm{f}_{u,s}^{\mathsf{IV},\mathsf{b},\kappa_0}(z) = -\frac{1}{3}K_0^{\mathsf{b},\kappa_0}z^3 + \mathrm{W}_2^{\mathsf{b}}(z) + \frac{1}{2}\cdot z^2(2uK_0^{\mathsf{b},\kappa_0}+\mathrm{Q}^{\mathsf{b}}(u)) + z\cdot(s-u^2K_0^{\mathsf{b},\kappa_0}-\mathrm{R}^{\mathsf{b}}(u))$$

Limiting kernel for the BBP regime

- If $b_2 = \infty$ then the marginals of the limit process coincides with a distribution of **[Baik-Rains '00]** up to rescaling.

- The case $b_N < b_{N+1} = \infty$ for some $N \in \mathbb{Z}_{>0}$ recovers the extended version of the **BBP kernel** **[Baik-Ben Arous-Péché '06], [Knizel-Petrov-Saenz '18], [Imamura-Sasamoto '07]**.

Limiting kernel for the Airy regime

- **Limit parameter:** Let $\kappa_0 \in (0,1]$. One can view that $b_1 = \infty$.

- The **limit kernel** $\widehat{\mathrm{K}}^{\mathsf{V},\kappa_0} : (\mathbb{R} \times \mathbb{R})^2 \to \mathbb{C}$ is of the form

$$\widehat{\mathrm{K}}^{\mathsf{V},\kappa_0}(u,s;v,t) = (\widehat{\phi}^{\mathsf{V},\kappa_0} + \widehat{\mathrm{I}}^{\mathsf{V},\kappa_0})(u,s;v,t)$$

  for $u, v, s, t \in \mathbb{R}$.

Limiting kernel for the Airy regime

- The **heat part** is given by $\widehat{\phi}^{\mathsf{V},\kappa_0}(u,s;v,t) =$

$$\frac{1_{\{u<v\}}}{\sqrt{2\pi}}\frac{1}{\sqrt{2(v-u)\kappa_0}}\exp\left\{-\frac{1}{2}\cdot\frac{((u^2-v^2)\kappa_0-s+t)^2}{2(v-u)\kappa_0}\right\}.$$

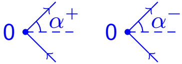
- $K_0^{\mathsf{b},\kappa_0} = \kappa_0 - \sum_{i=1}^{\infty}\frac{1}{b_i^3} = \kappa_0$ since $b_1 = \infty$.

- $\mathrm{Q}^{\mathsf{b}}(u) = 0$ and $\mathrm{R}^{\mathsf{b}}(u) = 0$ since $b_1 = \infty$.

Limiting kernel for the Airy regime

- The **integral part** is given by $\widehat{\mathrm{I}}^{\mathsf{V},\kappa_0}(u,s;v,t) =$

$$-\frac{1}{4\pi^2}\int\int \exp\{-\mathrm{f}^{\mathsf{V},\kappa_0}_{u,s}(z)+\mathrm{f}^{\mathsf{V},\kappa_0}_{v,t}(w)\}\frac{\mathrm{d}w\ \mathrm{d}z}{w-z}$$



where $\alpha^+ \in (\pi/6, \pi/2)$ and $\alpha^- \in (\pi/2, 5\pi/6)$.

- $\mathrm{f}^{\mathsf{V},\kappa_0}_{u,s}(z) = -\frac{1}{3}\kappa_0 z^3 + z^2 u\kappa_0 + z\cdot(s - u^2\kappa_0)$.

Limiting kernel for the Airy regime

- It is possible to **remove** κ_0 via rescaling.

- The limit process is the **extended Airy process** (**[Prähofer-Spohn '02]**) up to rescaling.

IV. Classifying boundary fluctuations for large levels

Estimating boundary

- Let $a = (a_i)_{i \in [n]}$ be an increasing sequence of length $n \in \mathbb{Z}_{>1}$.
- **Boundary function:** For $p \in (1, n)$, define

$$\gamma^p = \gamma^{p,a} = \sup_{z < a_1} \left\{ z + \frac{p}{\mathrm{A}(z)} \right\}.$$

- γ^p **approximates** $\mathrm{G}(p, 1)$, the position of the first particle on level $p \in [n-1]$.

Estimating boundary

- The **unique maximizer** is the critical point $\zeta = \rho^{-1}(p) \in (-\infty, a_1)$ where $\rho = \dfrac{\mathrm{A}^2}{\mathrm{A}'}$ is the level function.

- **Boundary function:** For $p \in (1, n)$,

$$\gamma^p = \sup_{z<a_1} \left\{ z + \frac{p}{\mathrm{A}(z)} \right\} = \zeta + \frac{p}{\mathrm{A}(\zeta)} = \zeta + \frac{\mathrm{A}(\zeta)}{\mathrm{A}'(\zeta)}.$$

Setup for fluctuation results

- Consider a sequence of **uniform GT patterns** $\mathrm{G}_n = \mathrm{G}^{\mathsf{a}_n}$ of depth $\ell_n \in \mathbb{Z}_{>1}$ and level zero $\mathsf{a}_n = (\mathsf{a}_n(i))_{i \in [\ell_n]}$.

- Consider a sequence of **levels** $p_n \in [\ell_n - 1]$.

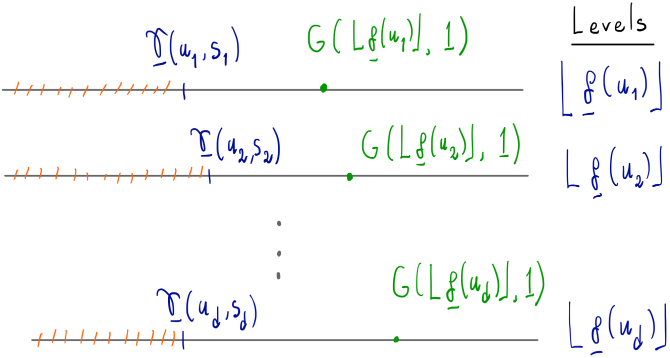
Setup for fluctuation results

- Fix a **dimension** $d \in \mathbb{Z}_{>0}$.
- Fix $\eta_0 > 0$ and $u_{\min}, u_{\max} \in \mathbb{R}$ with $u_{\min} \leqslant 0 \leqslant u_{\max}$.
- **Level variables:** Let $\mathrm{u} = (u_i)_{i\in[d]} \in [u_{\min}, u_{\max}]^d$ be a sequence with $u_i \geqslant u_{i+1} + \eta_0$ for $i \in [d-1]$.
- **Position variables:** Fix $T_0 > 0$ and let $\mathrm{s} = (s_i)_{i\in[d]} \in [-T_0, T_0]^d$.

Rescaled joint CDF of first particles

- **Rescaled CDF:**

$$\underline{\mathrm{F}}_{n,d}(\mathsf{u},\mathsf{s}) = \underline{\mathrm{F}}_{d}^{a_n,p_n}(\mathsf{u},\mathsf{s}) = \mathbb{P}\{\mathrm{G}_n(\lfloor\underline{\rho}(u_i)\rfloor, 1) \geqslant \underline{\gamma}(u_i, s_i), i \in [d]\}.$$



Limiting joint CDF of first particles

- **Large-level regimes:** Let $\square \in \{II, III, IV, V\}$.

- **Limit CDF:** Define $\widehat{\mathrm{F}}^{\square}_{d}(\mathrm{u}, \mathrm{s}) = \det[1 - \widehat{\mathrm{K}}^{\square}]_{L^2(\bigcup_{i\in[d]}\{u_i\}\times\mathbb{R}_{\geqslant s_i})}$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\mathrm{r}\in[d]^m} \int_{\mathbb{R}^m} \mathrm{dt} \prod_{i\in[m]} 1_{\{t_i > s_{r_i}\}} \cdot \det_{i,j\in[m]} [\widehat{\mathrm{K}}^{\square}(u_{r_i}, t_i; u_{r_j}, t_j)].$$

Classification theorem for large levels

II. Gaussian regime. Fix $\epsilon_0 > 0$. Assume that

- $(\mathsf{a}_n(1) - \zeta_n)\mathrm{A}_n(\zeta_n) \to \infty$.
- $(\mathsf{a}_n(i) - \zeta_n)\mathrm{A}'_n(\zeta_n)^{1/2} \overset{n\to\infty}{\to} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$.
- $b_1 < \infty$ and $b_1 \geqslant u_{\max} + \epsilon_0$.

Level ratio constraint: Furthermore, assume that $\dfrac{\underline{\rho}_n(u_{\min})}{\underline{\rho}_n(u_{\max})} \leqslant r_0$ where $r_0 > 1$ is an absolute constant (purely technical).

Then $\underline{\mathrm{F}}_{n,d}(\mathsf{u}, \mathsf{s}) \overset{n\to\infty}{\to} \widehat{\mathrm{F}}_d^{II,\mathsf{b}}(\mathsf{u}, \mathsf{s})$ uniformly in u, s and b.

Classification theorem for large levels

III. Weierstrass regime. Fix $\epsilon_0 > 0$. Assume that

- $(\mathsf{a}_n(1) - \zeta_n)(\mathrm{A}_n'(\zeta_n))^{1/2} \to \infty$.
- $(\mathsf{a}_n(i) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \overset{n\to\infty}{\to} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$.
- $b_1 < \infty$ and $b_1 \geqslant u_{\max} + \epsilon_0$.
- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to \displaystyle\sum_{i=1}^{\infty} \frac{1}{b_i^3}$.

Then $\underline{\mathrm{F}}_{n,d}(\mathsf{u}, \mathsf{s}) \overset{n\to\infty}{\to} \widehat{\mathrm{F}}_d^{III,\mathsf{b}}(\mathsf{u}, \mathsf{s})$ uniformly in u, s and b.

Classification theorem for large levels

IV. BBP regime. Fix $\epsilon_0 > 0$. Assume that

- $(\mathrm{a}_n(1) - \zeta_n)(\mathrm{A}_n'(\zeta_n))^{1/2} \to \infty$.
- $(\mathrm{a}_n(i) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \overset{n\to\infty}{\to} b_i \in [1, \infty]$ for $i \in \mathbb{Z}_{>0}$.
- $b_1 < \infty$ and $b_1 \geqslant u_{\max} + \epsilon_0$.
- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to \kappa_0 \geqslant \sum_{i=1}^{\infty} \dfrac{1}{b_i^3} + \epsilon_0$.

Then $\underline{\mathrm{F}}_{n,d}(\mathrm{u}, \mathrm{s}) \overset{n\to\infty}{\to} \widehat{\mathrm{F}}_d^{IV,\mathrm{b},\kappa_0}(\mathrm{u}, \mathrm{s})$ uniformly in u, s, b and κ_0.

Classification theorem for large levels

V. Airy regime. Fix $\epsilon_0 > 0$. Assume that

- $(\mathsf{a}_n(1) - \zeta_n)(\frac{1}{2}\mathrm{A}_n''(\zeta_n))^{1/3} \to \infty$.
- $\dfrac{\kappa_n(\zeta_n)}{\frac{1}{2}\mathrm{A}_n''(\zeta_n)} \to \kappa_0 \geqslant \epsilon_0$.

Then $\underline{\mathrm{F}}_{n,d}(\mathsf{u},\mathsf{s}) \overset{n\to\infty}{\to} \widehat{\mathrm{F}}_d^{V,\kappa_0}(\mathsf{u},\mathsf{s})$ uniformly in u, s and κ_0.

IV. Some specializations

Case of a limit shape

- Consider a sequence of uniform GT patterns $\mathrm{G}_n = \mathrm{G}^{\mathsf{a}_n}$ of depth $\ell_n \to \infty$ and level zero $\mathsf{a}_n = (\mathsf{a}_n(i))_{i \in [\ell_n]}$.

Assumptions.

(i) $\mathsf{a}_n(1) = a_0$ for $n \in \mathbb{Z}_{>0}$ for some fixed $a_0 \in \mathbb{R}$.

(ii) $\frac{1}{\ell_n} \sum_{i=1}^{\ell_n} \delta_{\mathsf{a}_n(i)} \overset{n \to \infty}{\to} \mu$ vaguely for some subprobability measure μ on $\mathbb{R}$ such that $\mu \neq 0$ and $\# \operatorname{supp} \mu > 1$.

Shape function

- Define the **shape function** $\widehat{\gamma} = \widehat{\gamma}^{a_0,\mu}$ by

$$\widehat{\gamma}(r) = \sup_{z \leqslant a_0} \left\{ z + \frac{r}{\widehat{\mathrm{A}}(z)} \right\} \quad \text{for } r \in (0, \mu(\mathbb{R}))$$

- $\widehat{\mathrm{A}} = \widehat{\mathrm{A}}^{\mu}$ denotes the negative of the **Cauchy transform** of μ:

$$\widehat{\mathrm{A}}(z) = \int_{\mathbb{R}} \frac{\mu(\mathrm{d}a)}{a - z} \quad \text{for } z \in \mathbb{C} \setminus \operatorname{supp} \mu.$$

Limit shape

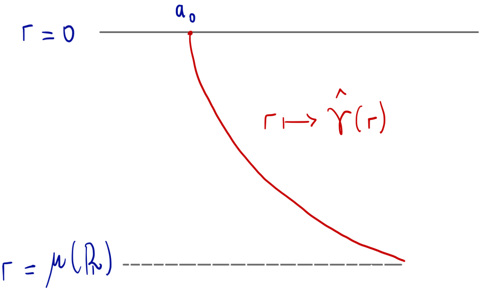
- Consider a sequence of levels $p_n \in [\ell_n - 1]$ such that $\dfrac{p_n}{\ell_n} \to r$ for some fixed ratio $r \in (0, \mu(\mathbb{R}))$.

- **(Weak) shape theorem.**

$$\frac{1}{\ell_n} \mathrm{G}_n(p_n, 1) \to \widehat{\gamma}(r) \qquad \text{in probability.}$$

Shape function is convex

- $r \mapsto \widehat{\gamma}(r) = \sup_{z \leqslant a_0} \left\{ z + \frac{r}{\widehat{\mathrm{A}}(z)} \right\}$ is **convex**.



Flat part of shape function

- **Limit level function:** Define $\widehat{\rho} = \widehat{\rho}^{\mu} = \dfrac{\widehat{\mathrm{A}}^2}{\widehat{\mathrm{A}}'}$.

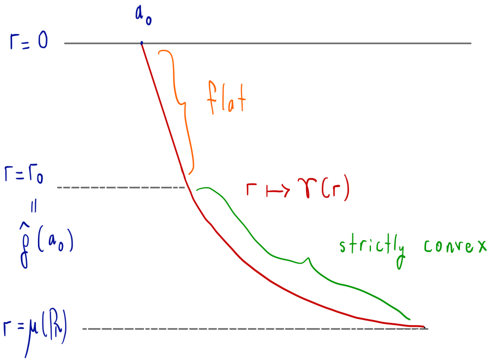
- Let $\underline{\mu} = \inf \operatorname{supp} \mu$.

- $\widehat{\rho}$ is a decreasing bijection from $(-\infty, \underline{\mu})$ onto $(\widehat{\rho}(\underline{\mu}), \mu(\mathbb{R}))$ for some $\rho(\underline{\mu}) \in [0, \mu(\mathbb{R}))$.

Flat part of shape function

- $a_0 = \mathrm{a}_n(1) \leqslant \underline{\mu}$.

- $\widehat{\gamma}$ has a **flat segment** if and only if $r_0 = \widehat{\rho}(a_0) > 0$.



Theorem. Assume that $r \in (r_0, \mu(\mathbb{R}))$. Let

$$\kappa_0 = \kappa_0^{\mu,r} = \frac{\widehat{\kappa}(\widehat{\zeta}(r))}{\widehat{\mathrm{A}}(\widehat{\zeta}(r))}$$

where $\widehat{\kappa} = \frac{1}{2}\widehat{\mathrm{A}}'' - \dfrac{(\widehat{\mathrm{A}}')^2}{\widehat{\mathrm{A}}}$ and $\widehat{\zeta} = (\widehat{\rho})^{-1}$.

Then $\underline{\mathrm{F}}_{n,d}(\mathsf{u},\mathsf{s}) \to \widehat{\mathrm{F}}_d^{V,\kappa_0}(\mathsf{u},\mathsf{s})$ uniformly in u and s.

A model with a limiting density

- **Level zero:** Fix $q \in (1,2)$. Assume that

  $$\mathrm{a}_n(i) = \left(\frac{i-1}{\ell_n}\right)^{1/(q+1)} \quad \text{for } i \in [\ell_n - 1] \text{ and } n \in \mathbb{Z}_{>0}.$$

- $\mathrm{a}_n(1) = a_0 = 0$.

- **Limit measure:** $\mu(\mathrm{d}a) = (q+1)1_{\{a\in[0,1]\}}a^q\mathrm{d}a$.

- The shape function $\widehat{\gamma}$ has a **flat segment:** $r_0 > 0$.

Fluctuations on the flat segment

Theorem. Fix $\epsilon_0 > 0$. Assume that $r \in (0, r_0)$. Let

$$b_1 = \left(1 - \frac{r}{r_0}\right)^{-1/2} \quad \text{and} \quad b_2 = \infty \quad \text{(Brownian motion)}.$$

Assume that $b_1 \geqslant u_{\max} + \epsilon_0$.

Furthermore, assume that $\dfrac{\underline{\rho}_n(u_{\min})}{\underline{\rho}_n(u_{\max})} \leqslant r_0$ where $r_0 > 1$ is an absolute constant (purely technical).

Then $\underline{\mathrm{F}}_{n,d}(\mathrm{u}, \mathrm{s}) \to \widehat{\mathrm{F}}_d^{II,\mathrm{b}}(\mathrm{u}, \mathrm{s})$ uniformly in u and s.

Fluctuations within the critical window

Theorem. Fix $\epsilon_0 > 0$. Assume that $r = r_0$ and

$$(p_n - p_n^{\mathrm{crit}}) \cdot \ell_n^{-2/(1+q)} \to x \in \mathbb{R}$$

where $p_n^{\mathrm{crit}} = \rho_n(-\ell_n^{-1/(1+q)})$.

Then $\underline{\mathrm{F}}_{n,d}(\mathsf{u},\mathsf{s}) \to \widehat{\mathrm{F}}_d^{III,\mathsf{b}}(\mathsf{u},\mathsf{s})$ uniformly in u and s provided that $b_1 \geqslant u_{\max} + \epsilon_0$ where

$$b_i = b_i^q(x) = ((i-1)^{1/(1+q)} + y) \cdot \left(\sum_{j=1}^{\infty} \frac{1}{((j-1)^{1/(1+q)} + y)^3} \right)^{1/3}$$

for $i \in \mathbb{Z}_{>0}$, and

Fluctuations within the critical window

$y = y^q(x) > 0$ is defined implicitly through

$$x = \frac{\hat{\mathrm{A}}(0)^2}{\hat{\mathrm{A}}'(0)^2} \sum_{j=1}^{\infty} \left\{ \frac{1}{((j-1)^{1/(1+q)} + 1)^2} - \frac{1}{((j-1)^{1/(1+q)} + y)^2} \right\}.$$

- $\sum_{i=1}^{\infty} \frac{1}{b_i^2} \approx \sum_{i=1}^{\infty} \frac{1}{i^{2/(1+q)}} = \infty$ since $q \in (1,2)$.

- Therefore, the limit process is specific to the **Weierstrass regime**.

Thanks!