

# Ergodicity and Synchronization of the Kardar-Parisi-Zhang Equation

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Based on joint works with  
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## Outline

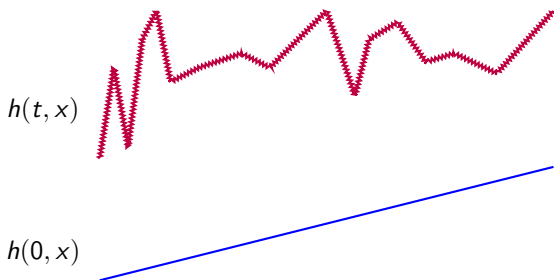
- 1 Introduce the model and the questions:
  - a Hopf-Cole solutions.
  - b Why stationary mod + c distributions?
  - c Synchronization and 1F1S.
- 2 Results
  - a Regularity of the solution semi-group and sharp characterization of non-explosive initial conditions. (AJRS)
  - b Ergodicity and synchronization (JRS)
- 3 Tools:
  - a Busemann process.
  - b Continuum directed polymer.
  - c Gibbs-DLR measures.
  - d Martingales.

The **KPZ equation** (Kardar-Parisi-Zhang 1986) is a prototypical example of **random planar growth**:

$$\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + W$$

$W$  is space-time white noise, “ $W(t, x) \sim N(0, \delta_{t,x})$ ”

$$\mathbb{E}[W(t, x) W(s, y)] = \delta_{t=s} \delta_{x=y}.$$



## KPZ/SHE

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + ZW, \quad (\text{SHE/PAM})$$

Formally,  $h = \ln Z$  solves

$$\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + W \quad (\text{KPZ})$$

This is the **Hopf-Cole** definition of solutions to KPZ.

**Fact:** (Bertini-Giacomin '97, Albers-Khanin-Quastel '14, Hairer-Quastel '16,...) The Hopf-Cole solutions where  $Z$  is the *mild solution* to (SHE/PAM), is a scaling limit of many models  $\implies$  physical.

**“Theorem.”** (Hairer '14 ( $\mathbb{T}$ ), Perkowski, Rosati '19 ( $\mathbb{R}$ )) The KPZ equation is well-posed for nice initial data. This solution agrees with the Hopf-Cole(\*) solution  $h(t, x) = \ln Z(t, x)$ .

## Stationary mod +c distributions:

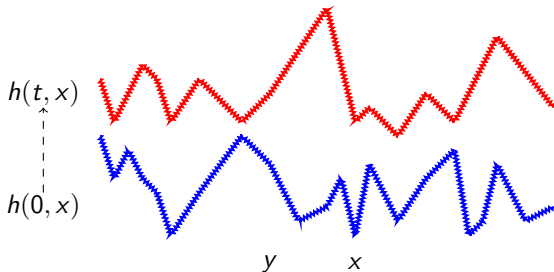
**Theorem:** (e.g. Amir, Corwin, Quastel '11) With “narrow-wedge” IC, in probability,

$$\lim_{t \rightarrow \infty} \frac{1}{t} h(t, 0 | 0, 0) = -\frac{1}{24} \implies \text{transience}$$

A (random) initial condition  $h_0$  is **stationary mod + c** if

$$\partial_t h = \frac{1}{2} \partial_{xx} h + \frac{1}{2} (\partial_x h)^2 + W, \quad h(0, x) = h_0(x)$$

$$(h(t, x) \bmod + c)_{x \in \mathbb{R}} \stackrel{d}{=} (h(0, x) \bmod + c)_{x \in \mathbb{R}}$$



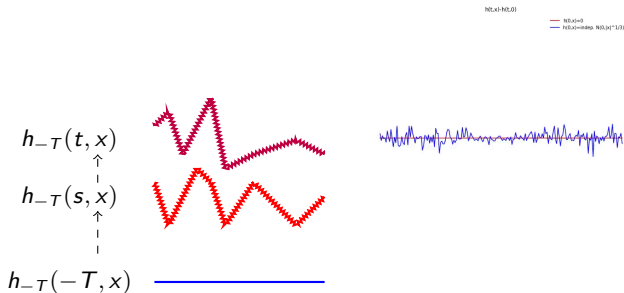
## Synchronization/1F1S:

**Theorem.** (Bertini, Giacomin '97) For  $\lambda \in \mathbb{R}$ , Brownian motion with drift,  $(B(x) + \lambda x)_{x \in \mathbb{R}}$ , is stationary mod  $+c$  for KPZ.

**Conjecture:** (1F1S, ex. Bakhtin, Khanin '18) If  $h_0(x) = \lambda x + \psi(x)$ ,  $\psi(x)$  sublinear

$$\begin{cases} \partial_t h_{-T} = \frac{1}{2} \partial_{xx} h_{-T} + \frac{1}{2} (\partial_x h_{-T})^2 + W \\ h_{-T}(-T, x) = h_0(x) \end{cases}$$

$$\lim_{T \rightarrow \infty} (h_{-T}(t, x) - h_{-T}(t, y)) = b^\lambda(t, x, t, y) \stackrel{d}{=} B(x) - B(y) + \lambda(x - y).$$



## Mild solutions:

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + ZW, \quad \rho(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathbf{1}_{(0, \infty)}(t)$$

$$\mu \in \mathcal{M}_{\text{HE}} = \left\{ \mu \in \mathcal{M}_+(\mathbb{R}) : \forall a > 0, \int_{\mathbb{R}} e^{-ax^2} \mu(dx) < \infty \right\}$$

Chen-Dalang '14/'15 show  $\exists!$  solution for fixed  $s \in \mathbb{R}, \mu \in \mathcal{M}_{\text{HE}}$  of

$$\begin{aligned} Z(t, x|s; \mu) &= \int_{\mathbb{R}} \rho(t-s, x-y) \mu(dy) \\ &\quad + \int_s^t \int_{-\infty}^{\infty} \rho(t-u, x-z) Z(u, z|s; \mu) W(du dz) \end{aligned}$$

The Green's function takes  $\mu = \delta_y$ :

$$Z(t, x|s, y) = Z(t, x|s; \delta_y)$$

**Other existence/uniqueness results:** Bertini and Cancrini '95, Bertini and Giacomin '97.

## Renormalized Green's function:

$$\mathcal{Z}(t, x|s, y) = \begin{cases} \frac{Z(t, x|s, y)}{\rho(t-s, x-y)} & s < t \\ 1 & s = t \end{cases} \quad \text{“} = \mathbf{E}_{(s, y), (t, x)}^{BB} \left[ : e^{\int_s^t W(u, B_u)} : \right] \text{”}$$

**Theorem.** (AJRS 22+)  $\exists$  a modification  $\mathcal{Z}(t, x|s, y) \in \mathcal{C}(\mathbb{R}^4, \mathbb{R})$ . A.s.  $\exists C = C(T, \omega) : \text{if } -T \leq s \leq t \leq T,$

$$C^{-1}(1 + |x|^4 + |y|^4)^{-1} \leq \mathcal{Z}(t, x|s, y) \leq C(1 + |x|^4 + |y|^4).$$

We **define**

$$Z(t, x|s, y) = \mathcal{Z}(t, x|s, y)\rho(t-s, x-y)$$

$$Z(t, x|s; \mu) = \int_{\mathbb{R}} Z(t, x|s, y)\mu(dy) = \int_{\mathbb{R}} \mathcal{Z}(t, x|s, y)\rho(t-s, x-y)\mu(dy).$$

**Previous work:** Alberts-Khanin-Quastel '14 constructed  $Z(t, x|s, y)$  for  $s < t$  previously.



## KPZ solution semi-group:

For  $f$  Borel, call

$$h(t, x|s; f) := \ln \int Z(t, x|s, y) e^{f(y)} dy, \quad h(s, x) = f(x).$$

$$\mathcal{M}_{\text{HE}} = \left\{ \mu \in \mathcal{M}_+(\mathbb{R}) : \forall a > 0, \int_{\mathbb{R}} e^{-ax^2} \mu(dx) < \infty \right\}$$

$$\mathcal{C}_{\text{KPZ}} = \left\{ f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \int_{\mathbb{R}} e^{f(x) - ax^2} dx \text{ for all } a > 0 \right\}.$$

**Theorem.** (AJRS '22+)

- 1  $e^f \in \mathcal{M}_{\text{HE}} \implies h(t, \cdot|s; f) \in \mathcal{C}_{\text{KPZ}}$ , and  $e^f \notin \mathcal{M}_{\text{HE}}, t \gg s \implies h(t, \cdot|s; f) \equiv \infty$ .
- 2  $e^f \in \mathcal{M}_{\text{HE}}, s \in \mathbb{R} \implies h(\cdot, \cdot|s; f)$  is the Hopf-Cole solution,  $\mathbb{P}$  a.s.
- 3 (Quenched continuous DS)  $(f, s, t) \mapsto h(t, \cdot|s; f)$  is continuous  $\mathcal{C}_{\text{KPZ}} \times \mathbb{R}_{s \leq t}^2 \rightarrow \mathcal{C}_{\text{KPZ}}$ .
- 4 (Conservation law)  $\lim_{x \rightarrow \pm\infty} f(x)/x = \lim_{x \rightarrow \pm\infty} h(t, x|s; f)/x$

## Ergodic distributions.

$$f \sim g \text{ if } f(\cdot) = g(\cdot) + c, \quad [f] \in \tilde{\mathcal{C}}_{\text{KPZ}} = \mathcal{C}_{\text{KPZ}} / \sim$$

**Corollary.** (AJRS 22+)  $[h(t, \cdot | s; f)] : \mathbb{R}_{s \leq t}^2 \times \tilde{\mathcal{C}}_{\text{KPZ}} \rightarrow \tilde{\mathcal{C}}_{\text{KPZ}}$  is continuous  $\implies$  Feller.

**Theorem.** (JRS 22+) The distribution of  $[B(\cdot) + \lambda \cdot]$  is (totally) ergodic. If  $P$  is *any* ergodic distribution, then either

- 1 There exists  $\lambda \in \mathbb{R}$  such that  $P$  is the distribution of  $[B(\cdot) + \lambda \cdot]$ .
- 2 There exists  $\lambda > 0$  such that  $P$  is supported on equivalence classes  $[f] \in \tilde{\mathcal{C}}_{\text{KPZ}}$  with

$$-\lambda = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lambda$$

for all  $f \in [f]$ .

**Open problem:** Do ergodic measures of the second type exist?

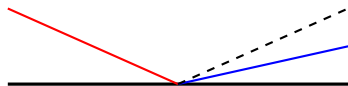
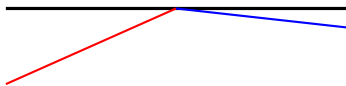
## Synchronization:

**(Simplified) Theorem.** (JRS 22+) There exists a random countable  $\Lambda^\omega$  with  $\lambda \in \mathbb{R}$ ,  $\mathbb{P}(\lambda \in \Lambda) = 0$  so that for  $\lambda \notin \Lambda$ , if  $\psi$  is sublinear and

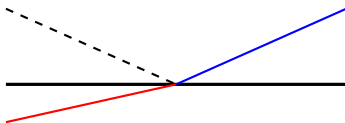
$$h_{-T}(-T, x) = (\mu x)1_{(-\infty, 0)}(x) + (\eta x)1_{(0, \infty)}(x) + \psi(x),$$

$$\mu \geq 0, \quad \eta \leq 0 \implies \lambda = 0$$

$$\mu < 0, \quad \eta < |\mu| \implies \lambda = \mu$$



$$\eta > 0, \quad -\mu < \eta \implies \lambda = \eta$$



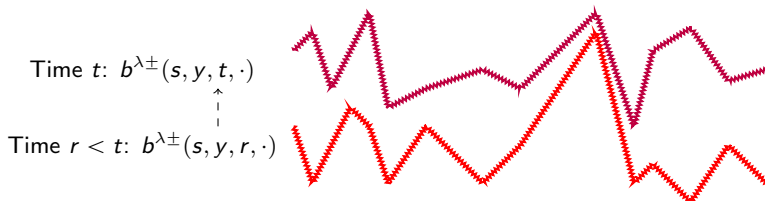
$$\lim_{T \rightarrow \infty} (h_{-T}(t, x) - h_{-T}(s, y)) = b^\lambda(s, y, t, x)$$

## Busemann process:

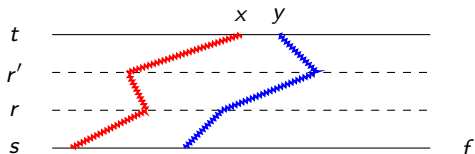
**Theorem.** (JRS 22+)  $\exists \{b^{\lambda\pm}(s, y, t, x) : \lambda, s, x, t, y \in \mathbb{R}\}$ , and a random countable  $\Lambda^\omega$  satisfying

- 1 For all  $\lambda \in \mathbb{R}$ ,  $\mathbb{P}(\lambda \in \Lambda) = 0$ .
- 2 (Equal off  $\Lambda$ ) If  $\lambda \notin \Lambda$ , then  $b^{\lambda+} = b^{\lambda-} (:= b^\lambda)$ .
- 3 (Continuous) For all  $\lambda$ ,  $b^{\lambda\pm}(\cdot, \cdot, \cdot, \cdot) \in \mathcal{C}(\mathbb{R}^4, \mathbb{R})$
- 4 (Brownian) For all  $t, \lambda \in \mathbb{R}$ ,  $b^\lambda(t, y, t, x) \stackrel{d}{=} B(x) - B(y) + \lambda(x - y)$ .
- 5 (Cocycle)  $b^{\lambda\pm}(s, y, r, z) + b^{\lambda\pm}(r, z, t, x) = b^{\lambda\pm}(s, y, t, x)$ .
- 6 (Eternal solution of KPZ) For all  $s, t, y, x, \lambda$  and  $r < t$ ,

$$b^{\lambda\pm}(s, y, t, x) = \ln \int Z(t, x | r, u) e^{b^{\lambda\pm}(s, y, r, u)} du = h(t, x | r; b^{\lambda\pm}(s, y, r, \cdot))$$



## Continuum Directed Random Polymer:



The CDRP is the measure  $Q_{(s;f),(t,x)}$  on  $\mathcal{C}([s, t], \mathbb{R})$  with transitions

$$\pi_{(s;f),(t,x)}(r, du | r', u') = Z(r', u' | r, u) \underbrace{\frac{Z(r, u | s; f)}{Z(r', u' | s; f)}}_{e^{h(r, u | s; \ln f) - h(r', u' | s; \ln f)}} du'$$

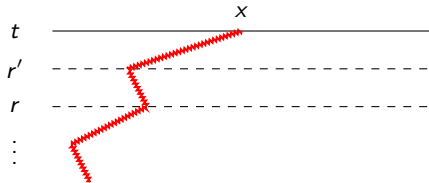
Originally introduced by Alberts-Khanin-Quastel '14 for fixed  $t, x, s, f$ .

**Theorem.**(AJRS '22+) A regular (continuous, monotone, etc.) coupling of all  $Q_{(s;\mu),(t,x)}$  exists.

## Infinite volume (Gibbs) polymers and eternal cocycles:

$b^{\lambda\pm}$  is an eternal (cocycle) KPZ solution: with  $f(y) = b^{\lambda\pm}(s, y, t, x)$ ,

$$\begin{aligned} \pi_{(s; e^{b^{\lambda\pm}(s, \cdot, t, x)}), (t, x)}(r', du' | r, du) &= Z(r', u' | r, u) \frac{Z(r, u | s; f)}{Z(r', u' | s; f)} du' \\ &= \underbrace{Z(r', u' | r, u) e^{b^{\lambda\pm}(r, u, r', u')}}_{\text{Does not depend on } s} du' \end{aligned}$$

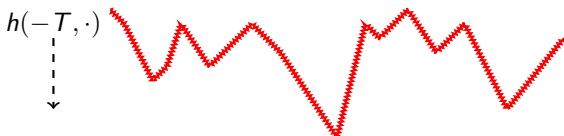
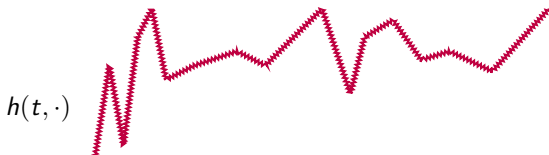


which defines  $Q_{(t,x)}^{\lambda\pm}$  on  $\mathcal{C}((-\infty, t], \mathbb{R})$ .

## Ergodicity, eternal cocycle solutions, and Gibbs polymers:

If  $P$  is ergodic take  $[h(-T, \cdot)] \sim P$ . Extension  $\implies$

$\exists$  global cocycle sol  $b^P(s, x, t, y)$  (on an extended space)



Global cocycle  $\implies \exists$  semi-infinite polymer  $Q_{t,x}^P$  with transitions

$$\pi_{t,x}^P(r', du' | r, du) = Z(r', u' | r, u) e^{b^P(r, u, r', u')} du'.$$

## Busemann limits and Gibbs martingales:

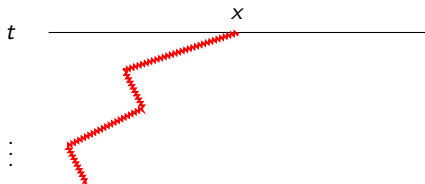
The Busemann limit says that if  $\lambda \notin \Lambda$ ,  $z_r/r \rightarrow -\lambda$  as  $r \rightarrow -\infty$

$$\frac{Z(s, y|r, z_r)}{Z(t, x|r, z_r)} \rightarrow e^{b^\lambda(s, y, t, x)}.$$

On the other hand, because  $Q_{t,x}^P$  is Gibbs,

$$M_r^{s, y, t, x} = \frac{Z(s, y|r, X_r)}{Z(t, x|r, X_r)} \text{ is a } Q_{t,x}^P \text{ bmg, } \mathbf{E}^{Q_{t,x}^P} \left[ \frac{Z(s, y|r, X_r)}{Z(t, x|r, X_r)} \right] = e^{b^P(s, y, t, x)},$$

$$\chi = \lim_{r \rightarrow -\infty} \frac{X_r}{r} \text{ exists } \quad Q_{t,x}^P - a.s. \quad (\chi \in [-\infty, \infty] \text{ is } Q_{t,x}^P \text{ random}).$$

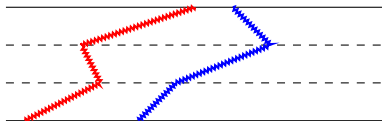




## LLN/slope duality:

By monotonicity, for large  $-r$  and  $x < 0$ ,  $Q_{t,x}^P$  a.s.

$$\frac{Z(t, x|r, -(\lambda + \varepsilon)r)}{Z(t, 0|r, -(\lambda + \varepsilon)r)} \cdot \mathbf{1}_{\{\chi \geq -\lambda\}} \leq \frac{Z(t, x|r, X_r)}{Z(t, 0|r, X_r)} \mathbf{1}_{\{\chi \geq -\lambda\}} \implies e^{b^{\lambda+\varepsilon}(t,0,t,x)} Q_{t,x}^P(\chi \geq -\lambda) \leq e^{b^P(t,0,t,x)}.$$



$$Q_{0,0}^P(\chi \geq -\lambda) > 0 \implies \overline{\lim}_{x \rightarrow -\infty} \frac{b^P(t, 0, t, x)}{x} \leq \lambda$$

$Q_{0,0}^P(\chi \geq -\lambda) > 0 \iff Q_{t,x}^P(\chi \geq -\lambda) > 0 \implies \{Q_{0,0}^P(\chi \geq -\lambda) > 0\}$  is translation invariant.

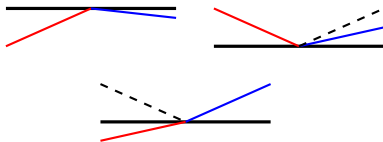
## LLN/slope duality continued:

If  $P$  is ergodic, then since  $[b^P(t, 0, t, \cdot)] \sim P$

$$\mathbb{P}(Q_{0,0}^P(\chi \geq -\lambda) > 0) > 0 \implies \mathbb{P}(\overline{\lim}_{x \rightarrow -\infty} f(x)/x \leq \lambda) = 1$$

Similar statements for  $\overline{\lim}_{x \rightarrow \pm\infty}$  and  $\underline{\lim}_{x \rightarrow \pm\infty}$  and a finiteness lemma  
 $\implies \exists$  finite  $\bar{\lambda}, \underline{\lambda}$ :

$$\mathbb{P}\left(\lim_{x \rightarrow -\infty} f(x)/x = \underline{\lambda}\right) = \mathbb{P}\left(\lim_{x \rightarrow \infty} f(x)/x = \bar{\lambda}\right) = 1.$$



Consistency with the Busemann limits now says that if we do not have  $\bar{\chi} > 0$  and  $\underline{\chi} = -\bar{\chi}$ , then  $b^P = b^\lambda$  for  $\lambda = \bar{\lambda} = \underline{\lambda}$ .

Thank you!