Random curves and surfaces

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Agenda

1. Uniformly random objects: Curves, functions, surfaces

2. Conformal welding of random surfaces

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2. Conformal welding of random surfaces

How can you sample a curve uniformly at random?

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First question: What is a curve?

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Curve

In mathematics, a **curve** (also called a **curved line** in older texts) is an object similar to a line, but that does not have to be straight.

Intuitively, a curve may be thought of as the trace left by a moving point. This is the definition that appeared more than 2000 years ago in Euclid's *Elements*: "The [curved] line[a] is [...] the first species of quantity, which has only one dimension, namely length, without any width nor depth, and is nothing else than the flow or run of the point which [...] will leave from its imaginary moving some vestige in length, exempt of any width."[1]

This definition of a curve has been formalized in modern mathematics as: *A curve is the image of an interval to a topological space by a continuous function*. In some contexts, the function that defines the curve is called a *parametrization*

Our set of curves:

$$\Omega := \{f : [0,1] \to \mathbb{R} \ : \ f \text{ is continuous}\}$$

How can you sample a curve uniformly at random?

$$\Omega := \{f : [0,1] \to \mathbb{R} \, : \, f \text{ is continuous}\}$$

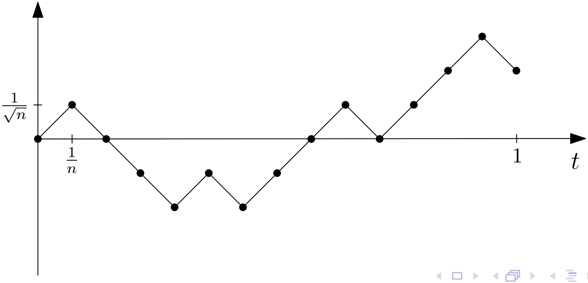
Uniformly at random: all possible outcomes have the same probability

Problem: Ω is infinite.

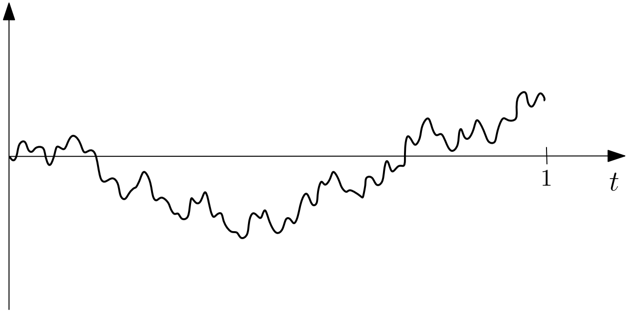
Solution: We **discretize** the space of curves Ω.

$\Omega_n = \{f \in \Omega \, : \, f \text{ linear with slope } \pm\sqrt{n} \text{ on } [\frac{k}{n}, \frac{k+1}{n}]; f(0) = 0\}$.

Pick a function uniformly at random from Ω_n; send $n \to \infty$.



Brownian motion



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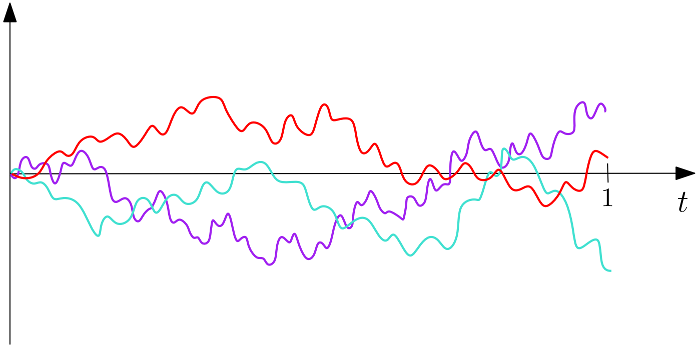
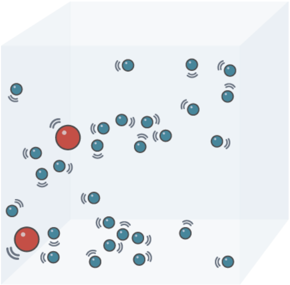
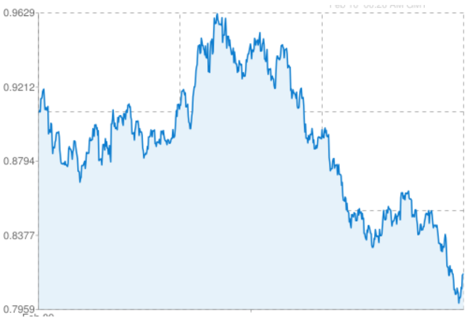


3 independent samples of Brownian motion

Brownian motion in applications



Pollen grains (red) in water



USD/EUR 2 year exchange rate

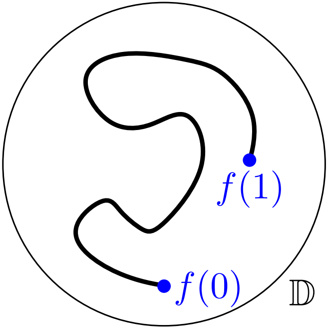
Broad range of applications in physics, chemistry, biology, economics, finance, etc.

Universality: scaling limit of large class of discrete models

How to sample uniform self-avoiding curve in unit disk?

How to sample uniform self-avoiding curve in unit disk?

$$\Omega = \{f : [0,1] \to \overline{\mathbb{D}} \,:\, f \text{ continuous and injective}\}.$$



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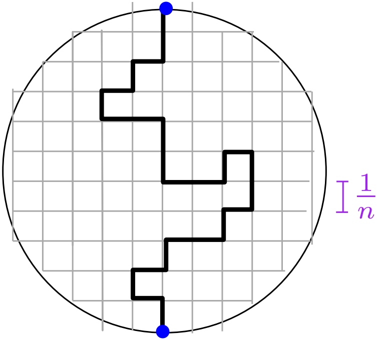
$$\Omega = \{f : [0,1] \to \overline{\mathbb{D}} \,:\, f \text{ continuous and injective}\}.$$

$$\Omega_n = \{f \in \Omega \,:\, f \text{ on } \tfrac{1}{n}\mathbb{Z}^2 \text{ connecting } \pm i;\ \text{speed } n^{1/3}\}.$$

Sample a curve uniformly at random from Ω_n.



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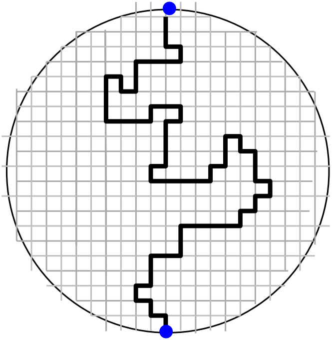

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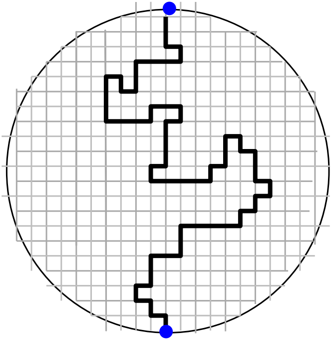
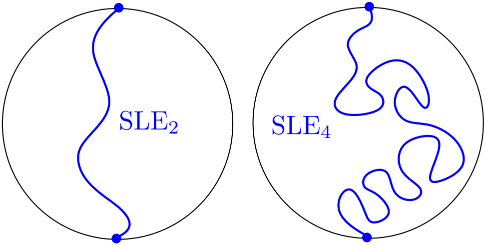
Sample a curve uniformly at random from Ω_n.

Conjecture: As $n \to \infty$, we get **Schramm-Loewner evolution (SLE)**



Schramm-Loewner evolution (SLE)

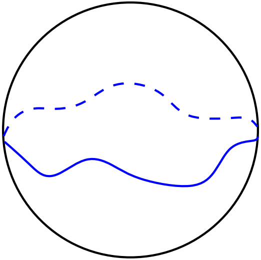


Schramm-Loewner evolution

- Random fractal curve introduced in Schramm'99.[1]
- Scaling limit of statistical physics models
    - Examples: Ising model, percolation, uniform spanning tree, etc.
- Uniquely characterized by conf. inv. and domain Markov property.
- Parameter $\kappa > 0$.

[1] The year here and later refers to the time of the initial arXiv preprint

Schramm-Loewner evolution (SLE)



SLE loop

- Random fractal curve introduced in Schramm'99.[1]
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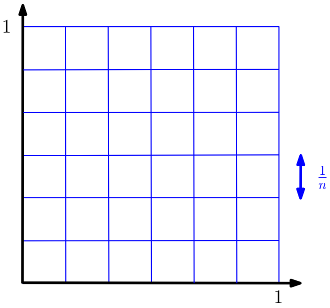
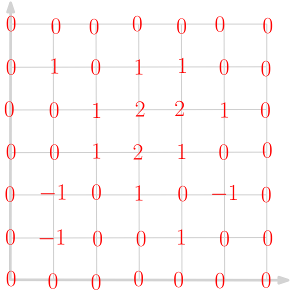
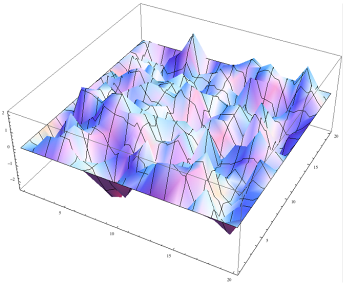
How to sample a function on $[0,1]^2$ uniformly at random?



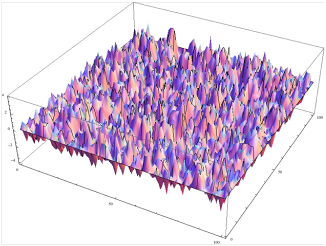


$h_n(i) - h_n(j) \in \{-1, 0, 1\}$ if $i \sim j$

How to sample a function on $[0,1]^2$ uniformly at random?



$n = 20$



$n = 100$

- $|h_n(z)|$ of order $\sqrt{\log n}$ for z in the bulk.
  - Glazman-Manolescu'18 proves this for the triangular lattice.
- Conjecture: $h_n \Rightarrow h$, where h is the Gaussian free field (GFF).

Gaussian free field (GFF)

- The GFF h is a **random distribution (generalized function)**. With probability one,
  - $h(z)$ is **not** well-defined for any fixed $z \in S := (0,1)^2$,
  - $\int_S hf$ is well-defined for f a smooth test function,
  - h is in the Sobolev space $H^{-\epsilon}(S)$ for any $\epsilon > 0$.
- $\int_S hf$ is normally distributed with mean 0 and

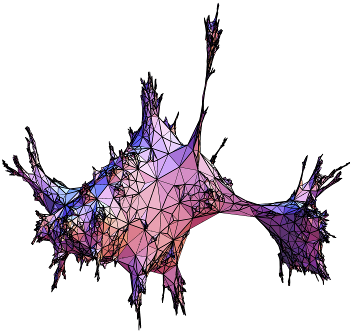
$$\mathrm{Cov}\left(\int_S hf, \int_S hg\right) = \int_{S\times S} f(x)G(x,y)g(y)\,dx\,dy,$$

  where $G : S \times S \to [0,\infty]$ is the Green's function.
- Universality: Scaling limit of fluctuations in a number of statistical physics models, e.g. domino tilings and random matrices.
- Generalization of Brownian motion to higher-dimensional time.
- Appears in constructions in conformal field theory.

How can you sample a surface uniformly at random?

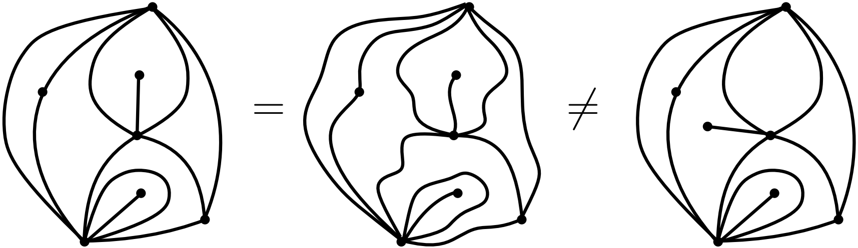
How can you sample a surface uniformly at random?

A uniform planar map

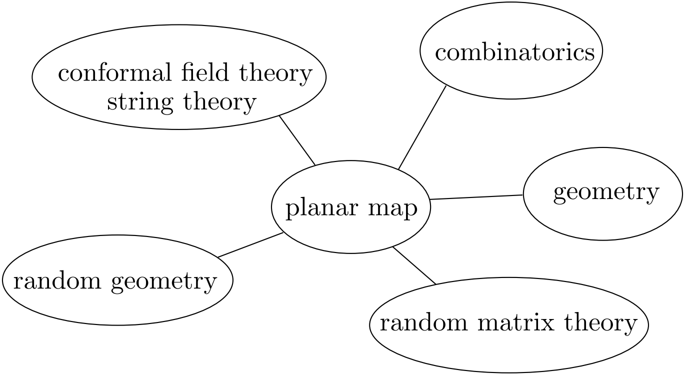


Random planar maps

- A **planar map** is a graph drawn on the sphere, viewed modulo continuous deformations.
- For $n \in \mathbb{N}$ sample M_n **uniformly** at random from the collection of planar maps with n edges.

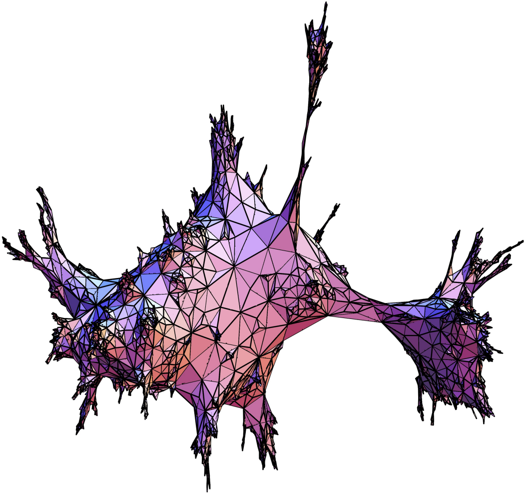


Planar maps


conformal field theory
string theory
combinatorics
planar map
geometry
random geometry
random matrix theory

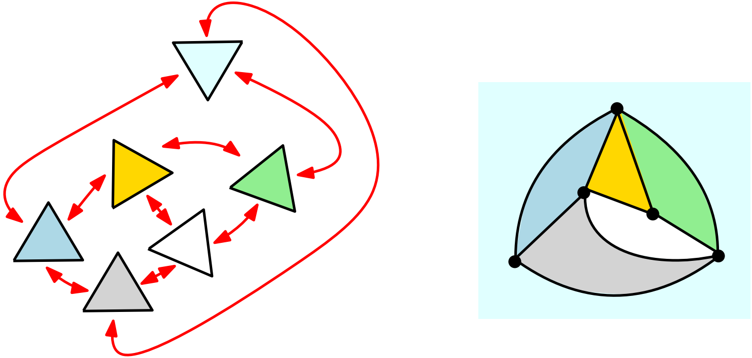
Scaling limits of uniform planar maps

M_n uniform planar map with n edges. Does M_n converge as $n \to \infty$?



Conformal embedding of random planar maps

Uniformization theorem: For any simply connected Riemann surface M there is a conformal map ϕ from M to either $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{S}^2$.



A triangulation viewed as a Riemannian manifold

Conformal embedding of random planar maps

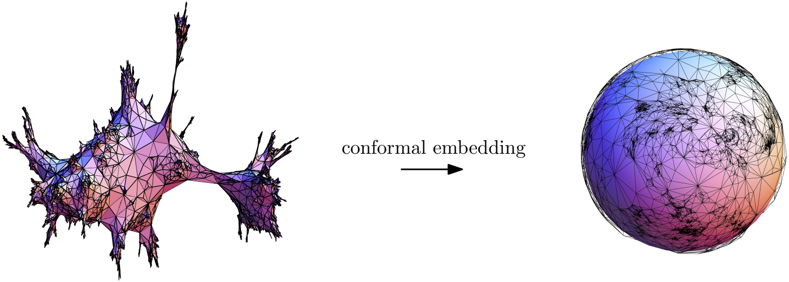
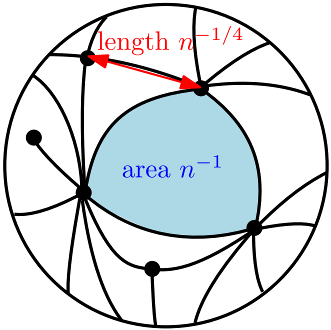


Figure by Nicolas Curien.

Conformal embedding of random planar maps

- Area measure μ_n on $\mathbb{S}^2$: each face has area n^{-1}
- Distance function (metric) D_n on $\mathbb{S}^2$: adjacent vertices distance $n^{-1/4}$



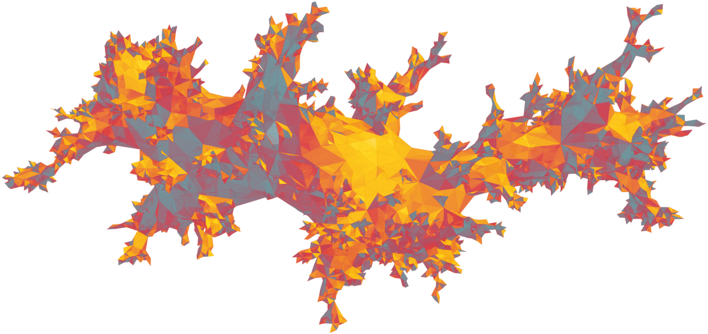
conformally embedded planar map
with the **Cardy-Smirnov embedding**

$\Rightarrow$

Liouville quantum gravity (LQG) surface (μ, D)

See H.-Sun'19, which builds on earlier works (Le Gall'11, Miermont'11, Garban-Pete-Schramm'08-'13, Duplantier-Miller-Sheffield'14, Mil.-Sheff.'16, Gwynne-Mil.'17) and works with Albenque, Bernardi, Garban, Gwynne, Lawler, Li, Sepulveda.

Liouville quantum gravity surface



2D Riemannian manifold $e^{\gamma h}(dx^2 + dy^2)$,
with $\gamma \in (0, 2)$ and h a 2D Gaussian free field

LQG surfaces introduced by Polyakov'81 in context of string theory

Liouville quantum gravity (LQG)

- Let $\gamma \in (0,2)$ and let h be the Gaussian free field in $(0,1)^2$.
- LQG surface: $e^{\gamma h}(dx^2 + dy^2)$

$$\text{Area measure: } \mu = \text{“}e^{\gamma h} d^2 z\text{”},$$

$$\text{Boundary measure: } \nu = \text{“}e^{\gamma h/2} dz\text{”},$$

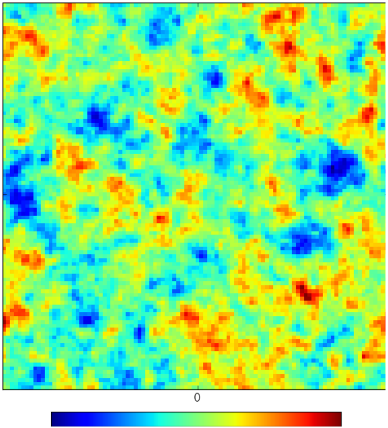
$$\text{Distance: } D = \text{“}e^{\gamma h/d_\gamma} |dz|\text{”}, \quad d_\gamma = \text{dimension} > 2.$$

- The definition of an LQG surface does not make literal sense since h is a distribution and not a function.
- μ, ν, D defined rigorously via regularized version h_ϵ of h, e.g.

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \quad U \subset (0,1)^2.$$

- References:
  - μ, ν: Hoegh-Krohn'71, Kahane'85, Duplantier-Sheffield'08, Rhodes-Vargas'13, Berestycki'15, etc.
  - D: Ding-Dubedat-Dunlap-Falconet'19, Gwynne-Miller'19

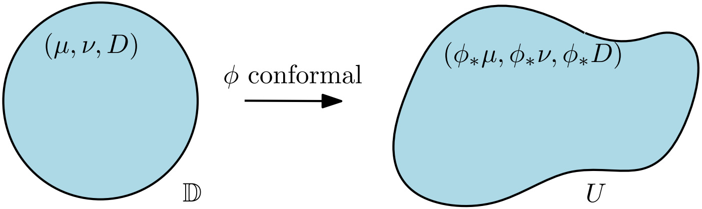
LQG area measure



Random area measure $\mu =$ "$e^{\gamma h} d^2 z$" (figure by M. Park)

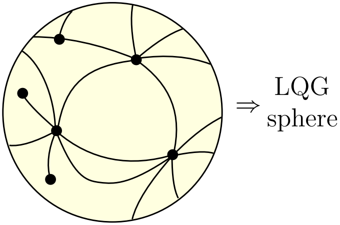
Liouville quantum gravity (LQG) surface

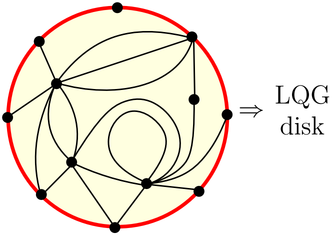
The tuple (μ, ν, D) describes the geometry of the γ-**LQG surface** $(\mathbb{D}, h)$.



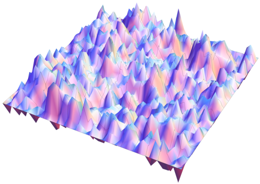
Two different embeddings of the same γ-LQG surface

Random planar maps converging to variants of LQG

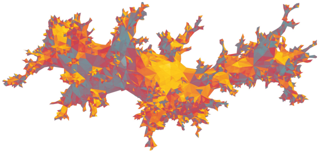




GFF, LQG and SLE: Three random planar objects



Gaussian Free Field
(generalized function)



Liouville quantum gravity
(2D Riemannian manifold)

Schramm-Loewner evolution
(non-crossing curve)

All three objects
have intriguing **conformal symmetries** and
describe the scaling limit of many discrete models (universality).

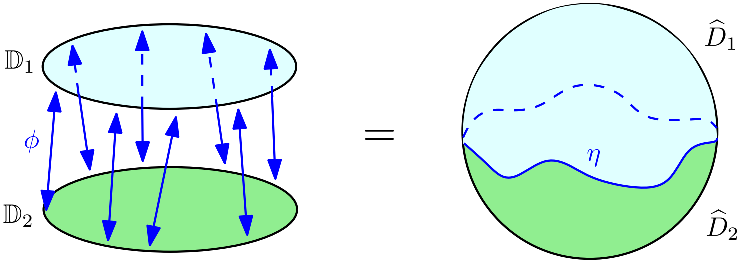
Agenda

1. Uniformly random objects: Curves, functions, surfaces

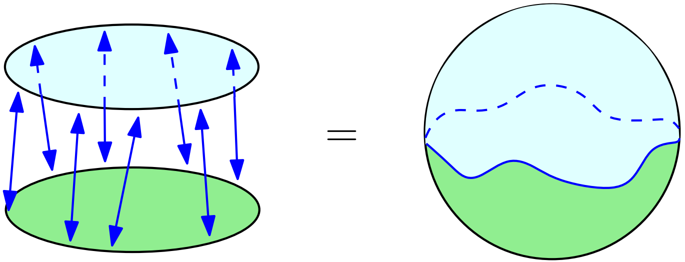
2. Conformal welding of random surfaces

The conformal welding problem

- $\mathbb{D}_1, \mathbb{D}_2$ copies of the unit disk; $\phi : \partial\mathbb{D}_1 \to \partial\mathbb{D}_2$ a homeomorphism.
- Conformal welding: a conformal structure on the sphere $\mathbb{S}^2$ obtained by identifying $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_2$ according to ϕ.
  - More precisely, we are interested in a curve η and conformal maps $\psi_j : \mathbb{D}_j \to \widehat{D}_j$, $j = 1, 2$, such that $\phi = \psi_2^{-1} \circ \psi_1|_{\partial\mathbb{D}_1}$.
- Does there exist a conformal welding? If so, is it unique?
- Existence and uniqeness may fail, but sufficient regularity of ϕ or η guarantees the existence of a unique solution.



Disk + disk = sphere + SLE loop

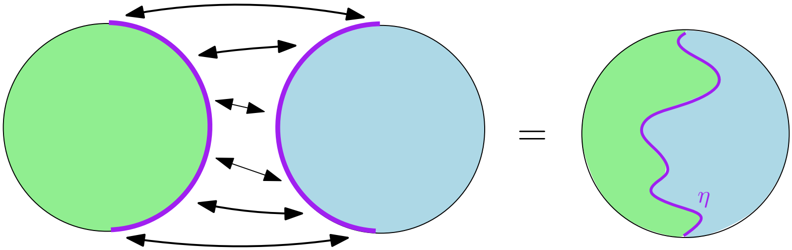


Two γ-LQG disks

γ-LQG sphere and SLE_{γ^2} loop

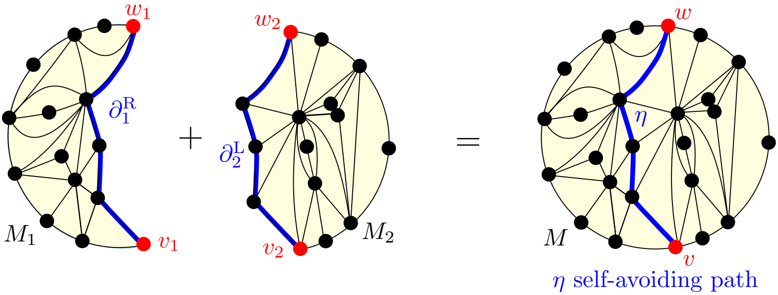
- Green & blue disks **independent** cond. on matching bdy lengths
- Welding homeomorphism given by LQG boundary length
- SLE loop and sphere in right figure **independent**
- Ang-H.-Sun'21, building on several earlier works (see next slide)

Disk + disk = disk + chordal SLE



- Green & blue disks **independent** cond. on matching bdy lengths
- Welding homeomorphism given by LQG boundary length
- SLE and disk in right figure **independent**
- Ang-H.-Sun'20, building on Sheffield'10 & Duplantier-Miller-Sheff.'14

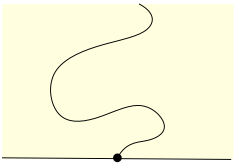
Discrete motivation for conformal welding



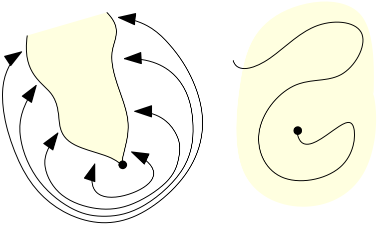
Bijection: $((M_1, v_1, w_2), (M_2, v_2, w_2)), \#\partial_1^{\mathsf{R}} = \#\partial_2^{\mathsf{L}}$ and (M, v, w, η)

Continuum result inspired by planar maps, but proof is purely continuum.

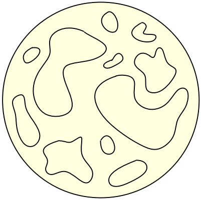
More examples



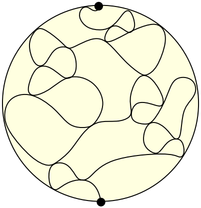
simple chordal SLE



whole-plane SLE from 0 to ∞



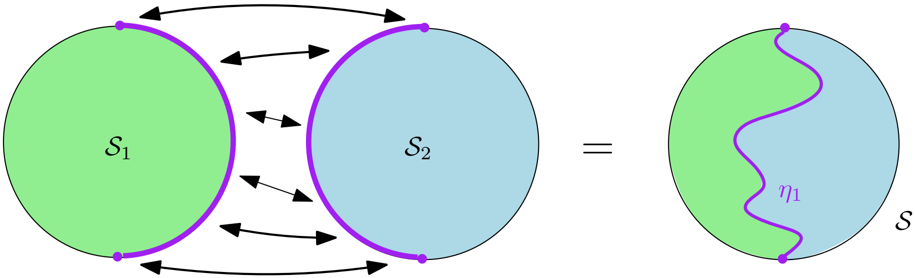
conformal loop ensemble



non-simple chordal SLE

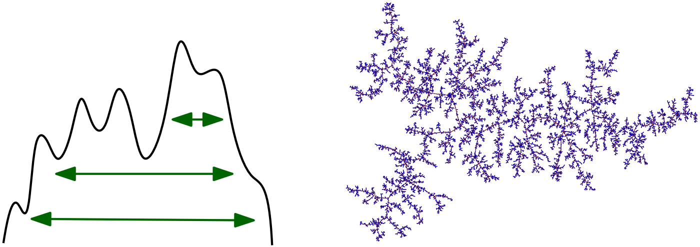
Sheffield'10, Duplantier-Miller-Sheffield'14, Miller-Sheffield-Werner'20

General principles



- Cutting a γ-LQG surface $\mathcal{S}$ by the "right" independent SLE_κ-type curve(s) $\eta_1, \eta_2, \ldots$ gives **independent** surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots$ in the complementary components.
- $\mathcal{S}_1, \mathcal{S}_2, \ldots$, plus info about how the surfaces are glued together, determine $\mathcal{S}$ and $\eta_1, \eta_2, \ldots$.
- Discrete analogues on planar maps, although proof continuum.

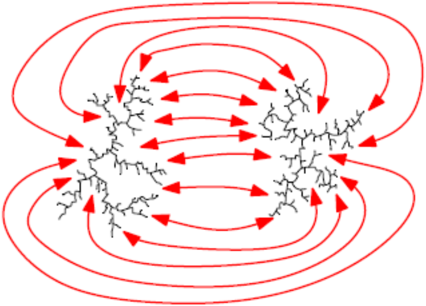
Mating of trees



Brownian excursion and continuum random tree

Right figure by Kortchemski

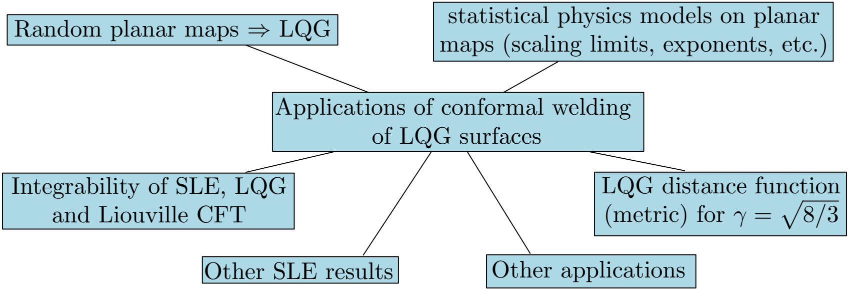
Mating of trees



Duplantier-Miller-Sheffield'14:
Mating/welding two continuum random trees gives
a γ-LQG sphere with a space-filling $\mathrm{SLE}_{16/\gamma^2}$

Allows to study LQG and SLE with Brownian motion

Applications of conformal welding of LQG surfaces

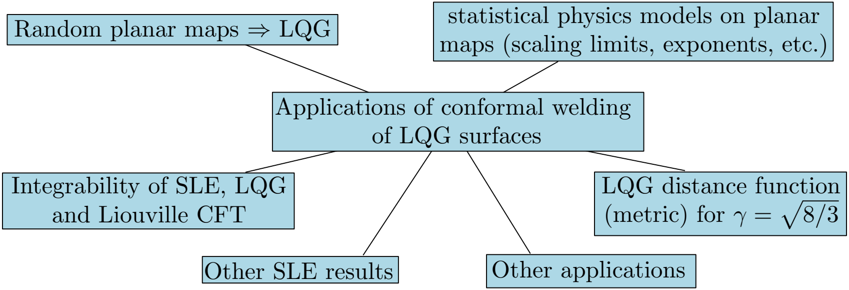


Why is conformal welding so powerful in the study of LQG and SLE?

- exploit interplay between LQG and SLE
- study complicated surfaces by decomposing into smaller indep. pieces

See works of Ang, Borga, Duplantier, Gwynne, H., Kavvadias, Lehmkuehler, Miller, Pfeffer, Schoug, Sheffield, Sun, Werner, Yu, etc.

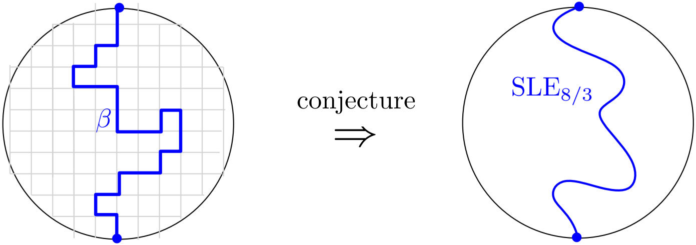
Applications of conformal welding of LQG surfaces



Two examples:

1. Self-avoiding loop on random planar map ⇒ $\mathrm{SLE}_{8/3}$ loop
2. Random permutations and density of the Baxter permuton

Conjecture: Self-avoiding walk $\Rightarrow$ SLE$_{8/3}$

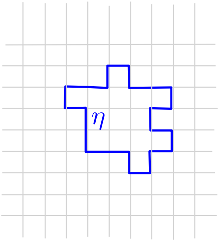


self-avoiding walk

weight $= \mu^{-\mathrm{length}(\beta)}$

$\mu =$ connective constant

Conjecture: Self-avoiding loop $\Rightarrow$ $\mathrm{SLE}_{8/3}$ loop



self-avoiding loop

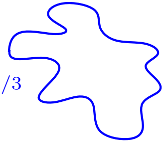
weight $= \mu^{-\mathrm{length}(\eta)}$

$\mu =$ connective constant

conjecture

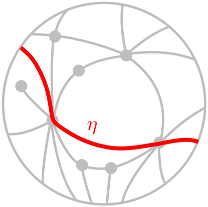
$\Longrightarrow$

$\mathrm{SLE}_{8/3}$



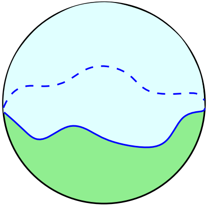
Self-avoiding loop on planar map $\Rightarrow$ SLE$_{8/3}$ loop

Ang-H.-Sun'21:



weight $= 12^{-\#\text{faces}} \cdot 54^{-\text{length}(\eta)}$
self-avoiding loop on spherical planar map

$\Rightarrow$



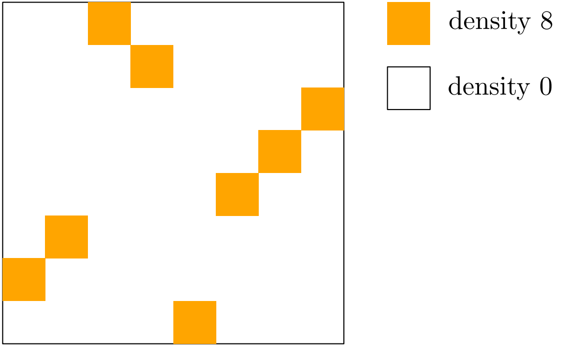
$\sqrt{8/3}$-LQG sphere and SLE$_{8/3}$ loop

Proof builds on

- Conformal welding "disk + disk = sphere + SLE loop"
- Discrete counterpart of welding on planar maps (bijection)
- Technical inputs: Brown'65, Gwynne-Miller'16

Analogous results chordal SLE$_{8/3}$: Gwynne-Miller'16 & Ang-H.-Sun'20

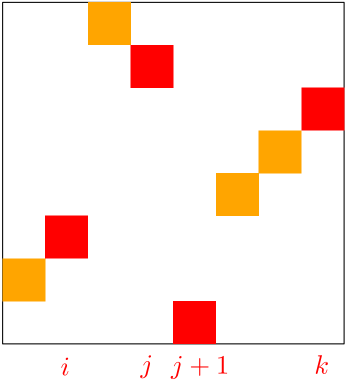
Permutons



Permuton of permutation $\sigma = 23871456$

- Permuton: probability measure on $[0,1]^2$ with uniform marginals.
- Permutons describe the scaling limit of many random permutations: $(\sigma_n)_{n\in\mathbb{N}}$ conv. to a permuton if associated permutons converge.

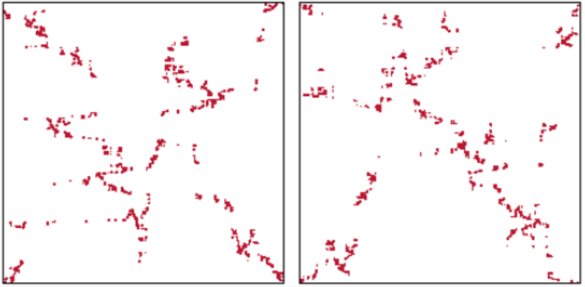
The Baxter permuton



Not a Baxter permutation

- Baxter permutations: no $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$.
- Introduced in Baxter'64.
- Connections with a number of other combinatorial objects.
- Borga-Maazoun'20: Uniform Baxter permutation $\Rightarrow$ Baxter permuton

The Baxter permuton



Two large uniform Baxter permutations

- Baxter permutations: no $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$.
- Introduced in Baxter'64.
- Connections with a number of other combinatorial objects.
- Borga-Maazoun'20: Uniform Baxter permutation $\Rightarrow$ Baxter permuton

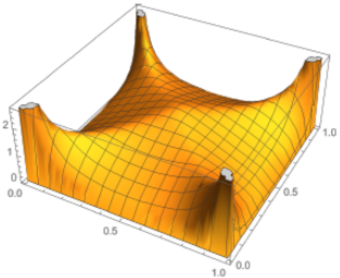
Figure due to Borga

Density of the Baxter permuton

Borga-H.-Sun-Yu'22: Density of $\mathbb{E}[\mu_B]$, with μ_B the Baxter permuton, is

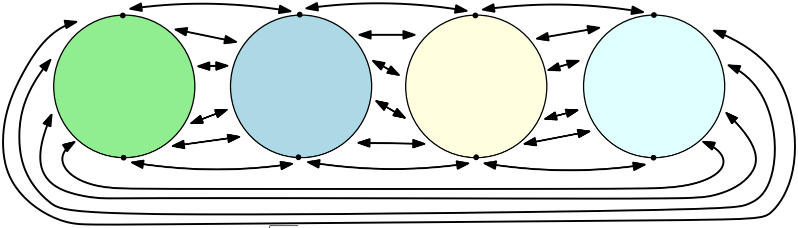
$$p_B(x,y) = c \int_{\max\{0,x+y-1\}}^{\min\{x,y\}} \int_{\mathbb{R}_+^4} \rho(y-z,\ell_1,\ell_2)\rho(z,\ell_2,\ell_3)\rho(x-z,\ell_3,\ell_4) \rho(1+z-x-y,\ell_4,\ell_1)\; d\ell_1 d\ell_2 d\ell_3 d\ell_4\, dz,$$

where $$\rho(t,x,r) := \frac{1}{t^2}\left(\left(\frac{3rx}{2t}-1\right)e^{-\frac{r^2+x^2-rx}{2t}} + e^{-\frac{(x+r)^2}{2t}}\right).$$



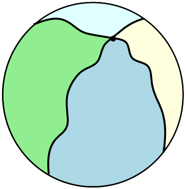
Density of the Baxter permuton: proof

- SDE defining Baxter permuton also arises in LQG conformal welding.
- Baxter density corresponds to computable LQG observable.



Four $\sqrt{4/3}$-LQG disks with $4/\sqrt{3}$-singularities

$=$



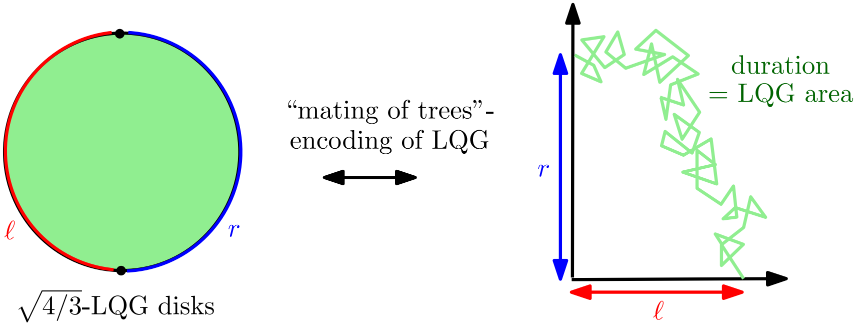
$\sqrt{4/3}$-LQG sphere with $\mathrm{SLE}_{4/3}$-type curves

Density of the Baxter permuton: proof

- SDE defining Baxter permuton also arises in LQG conformal welding.
- Baxter density corresponds to computable LQG observable.

For $I_1, I_2 \subset [0,1]$ and with A_1, A_2, A_3, A_4 denoting the LQG disk areas,

$$\mathbb{E}[\mu_B(I_1 \times I_2)] = \mathbb{P}(A_2 + A_3 \in I_1, A_1 + A_2 \in I_2).$$



$\sqrt{4/3}$-LQG disks with $4/\sqrt{3}$-singularities

Thanks for the attention