Random Monomial Ideals and their Free Resolutions

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Joint Work

joint work with

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Motivating Question

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Folklore: As long as it can be !

Erdős-Rényi-type Model for Random Monomial Ideals

Pioneered by De Loera-Petrović-Silverstein-Stasi-Wilburne in "Random Monomial Ideals" [2017]:

- fix positive integer D;
- sample a generating set G

$$\mathbb{P}[x^\alpha \in G] = \begin{cases} p & |\alpha| \leq D \\ 0 & \text{otherwise,} \end{cases}$$

for all $x^\alpha \in S = k[x_1, \ldots, x_n]$, and set $M = \langle G \rangle$.

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- inspired by the Erdős-Rényi model for generating random graphs;
- if restricted to *squarefree* monomials, it will yield a random simplicial complex.

Threshold Behavior for Dimension

Theorem (DPSSW)

If $D^{-(t+1)} \ll p = p(D) \ll D^{-t}$ *as* $D \to \infty$, *then* $\dim(S/M) = t$ *asymptotically almost surely.*

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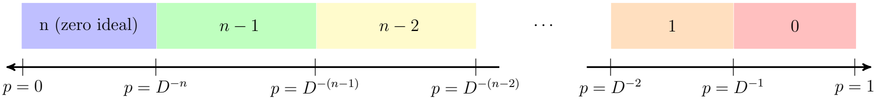

Other Work

- random flag complexes and their syzygies [Erman-Yang '17]
- characteristic dependence of syzygies of random monomial ideals [Booms-Erman-Yang '20]
- asymptotic degree of random monomial ideals [Silverstein-Wilburne-Yang '20]
- edge ideals of random graphs [Banerjee-Yogeshwaran '21]

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$M \sim \mathcal{M}(n, D, p)$.

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Theorem

Let $M \sim \mathcal{M}(n, D, p)$ *and* $p = p(D)$. *As* $D \to \infty$, $p = D^{-n+1}$ *is a threshold for the projective dimension of* S/M. *If* $p \ll D^{-n+1}$ *then* $\mathrm{pdim}(S/M) = 0$ *asymptotically almost surely and if* $p \gg D^{-n+1}$ *then* $\mathrm{pdim}(S/M) = n$ *asymptotically almost surely.*

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Corollary

If $D^{-n+1} \ll p \ll 1$, *then asymptotically almost surely* S/M *is not Cohen-Macaulay.*

Witness Sets and Witness LCMs

Theorem (Alesandroni 2017)

Let $M = \langle G \rangle$. Then $\operatorname{pdim}(S/M) = n$ if and only if there is a subset L of G with the following properties:

1. *L is dominant.*
2. $|L| = n$.
3. *No element of G strongly divides $\operatorname{lcm}(L)$.*

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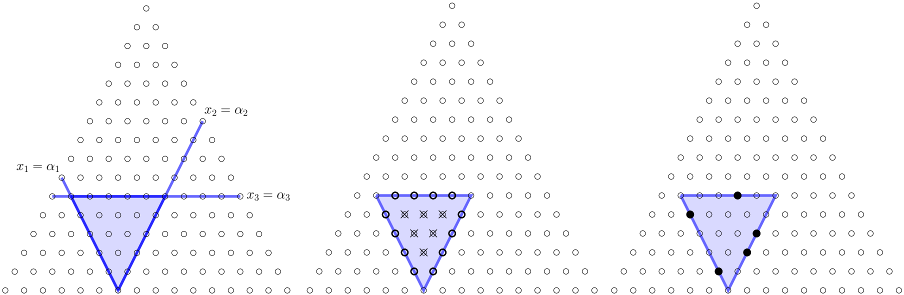
Definition

When L is as above then L witnesses $\operatorname{pdim}(S/M) = n$, and we say L is a *witness set*. The monomial $x^\alpha \in S$ is a *witness lcm* if L is a witness set and $x^\alpha = \operatorname{lcm}(L)$.



(a) Δ_α associated with the witness lcm $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$

(b) x^α is a witness lcm.

Outline of Proof

- $p \ll D^{-n+1}$:

$$\lim_{D\to\infty} \mathbb{P}\left[\text{there is } \geq 1 \text{ min generator}\right] \leq \lim_{D\to\infty} \binom{D+n-1}{n-1} p = 0,$$

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- $p \gg D^{-n+1}$: First show

$$\mathbb{E}\left[\sum_{a=n-1}^{A} \sum_{\substack{|\alpha|=D+a \\ \alpha_i \geq a\, \forall i}} w_\alpha\right] \to \infty \text{ as } D \to \infty$$

Outline of Proof

- Then show $\mathrm{Var}[W] = o\left(\mathbb{E}[W]^2\right)$ and use Chebyshev's inequality

$$\mathbb{P}[W = 0] \leq \frac{\mathrm{Var}[W]}{\mathbb{E}[W]^2}$$

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- Hence, $\mathbb{P}[W > 0] \to 1$, i.e., $M \sim \mathcal{M}(n, D, p)$ has at least one witness to $\mathrm{pdim}(S/M) = n$ with probability converging to 1 as $D \to \infty$.

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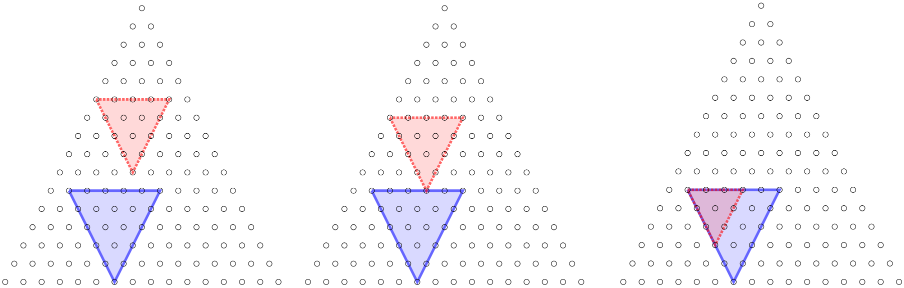
- Non-Cohen-Macaulay-ness: M is almost never 0-dimensional.

Estimating Covariances Cov $[w_\alpha, w_\beta]$



(a) $\gcd(x^\alpha, x^\beta)$ has degree $\leq D$, so Cov $[w_\alpha, w_\beta] = 0$.

(b) If $x^\alpha | x^\beta$ then Cov $[w_\alpha, w_\beta] < 0$.

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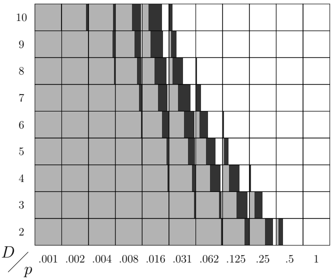
Theorem

As $D \to \infty$, $p = D^{-n+3/2}$ is a threshold for M being generic and for M being strongly generic. If $p \ll D^{-n+3/2}$ then M is generic or strongly generic asymptotically almost surely, and if $p \gg D^{-n+3/2}$ then M is neither generic nor strongly generic asymptotically almost surely.

Scarfness and Genericity



(a) $n = 4$

Generic and Scarf

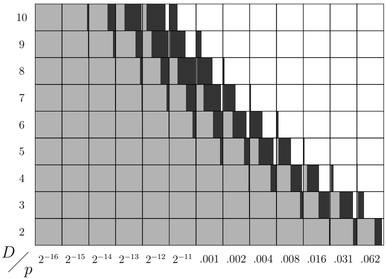
Scarf but not generic

Not Scarf

Ex. for $n = 4$, $D = 10$, and $p = 0.016$, 24% of random samples were generic, 50% were Scarf but not generic, and 26% were neither.

24 26

50



(b) $n = 8$

Scarf but not generic

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Example

$I = \langle x_1^4 x_3 x_5^5,\ x_1 x_2^2 x_3^2 x_6^4 x_8,\ x_2^3 x_5^2 x_6^3 x_7 x_8,\ x_1^3 x_5^2 x_7^2 x_8^3,\ x_2 x_3 x_4^3 x_6 x_8 x_9^3,\ x_1 x_3^4 x_4 x_6^2 x_8 x_{10},\ x_1 x_3 x_4^2 x_5 x_6 x_8^3 x_{10},\ x_2 x_3 x_6^3 x_8^4 x_{10},\ x_4 x_5^5 x_7 x_{10}^3,\ x_1 x_5^4 x_{10}^5 \rangle \subseteq k[x_1, \ldots x_{10}]$, which has the following total Betti numbers:

i	0	1	2	3	4	5	6	7	8
β_i	1	10	45	114	168	147	75	20	2