GRADED TRACES, MODULARITY AND PSEUDOTRACES

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BASED ON JOINT WORK IN PROGRESS WITH K. BARRON, K. BATISTELLI AND G. YAMSKULNA.

+ AN OVERVIEW OF FUTURE WORK WITH OUR WOMAP II TEAM

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PLAN FOR THE TALK

1. Motivation
2. VOA Example 1: Free Bosons
3. Modularity and graded dimensions
4. Graded traces and pseudotraces
5. Our results and future work

VERTEX ALGEBRAS: WHY?

- Vertex Operators are local operators that describe the propagation of string states in string theory.
- The chiral algebra of symmetries of a conformal field theory is a vertex algebra.
- The Moonshine module $V^\natural$ acted on by the Monster group is a vertex algebra.
- Representations of vertex algebras provide examples of modular tensor categories.
- Vertex algebras are connected to modular forms and mock modular forms.

VERTEX OPERATOR ALGEBRAS

A vertex operator algebra consists of a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_n$ such that

- $\dim V_n < \infty$ and $V_n = 0$ for n sufficiently small
- A linear map $Y(\,.\,,z) : V \longrightarrow \mathrm{End}\, V[[z^{\pm 1}]]$

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{\underline{-n-1}}$$

Weird expansion

$\mathrm{Res}_z z^n Y(a,z) = a_n$

where $Y(a,z)b = \sum_{n\in\mathbb{Z}} a_n b z^{-n-1} \in V((z)) \iff a_n b = 0$ for $n >> 0$

$$Y(a,z)b = a_{99} b z^{-100} + a_{98} b z^{-99} + \dots + a_1 b z^{-2} + a_0 b z^{-1} + a_{-1} b + a_{-2} b z + \cdots$$

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SEVERAL AXIOMS

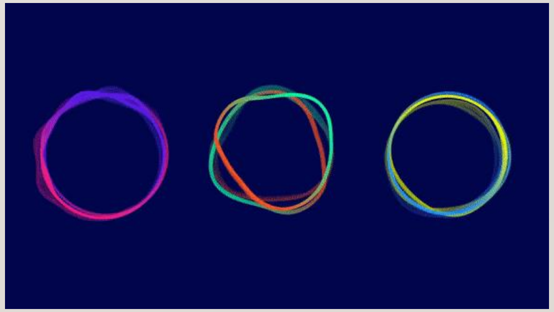

Image Credit: String Theory Explained – What is The True Nature of Reality? https://youtu.be/Da-2h2B4faU

VERTEX OPERATOR ALGEBRAS

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- A linear map $Y(\ .\ , z) : V \longrightarrow \mathrm{End}\ V[[z^{\pm 1}]]$

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

- $|0\rangle \in V_0$, the vacuum vector
- $\omega \in V_2$ (the conformal vector) which satisfies $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$
- $[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12} c\delta_{n,-m}$ ← Virasoro action on V
- $L_0 v = nv = (\mathrm{wt} v)v$ for $n \in \mathbb{Z}$, $v \in V_n$ ← $V = \coprod_{n\in\mathbb{Z}} V_n$ is graded by L_0-eigenvalues.

Miguel Ángel Virasoro
(1940-2021)
Image credit:
M. Sterle/ICTP Photo Archives

EXAMPLES OF VERTEX (OPERATOR) ALGEBRAS (=VOAS)

1. Free bosons = Fock Representation π

2. The Virasoro VOAs M_c

EXAMPLE 1: THE FREE BOSONS

Consider the Heisenberg Lie algebra H with generators $x_n, n \in \mathbb{Z}, \mathfrak{c}$ where $\mathfrak{c}$ is central and

$$[x_n, x_m] = n\delta_{n,-m}\mathfrak{c}$$

- $x_1 x_{-2} - x_{-2} x_1 = 0$
- $x_2 x_{-2} - x_{-2} x_2 = 2\mathfrak{c}$
- $x_2 \mathfrak{c} - \mathfrak{c} x_2 = 0$

We fix a triangular decomposition for H:

$$H^+ := \bigoplus_{n>0} \mathbb{C}x_n, \qquad H^0 := \mathbb{C}x_0 \oplus \mathbb{C}\mathfrak{c}, \qquad H^- := \bigoplus_{n>0} \mathbb{C}x_{-n},$$

and build a one dimensional representation $\mathbb{C}\mathbf{1}$ for $H^{\geq} := H^+ \oplus H^0$ with action given by

- $x_n \mathbf{1} = 0 \text{ for } n \geq 0$
- $\mathfrak{c}\mathbf{1} = \mathbf{1}$

EXAMPLE 1: THE FREE BOSONS (continued)

$$[x_n, x_m] = n\delta_{n,-m} z$$

- $x_1 x_{-2} - x_{-2} x_1 = 0$
- $x_2 x_{-2} - x_{-2} x_2 = 2\mathfrak{c}$
- $x_2 \mathfrak{c} - \mathfrak{c} x_2 = 0$

The Fock representation π is defined as the induced representation

$$\pi = U(H) \otimes_{U(H^+ \oplus H^0)} \mathbb{C}\mathbf{1}$$

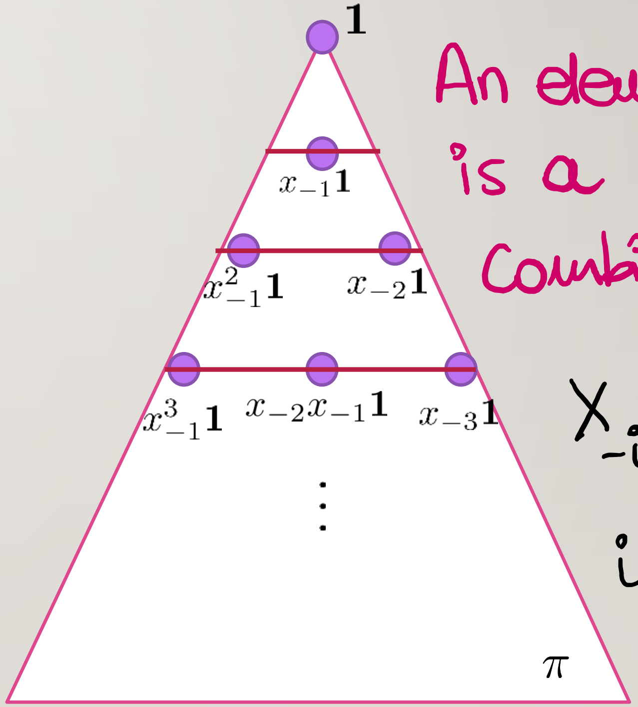
- $x_n \mathbf{1} = 0$ for $n \geq 0$
- $\mathfrak{c}\mathbf{1} = \mathbf{1}$

Linearly $\pi \cong \mathbb{C}[x_{-1}, x_{-2}, x_{-3}, \cdots]$

X_{-i} creation operators $i>0$

- $x_1 x_{-2} \mathbf{1} = x_{-2} x_1 \mathbf{1} = 0$
- $x_2 x_{-2} \mathbf{1} = x_{-2} x_2 \mathbf{1} + 2\mathfrak{c}\mathbf{1} = 2\,\mathbf{1}$ $\rightsquigarrow$ $X_i = i\dfrac{\partial}{\partial X_{-i}}$ for $i>0$
- $x_{-1} x_{-2} \mathbf{1} = x_{-2} x_{-1} \mathbf{1}$

annihilation operators

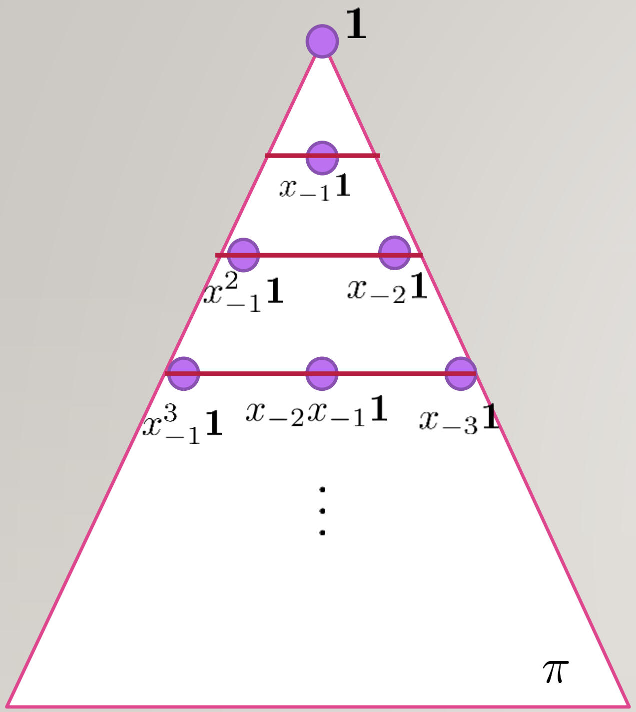


An element in π is a linear combination of

$X_{-i_1} \cdots X_{-i_n} \mathbf{1}$

$i_1 \geq \cdots \geq i_n > 0$

VOA STRUCTURE ON π =FREE BOSONS



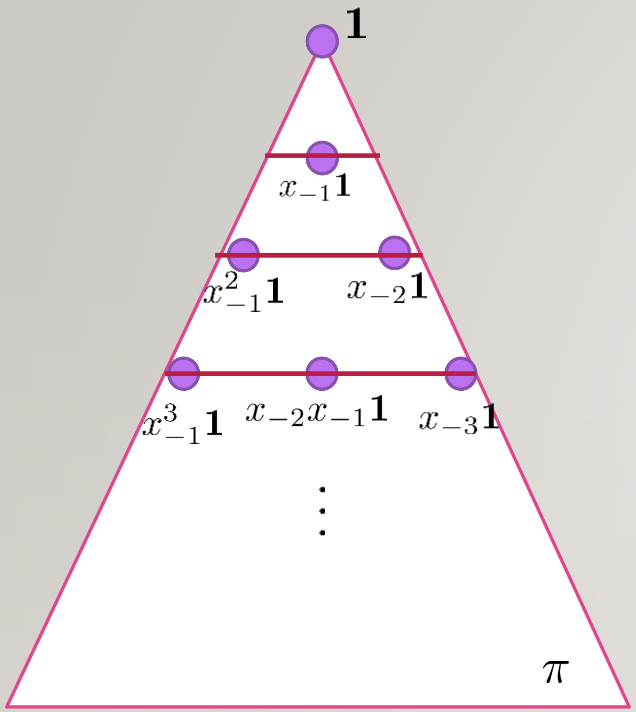
- $|0\rangle = \mathbf{1}$
- $\boxed{\omega_a = \frac{1}{2}x_{-1}^2\mathbf{1} + a x_{-2}\mathbf{1}}$ is a conformal vector $\forall a \in \mathbb{C}$

We denote that VOA structure by π_a

- $Y(x_{-1}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} x_n z^{-n-1} \Longrightarrow X(z) := \sum_{n\in\mathbb{Z}} x_n z^{-n-1}$
- $Y\left(\frac{1}{2}x_{-1}^2\mathbf{1}, z\right) = \frac{1}{2} : X(z)X(z) :$
- $Y(x_{-j_1}\cdots x_{-j_k}\mathbf{1}, z) = \frac{1}{(j_1-1)!\cdots(j_k-1)!} : \partial^{j_1-1}X(z)\cdots\partial^{j_k-1}X(z) :$

$$\pi = \pi_0$$

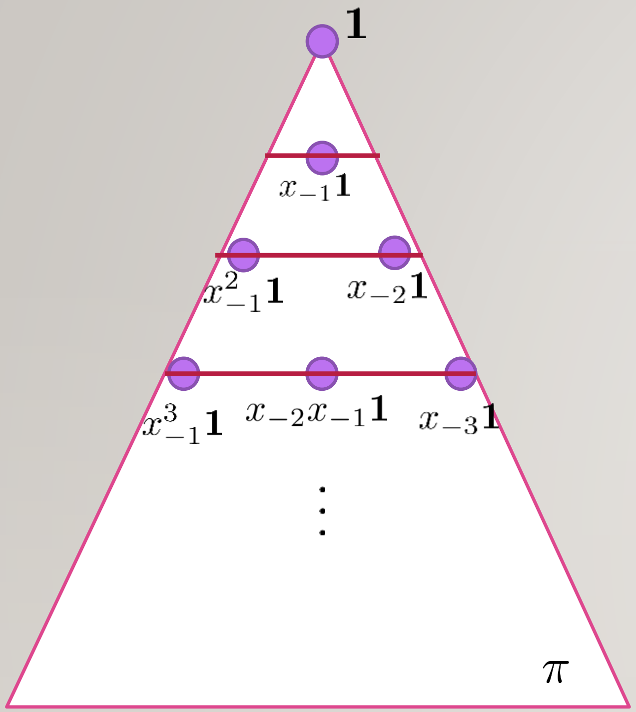
VOA STRUCTURE ON π =FREE BOSONS



- $|0\rangle = \mathbf{1}$
- $\omega = \frac{1}{2}x_{-1}^2\mathbf{1}$
- $Y\left(\frac{1}{2}x_{-1}^2\mathbf{1}, z\right) = \frac{1}{2} : X(z)X(z) :$

$$L_0 = \frac{1}{2}\left(\sum_{k<0} x_k x_{-k} + \sum_{k\geq 0} x_{-k}x_k\right) = \frac{1}{2}\left(2\sum_{\ell>0} x_{-\ell}x_\ell\right) + \frac{1}{2}x_0^2 = \frac{1}{2}\left(2\sum_{\ell>0}\ell\, x_{-\ell}\frac{\partial}{\partial x_{-\ell}}\right) + \frac{1}{2}x_0^2$$

VOA STRUCTURE ON π =FREE BOSONS



$$L_0 = \frac{1}{2}\left(2\sum_{\ell>0} \ell\ x_{-\ell}\frac{\partial}{\partial x_{-\ell}}\right) + \frac{1}{2}x_0^2$$

- $L_0|0\rangle = \frac{1}{2}\left(2\sum_{\ell>0} \ell\ x_{-\ell}\frac{\partial}{\partial x_{-\ell}}\right)\mathbf{1} + \frac{1}{2}x_0^2\mathbf{1} \ = 0|0\rangle$

- $L_0x_{-3}\mathbf{1} = \frac{1}{2}\left(2\sum_{\ell>0} \ell\ x_{-\ell}\frac{\partial}{\partial x_{-\ell}}\right)x_{-3}\mathbf{1} + \frac{1}{2}x_0^2x_{-3}\mathbf{1}$

$$= 3x_{-3}\frac{\partial}{\partial x_{-3}}x_{-3}\mathbf{1}$$

$$= 3x_{-3}\mathbf{1}$$

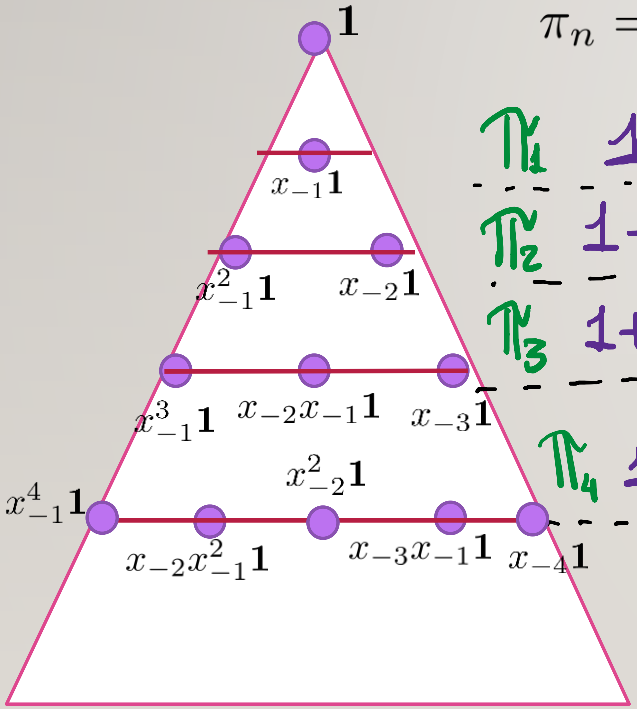
More generally, we have

- $L_0(x_{-j_1}\cdots x_{-j_k}\mathbf{1}) = (j_1 + j_2 + \cdots + j_k)x_{-j_1}\cdots x_{-j_k}\mathbf{1}$

GRADED DIMENSION OF π

Therefore, we have a decomposition into L_0-eigenspaces

$$\pi = \bigoplus_{n \geq 0} \pi_n$$



$$\pi_n = span_{\mathbb{C}}\{x_{-j_1} \cdots x_{-j_k}\mathbf{1} \mid j_1 \geq \cdots \geq j_k, \;\; j_1 + \cdots + j_k = n\}$$

π_1 1

π_2 1+1 2

π_3 1+1+1 2+1 3

π_4 1+1+1+1, 2+1+1, 2+2, 3+1, 4

dim 1 = P(1)

dim 2 = P(2)

dim 3 = P(3)

dim 5 = P(4)

GRADED DIMENSION OF π

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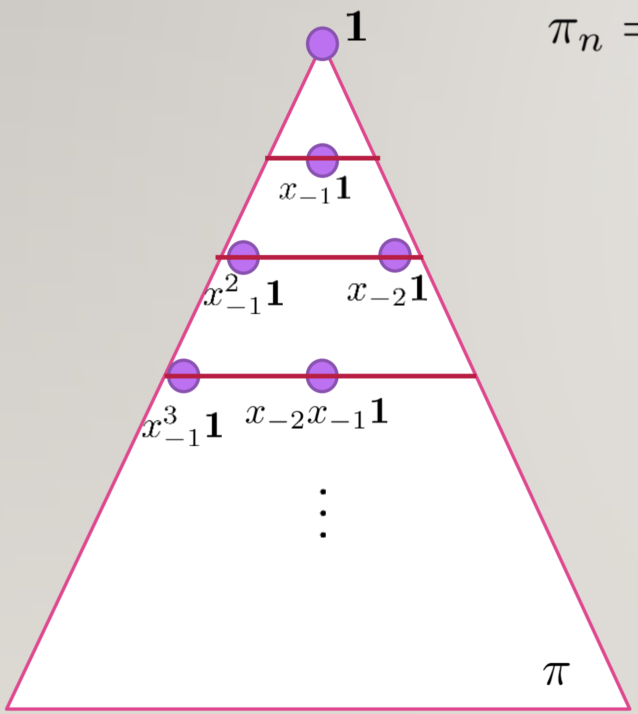


$$\pi_n = span_{\mathbb{C}}\{x_{-j_1} \cdots x_{-j_k}\mathbf{1} \mid \; j_1 \geq \cdots \geq j_k, \quad j_1 + \cdots + j_k = n\}$$

$\dim \pi_n = p(n)$ = # of partitions of n

$$dim_q(\pi) := q^{-\frac{1}{24}} \sum_{n \geq 0} dim(\pi_n) q^n \longrightarrow \text{Graded dimension of } \pi$$

$$= q^{-\frac{1}{24}} \sum_{n \geq 0} p(n) q^n$$

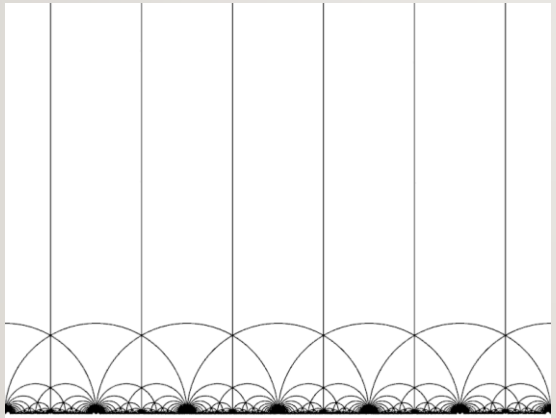
$$= [\eta(q)]^{-1}$$

$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ = Dedekin eta function, a modular form (compatible with an action of $SL(2, \mathbb{Z})$ if $q = e^{2i\pi\tau}$)

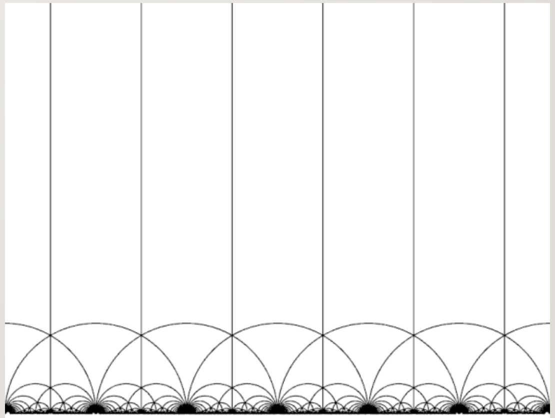
THE MODULAR GROUP

$SL(2,\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid Re(\tau) > 0\}$

It is generated by the following two transformations:



$$T(\tau) = \tau + 1$$

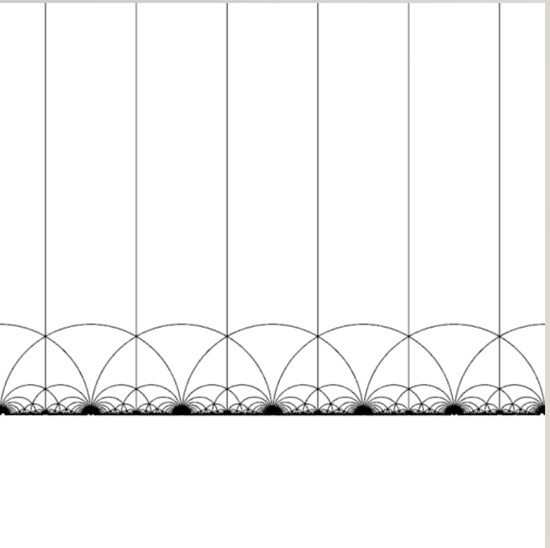


$$S(\tau) = -\frac{1}{\tau}$$

THE MODULAR GROUP

$SL(2,\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid Re(\tau) > 0\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$



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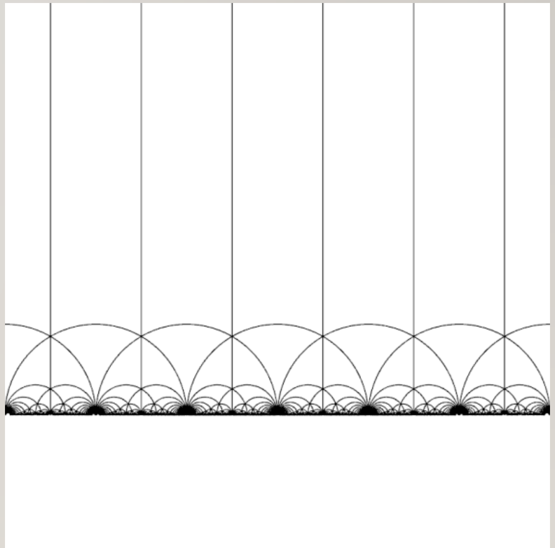
$$T(\tau) = \tau + 1$$

$$[T] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S(\tau) = -\frac{1}{\tau}$$

$$[S] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



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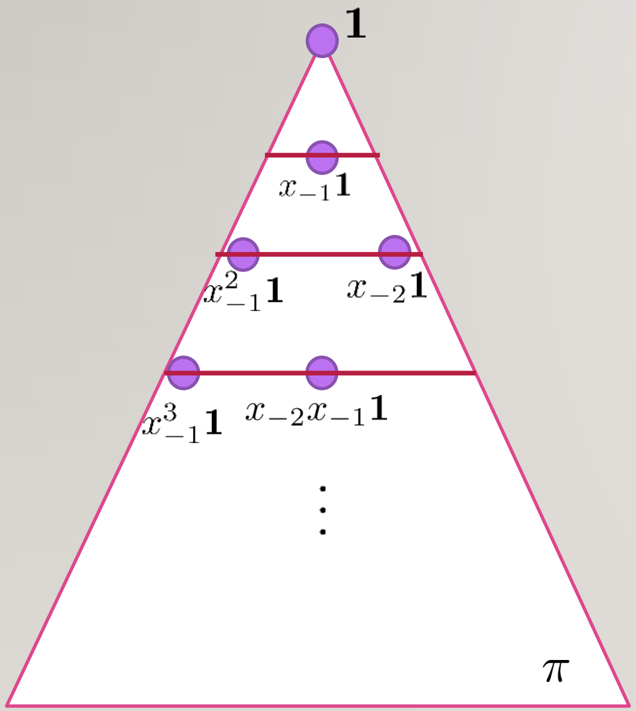
MODULAR FORMS

A modular form of weight k is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{H}$ satisfying



- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

- f is bounded as $\tau \longrightarrow i\infty$

$$dim_q(\pi) = [\eta(q)]^{-1}$$

$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ = Dedekin eta function, a modular form if $q = e^{2i\pi\tau}$ of weight $k = \frac{1}{2}$

MODULE FOR A VERTEX OPERATOR ALGEBRA

A V-module is a vector space $W = \coprod_{n \in \mathbb{R}_+} W_n$ together with a linear map

$$Y_W(\, . \, , z) : V \longrightarrow \mathrm{End}\, W[[z^{\pm 1}]]$$

$$a \mapsto Y_W(a, z) = \sum_{n \in \mathbb{Z}} a_n^W \, z^{-n-1}$$

+ SEVERAL AXIOMS

“NICE” VERTEX ALGEBRAS

- A vertex operator algebra is RATIONAL if it has semisimple representation theory

- A vertex operator algebra is C_2-cofinite if $\dim V/C_2(V) < \infty$

There are finitely many inequivalent irreducible modules $(W^1, \ldots W^M)$

$$C_2(V) = \langle \{u_{-2}V \; | u, v \in V\} \rangle_{\mathbb{C}}$$

$$Y(u,z)v = \sum_{n\in\mathbb{Z}} u_n v z^{-n-1}$$

Theorem (Zhu-96) "The space of graded traces for simple modules of nice VOAs is modular invariant"

Let V be a rational C_2 cofinite vertex operator algebra $V = \bigoplus_{n\geq 0} V_n$.

Let $k \in \mathbb{Z}_+$ and $u \in V_k$ and let W be an irreducible positive energy module.

Then, the trace function

$$Tr_W(u,\tau) := Tr|_W \; o^W(u) q^{L_0 - c/24}$$

converges to a holomorphic function on $\mathbb{H}$, where $o^W(u) = u^W_{k-1}$.

Moreover, the space spanned by $\{Tr_{W^i}(u,\tau) \; 1 \leq i \leq M\}$ $(W^1, \ldots W^M$ all of the irreducibles$)$

is a finite dimensional representation of $SL(2,\mathbb{Z})$.

Taking $u = |0\rangle$

$$dim_q(W^i) = Tr|_{W^i}(|0\rangle, \tau) \; = q^{-c/24} \sum_{\lambda \in \mathbb{R}_+} tr(Id_{W^i_\lambda}) q^\lambda$$

In particular, the space of graded dimensions for simple modules of nice VOAs is modular invariant.

OTHER VERTEX ALGEBRAS AND MODULARITY

Many interesting vertex algebras, such as the free bosons are not rational, i.e. they admit indecomposable and not irreducible modules.

In 2004, Miyamoto defined **graded pseudo-traces for indecomposable modules and proved their modularity** for irrational **and** C_2 **cofinite** vertex operator algebras.

Informally:
Once we have indecomposable modules, degree-preserving operators are not diagonalizable on each graded component of a module. For a nice enough module (an interlocked module)

$$o^W(v)|_{W_\lambda} = \begin{bmatrix} A & C & B \\ 0 & E & {}^*C \\ 0 & 0 & A \end{bmatrix}$$

Pseudo-traces capture some of the new off-diagonal information

GRADED PSEUDO-TRACES

For an "interlocked module" $W = \prod_{\lambda \in \mathbb{C}} W_{[\lambda]}$ $\longrightarrow$ Generalized eigenvectors for L_0

one can choose a "Miyamoto basis" for each graded component such that

$$o^W(v)|_{W_{[\lambda]}} = \begin{bmatrix} A & C & B \\ 0 & E & {}^*C \\ 0 & 0 & A \end{bmatrix}$$

$$\mathrm{pstr}(o^W(v)|_{W_{[\lambda]}}) := \mathrm{tr}(B)$$ $\longrightarrow$ Pseudo-trace at degree λ

$L_0|_{W_{[\lambda]}} = (L_0^S)_\lambda + (L_0^N)_\lambda$ (because L_0 acts not diagonalizably on W)

Modified Pseudo-trace at degree λ $\longrightarrow$ $\mathrm{pstr}(o^W(v)|_{W_{[\lambda]}} \boxed{q^{(L_0^N)_\lambda}}) = \sum_{j \in \mathbb{Z}_{\geq 0}} \mathrm{pstr}(o^W(v)|_{W_{[\lambda]}} \frac{1}{j!} (L_0^N)_\lambda^j (\log q)^j)$

$$\| \\ e^{(L_0^N)_\lambda \log(q)}$$

GRADED PSEUDO-TRACES

For an "interlocked module" $W = \prod_{\lambda \in \mathbb{C}} W_{[\lambda]} \longrightarrow$ Generalized eigenvectors for L_0

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$$o^W(v)|_{W_\lambda} = \begin{bmatrix} A & C & B \\ 0 & E & {}^*C \\ 0 & 0 & A \end{bmatrix}$$

$$\mathrm{pstr}_W(v,\tau) = \mathrm{pstr}_W(o^W(v)q^{L_0 - c/24}) = \sum_{\lambda \in \mathbb{C}} \mathrm{pstr}(o^W(v)|_{W_{[\lambda]}} q^{(L_0^N)_\lambda}) q^{(L_0^S)_\lambda} q^{-c/24}$$

$$= \sum_{\lambda \in \mathbb{C}} (\text{ modified pseudo-trace})_\lambda q^{(L_0^S)_\lambda} q^{-c/24}$$

$$= \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{Z}_{\geq 0}} \mathrm{pstr}(o^W(v)|_{W_{[\lambda]}} \tfrac{1}{j!} (L_0^N)_\lambda^j (\log q)^j) q^{\lambda - c/24}$$

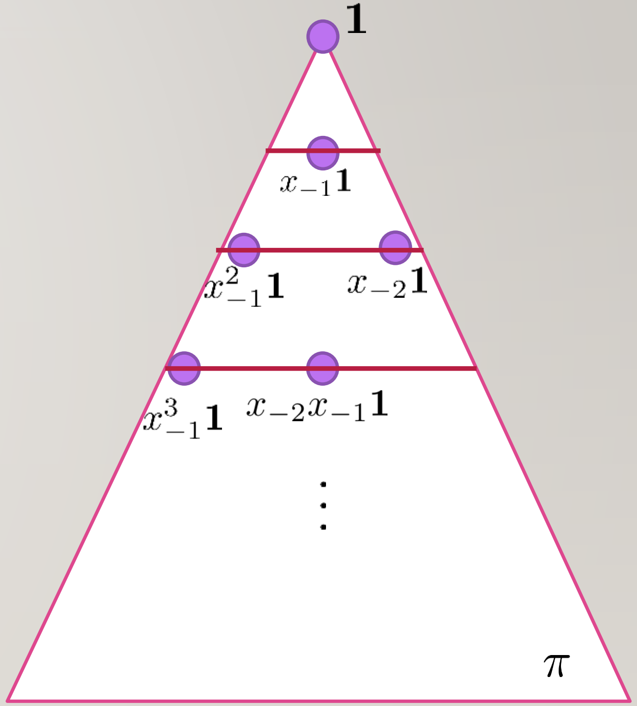
GRADED PSEUDO-TRACES: AN EXAMPLE

Let $W(\lambda, k)$ be the indecomposable π-module with $W(\lambda,k) = \bigoplus_{n\in\mathbb{Z}_+} W(\lambda,k)_n$

$$W(\lambda,k)_0 = span\{u_1, \ldots u_k\}$$

$$(x_0 - \lambda)u_1 = 0, \quad (x_0 - \lambda)u_j = u_{j-1}, \qquad j = 2, \ldots, k$$

$$x_0|_{W(\lambda,k)_0} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$



$x_0 = o^{W(\lambda,k)}(x_{-1}\mathbf{1})$ is the degree-preserving endomorphism associated to $x_{-1}\mathbf{1} \in \pi$

GRADED PSEUDO-TRACES: AN EXAMPLE

Let $W(\lambda,k)$ be the indecomposable π-module with $W(\lambda,k)=\bigoplus_{n\in\mathbb{Z}_+}W(\lambda,k)_n$

$$W(\lambda,k)_0 = span\{u_1,\ldots u_k\}$$

$$(x_0-\lambda)u_1=0,\quad (x_0-\lambda)u_j=u_{j-1},\qquad j=2,\ldots,k$$

$$L_0=\tfrac{1}{2}\left(2\sum_{\ell>0}\ell\, x_{-\ell}\frac{\partial}{\partial x_{-\ell}}\right)+\tfrac{1}{2}x_0^2$$

$$L_0|_{W(\lambda,k)_0}=\underbrace{\begin{bmatrix}\frac{\lambda^2}{2} & 0 & 0 & \cdots & 0 & 0\\ 0 & \frac{\lambda^2}{2} & 0 & \cdots & 0 & 0\\ 0 & 0 & \frac{\lambda^2}{2} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{\lambda^2}{2} & 0\\ 0 & 0 & 0 & \cdots & 0 & \frac{\lambda^2}{2}\end{bmatrix}}_{L_0^S}+\underbrace{\begin{bmatrix}0 & \lambda & \frac{1}{2} & 0 & \cdots & 0 & 0\\ 0 & 0 & \lambda & \frac{1}{2} & \cdots & 0 & 0\\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \lambda & \frac{1}{2}\\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}}_{L_0^N}$$

GRADED PSEUDO-TRACES: AN EXAMPLE

If we take $k=2$, $\quad W(\lambda,2)=\bigoplus_{n\in\mathbb{Z}_+} W(\lambda,2)_n$

$$W(\lambda,2)_0 = span\{u_1,u_2\} \qquad (x_0-\lambda)u_1=0,\ \ (x_0-\lambda)u_2=u_1,$$

$$L_0|_{W(\lambda,2)_0} = \underbrace{\begin{bmatrix} \frac{\lambda^2}{2} & 0 \\ 0 & \frac{\lambda^2}{2} \end{bmatrix}}_{L_0^S} + \underbrace{\begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}}_{L_0^N} \Rightarrow q^{L_0^N} = e^{L_0^N \log(q)} = \begin{bmatrix} 1 & \lambda\log(q) \\ 0 & 1 \end{bmatrix}$$

Modified Pseudo-trace at degree 0 $\longrightarrow$ $\mathrm{pstr}(o^W(\mathbf{1})|_{W_0} q^{L_0^N}) = pstr\left(Id|_{W_0}\begin{pmatrix} 1 & \lambda\log q \\ 0 & 1\end{pmatrix}\right) = tr(\lambda\log(q)) = \lambda\log(q)$

SOME GRADED PSEUDO-TRACES

The free boson admits indecomposable non-irreducible modules. We computed its pseudo-traces for certain vectors in π_a [Barron, Batistelli, OH, Yamskulna, 2023]

$$pstr_W(\mathbf{1}, \tau)$$

$$pstr_W(x_{-1}\mathbf{1}, \tau)$$

$$pstr_W(\omega^a, \tau)$$

The free bosons are not C_2-cofinite. However, we can still apply Miyamoto's construction to interlocked modules.

Theorem [Barron, Batistelli, OH, Yamskulna, 2023]
All indecomposable modules for π_a are interlocked in the sense of Miyamoto.

THEOREM: (GRADED PSEUDO TRACES)

[Barron, Batistelli, OH, Yamskulna, 2023]

Let $W(a,\lambda,k)$ be the indecomposable π_a -module with $W(a,\lambda,k) = \bigoplus_{n\in\mathbb{Z}_+} W(a,\lambda,k)_n$

$$W(a,\lambda,k)_0 = span\{u_1,\ldots u_k\}$$

$$(x_0-\lambda)u_1 = 0, \quad (x_0-\lambda)u_j = u_{j-1}, \qquad j = 2,\ldots,k$$ for fixed $\lambda, a \in \mathbb{C}, k \in \mathbb{N}$.

Then,

$$\mathrm{pstr}_W(\mathbf{1},\tau) = \delta_{a,\lambda,k}(\log q)\ dim_q(W(a,\lambda,k))$$

for a function $\delta_{a,\lambda,k}(\log q)$ that we can describe explicitly.

- $k=1$ ⇨ $\delta_{a,\lambda,k=1} = 1$
- $k=2$ ⇨ $\delta_{a,\lambda,k=2} = (\lambda-a)\log q$
- $k=3$ ⇨

$$\delta_{a,\lambda,k=3} = \frac{1}{2}\log q + \frac{1}{2!}(\lambda-a)^2(\log q)^2$$

EXAMPLE 2: THE VIRASORO VERTEX ALGEBRA

Vir is the Lie algebra with generators $\{L_n\}_{n\in\mathbb{Z}}$ and C where C is central and

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{n^3-n}{12}C\delta_{n,-m}$$

VERMA MODULES

Let $Vir^+ := \oplus_{n\geq 0}\mathbb{C}L_n \oplus \mathbb{C}C$ and let $\mathbb{C}\mathbf{1}_c^h$ denote the Vir^+-module given by

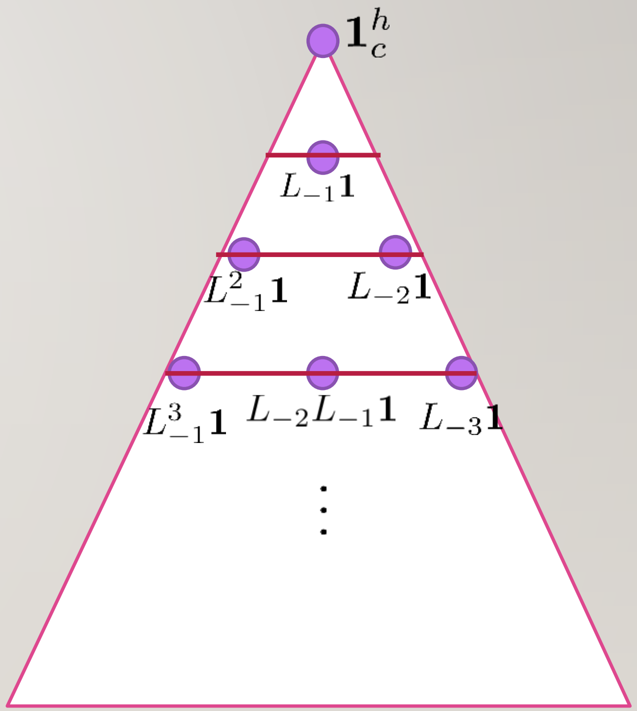
$$L_n\mathbf{1}_c^h = 0, n > 0$$

$$L_0\mathbf{1}_c^h = h\mathbf{1}_c^h$$ Conformal weight $h \in \mathbb{C}$

$$C\mathbf{1}_c^h = c\mathbf{1}_c^h$$ Central charge $c \in \mathbb{C}$

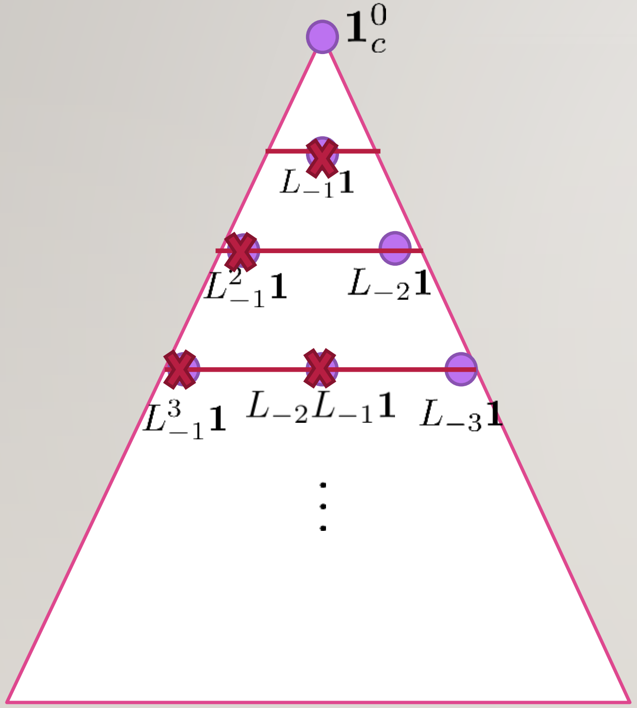
Then, the Verma module is

$$M(c,h) := U(Vir) \otimes_{U(Vir^+)} \mathbb{C}\mathbf{1}_c^h$$



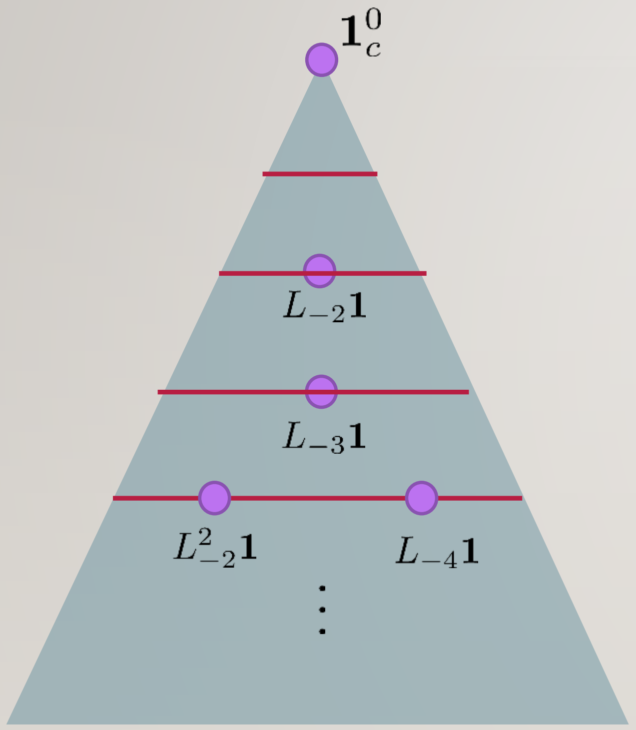
VIRASORO VERTEX OPERATOR ALGEBRA

Frenkel, Zhu $\quad M_c = M(c,0)/ < L_{-1}\mathbf{1} >$ is a VOA for any $c \in \mathbb{C}$



$$\frac{M(c,0)}{< L_{-1}\mathbf{1} >} = M_c$$

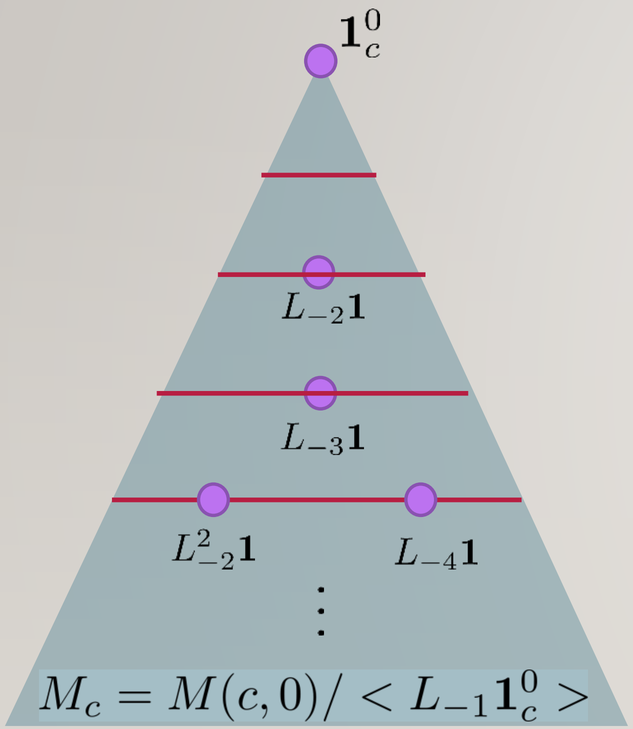
VOA STRUCTURE ON M_c



- $|0\rangle = \mathbf{1}_c^0$
- $\omega = L_{-2}\mathbf{1}$
- $Y(L_{-2}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \Rightarrow L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$
- $Y(L_{-j_1-2}\cdots L_{-j_k-2}\mathbf{1}, z) = \frac{1}{j_1!\cdots j_k!} : \partial^{j_1} L(z) \cdots \partial^{j_k} L(z) :$

$M_c = M(c,0)/ < L_{-1}\mathbf{1}_c^0 >$

M_c IS IRRATIONAL



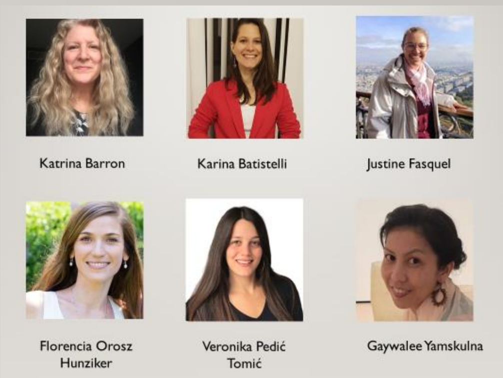
We have described the graded pseudo-traces for certain indecomposable modules for the universal virasoro vertex algebra M_c
[Barron, Batistelli, OH, Yamskulna, 2023]

M_c is neither rational nor C_2-cofinite

For the Virasoro vertex algebra, we proved that there are indecomposable modules that are "not nice" (i.e. NOT INTERLOCKED).
(Unlike in the Free boson case, where all indecomposables are "nice", i.e. interlocked)

[Barron, Batistelli, OH, Yamskulna, 2023]
We described which indecomposables (induced from the level 0 Zhu algebra) are interlocked and computed their pseudotraces.

FUTURE WORK



- M_c and π are neither rational nor C_2-cofinite
- They both have, however, "interlocked modules" and interesting graded pseudo-traces.
- These pseudo-traces satisfy an important logarithmic derivative property

  (key for modularity in the irrational and C_2 cofinite case [Miyamoto])
- We computed [Barron, Batistelli, OH, Yamskulna, 23]

$pstr_W(\mathbf{1},\tau)$

$pstr_W(x_{-1}\mathbf{1},\tau)$

$pstr_W(\omega^a,\tau)$

π_a

$pstr_W(\mathbf{1},\tau)$

$pstr_W(\omega,\tau)$

M_c

WE WILL STUDY OTHER PSEUDOTRACES

$$pstr(v,\tau) \quad \text{for other } v \in \pi_a$$

And their (pseudo)-modular behavior

THANK YOU!

¡GRACIAS!