The ubiquity of the directed landscape

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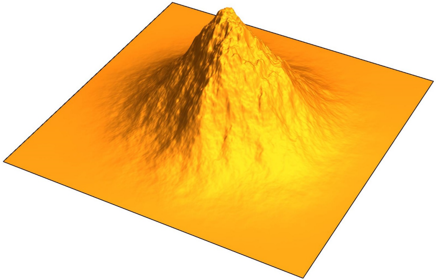
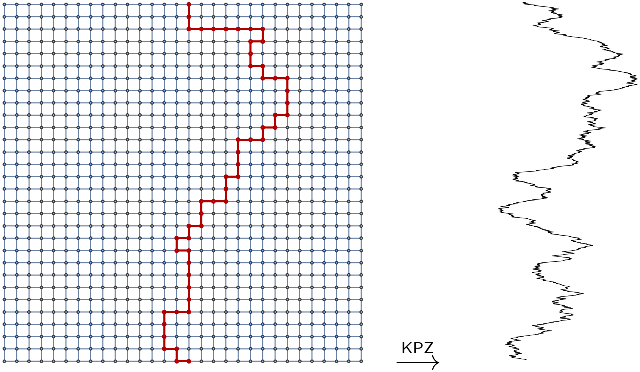


The fox and the rabbit

0:1:2 to 1:2:3

$\sigma^{4/3}$

First passage percolation



Scale $\updownarrow n$, $\leftrightarrow n^{2/3}$. Limit is a graph of $\gamma : [0,1] \to \mathbb{R}$

The **directed geodesic**.

The directed landscape

$$d(n^{2/3}x, ns; n^{2/3}y, nt) = n(t-s) - n^{1/3}\mathcal{L}(x, s; y, t) + \text{error}$$

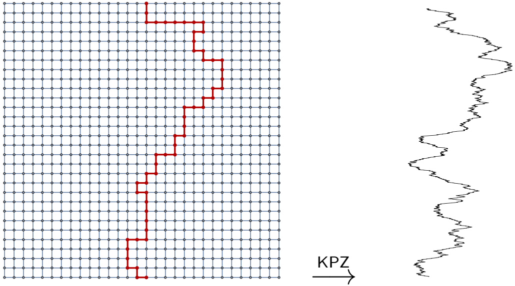
- $\mathcal{L}$: the **directed landscape**, a *universal* random plane geometry
- $-\mathcal{L}$: Δ inequality, $\mathcal{L}(p, p) = 0$
-

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \begin{cases} -\frac{(y-x)^2}{t-s} + (t-s)^{1/3} TW, & s < t \\ 0 & (x, s) = (y, t) \\ -\infty & \text{else} \end{cases}$$

Dauvergne Ortmann V (2023)

DL: the full scaling limit

The same structure as last passage percolation

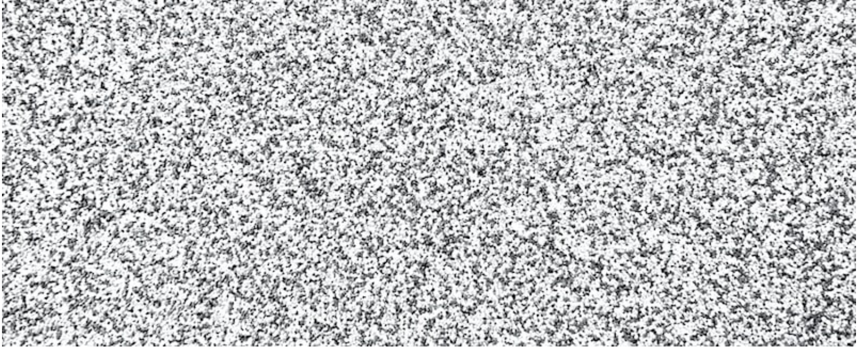


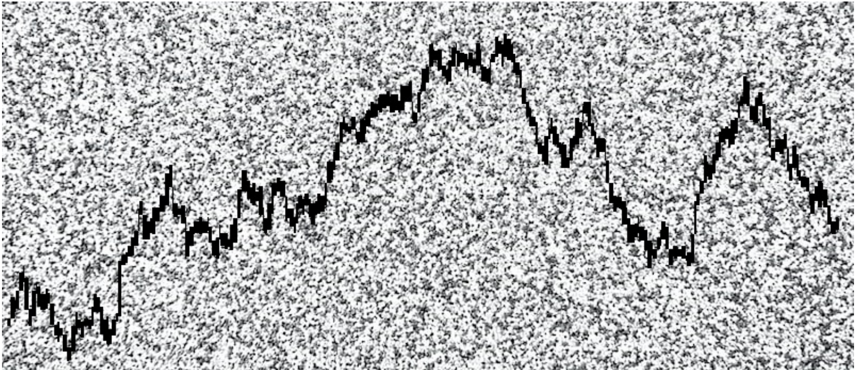
description	geodesic shape information
Tracy-Widom law	none
Airy process	1d marginal, point-to-point
KPZ fixed point	1d marginal, general
directed landscape	full law, general

The fox and the rabbit

w Jeremy Quastel and Alejandro Ramirez

FOX.





Absolute continuity

ξ planar white noise
B: occupation measure on graph of BM on $[0,1]$

Find $\xi_n \to \xi$, and $B_n \to B$ independent so that

$$Z_n = \frac{\text{law}(\xi_n + B_n)}{\text{law}(\xi_n)} \qquad \text{is } L^1\text{-tight}$$

ξ_n, B_n: projection to $\mathrm{e}_1, \ldots, \mathrm{e}_n$, basis of $L^2(\mathbb{R}^2)$.
But Z_n is bounded in L^2 since

$$EZ_n^2 = E \exp\langle B_n, B_n' \rangle \leq E \exp \alpha(B, B') = E \exp |N| < \infty. \quad \square$$

The continuum directed random polymer

Alberts, Khanin, Quastel (2014)

$$P(CDRP \in A|\xi) = \frac{\text{law}(\xi) * (\text{law}(B)|_A)}{\text{law}(\xi)}$$

Class of **planar** models

$$\Delta + \xi$$

- RWIRE
- Random conductances
- Parabolic Anderson model
- Brownian motion with obstacles
- planar stochastic heat equation

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

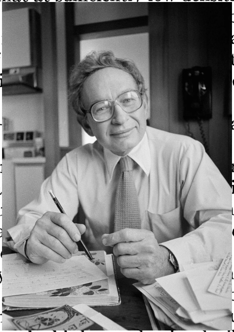
This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

I. INTRODUCTION

A NUMBER of physical phenomena seem to involve quantum-mechanical motion, without any particular thermal activation, among sites at which the mobile entities (spins or electrons, for example) may be localized. The clearest case is that of spin diffusion[1,2]; another might be the so-called impurity band conduction at low concentrations of impurities. In such situations we suspect that transport occurs not by motion of free carriers (or spin waves), scattered as they move through a medium, but in some sense by quantum-mechanical jumps of the mobile entities from site to site. A second common feature of these phenomena is randomness: random spacings of impurities, random interactions with the "atmosphere" of other impurities, random arrangements of electronic or nuclear spins, etc.

Our eventual purpose in this work will be to lay the foundation for a quantum-mechanical theory of trans-

reasonably well, and to prove a theorem about the model. The theorem is that at sufficiently low densities,
transport does not take ... ons
are localized in a small ... ain
a fairly good estimate of ... the
theorem fails. An additi... ces
be of sufficiently short ... as
$r \to \infty$ faster than $1/r^3$— ... ate
of the rate of transport ...



Such a theorem is of i... ns:
first, because it may a... ion
among donor electrons i... er[3]
has shown experimenta... eg-
ligible; second, and pr... an
example of a real phy... ite
number of degrees of ... ous
oversimplification, in wh... um
is simply impossible; and third, as the irreducible minimum from which a theory of this kind of transport,

Class of **planar** models for today

$$\Delta + \xi$$

- RWIRE
- Random conductances
- Parabolic Anderson model
- Brownian motion with obstacles
- planar stochastic heat equation

We chose $\Delta = \partial_{xx} + \partial_{yy}$, $\xi =$ planar white noise.

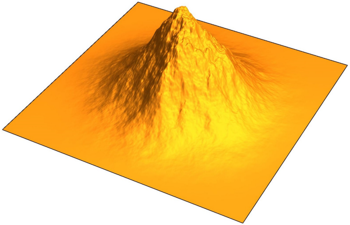
Main Theorem

Let u satisfy the Wick-ordered planar SHE

$$\partial_t u = \frac{1}{2}\Delta u + u\,\xi, \qquad u(\cdot, 0) = \delta_0.$$

Then for any $t > 0$ and $a \in \mathbb{R}$ as $N \to \infty$,

$$P(u((0, N^{3/2}t), Nt) \times Ne^{N^2 t/2}\sqrt{2\pi t} \leq a) \to F_{KPZ}(t, a).$$



When chaos expansion fails: proof idea

Defining 2d SHE. By analogy, in 2d

$$u = \frac{\text{law}(B_{2d} + \xi)}{\text{law}(\xi)}$$

B_{2d}: occ. measure on the path planar BM.

L^2 iff $t < t_c$. Gagliardo–Nirenberg. But still L^1 tight.

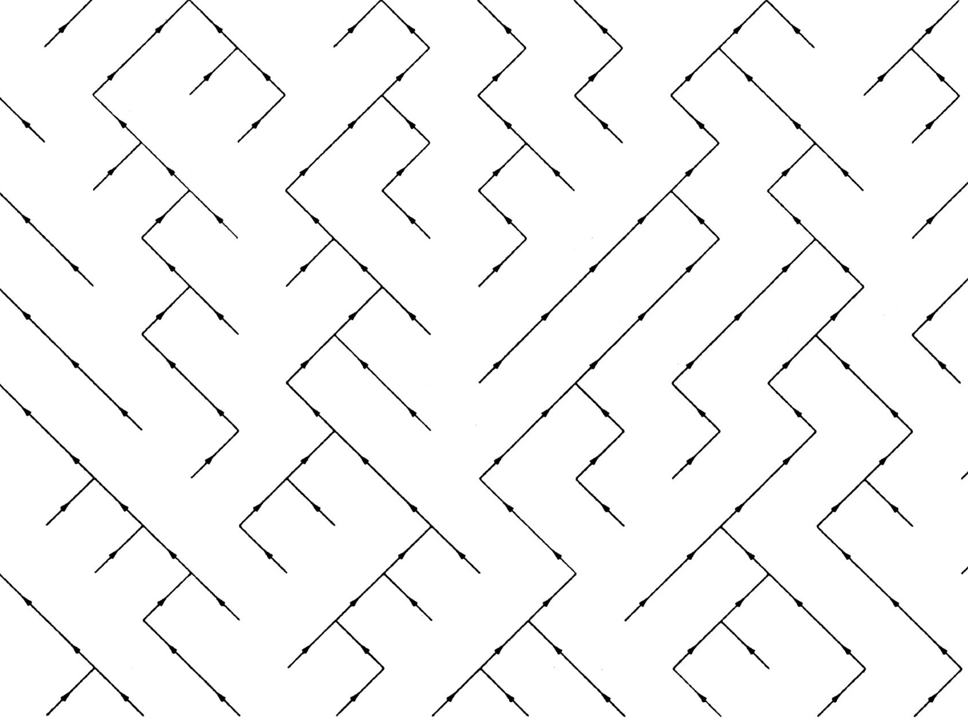
Convergence to KPZ. In our scaling:

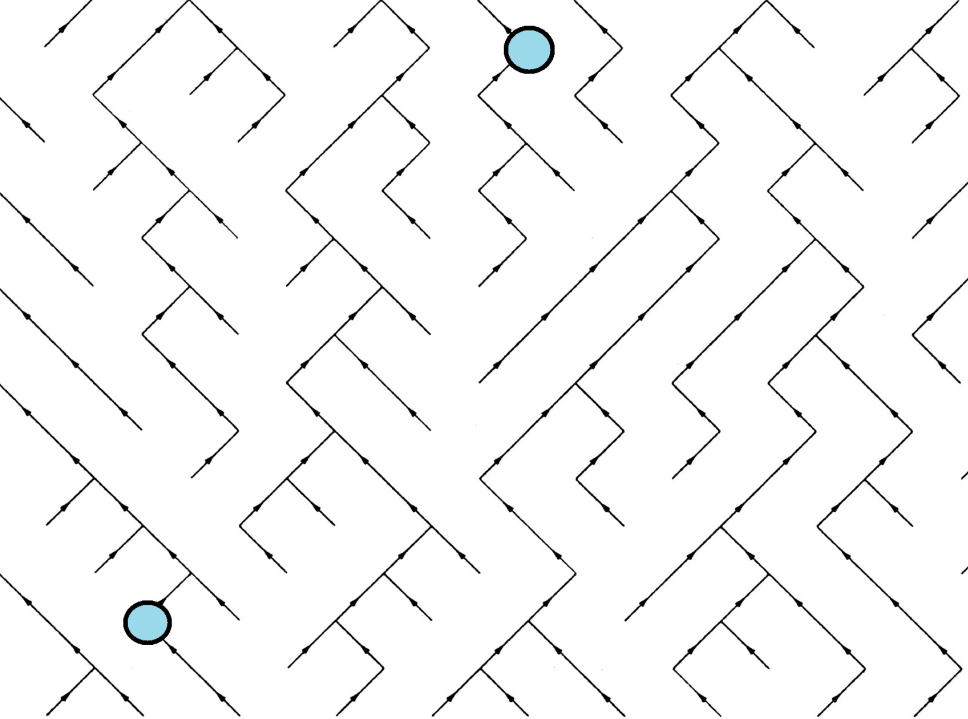
$$\text{path (2d BM)} \to \text{graph (1d BM)}$$

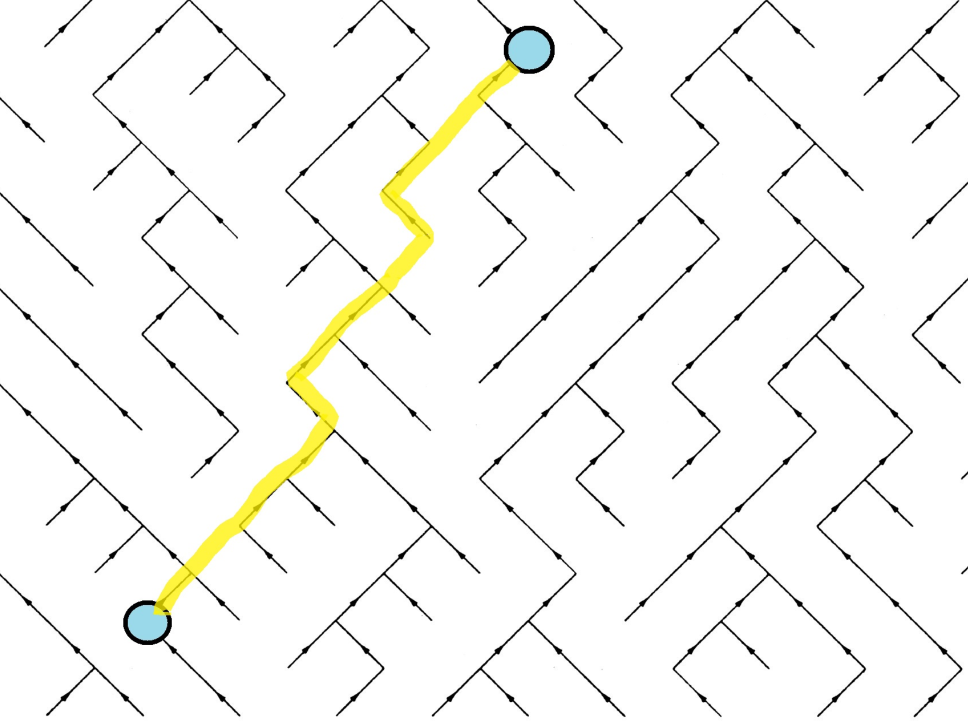
L^1 tightness: technical. $\square$

0:1:2 to 1:2:3

w Bálint Vető





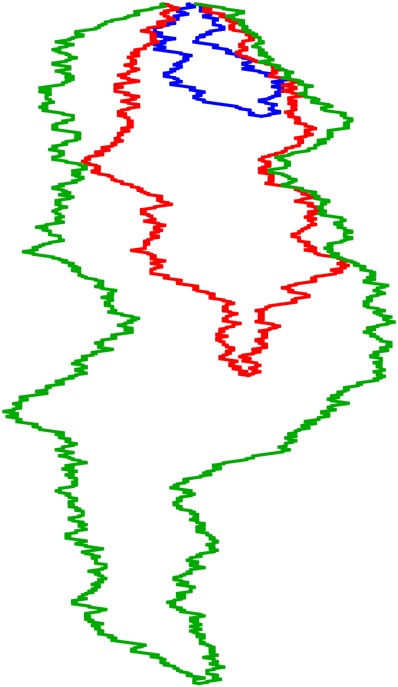


The Brownian web distance

Theorem.(Vető, V.) After $0:1:2$ scaling, the discrete web distance converges to the Brownian web distance.

Arratia (1981), Tóth Werner (1998), Newman Ravishankar Schertzer (2010), Dumaz Tóth (2013)

- integer-valued
- number of times you have to switch paths
- $0:1:2$-scale invariant
- no time-reversal symmetry:
- for distinct points $d(x,s;,y,t)<\infty$ iff $s<t$ and (y,t) is on the skeleton.



KPZ limit

Theorem. As $m \to \infty$,

$$\frac{tn + 2zn^{2/3} - d_{br}(2tn + 2zn^{2/3}, -tn; \mathbb{R}_-, 0)}{n^{1/3}} \to \mathcal{L}(0, 0; z, t)$$

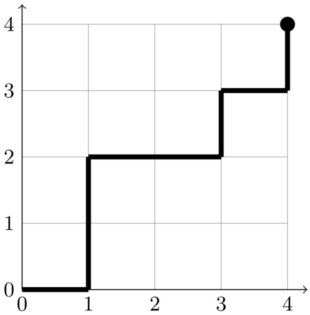
compactly in law.

Future directions

- Random limit shapes
- Nonunique geodesics

$\sigma^{4/3}$: universality of directed polymers $\beta = n^{-\alpha},\ \alpha < \frac{1}{5}$

solo by Julian Ransford



$$Z_{n,\beta} = \sum_{\pi:(0,0)\to(n,n)} \prod_{i=0}^{2n} e^{\beta \xi_{\pi(i)}}.$$

$\xi_{i,j}$ i.i.d with $\psi(\lambda) = Ee^{\lambda \xi_{i,j}}$, $\psi(\epsilon) < \infty$, $Var(\xi_{i,j}) = \sigma^2$.

Theorem (Ransford) Let $\beta_n = n^{-\alpha}$, $\alpha \in (1/5, 1/4)$. Then

$$\frac{\log Z_{n,\beta_n} - a_n}{(4\beta_n^4 n)^{1/3}} \; {}^{d}\!\to \sigma^{4/3} TW_{GUE}$$

$$a_n = 2n\Big(\log \psi(\beta_n) + \log 2 + \frac{\sigma^2 \beta_n^4}{3} \Big)$$

Alberts Quastel Khanin 2014

Borodin Corwin Remenik 2013, Krishnan Quastel 2018

For k moments matching, need

$$\alpha > \frac{2}{3k + 11}$$

Two moments should suffice for

$$\alpha > \frac{2}{17}$$

A different obstacle at $1/5$.

Happy birthday Timo!

Balázs, Quastel, **Seppäläinen.** Fluctuation exponent of the KPZ/stochastic Burgers equation. JAMS 2011.

Seppäläinen. Exact limiting shape for a simplified model of first-passage percolation on the plane, AOP 1998.

Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. AOP 2012.

Thank you.