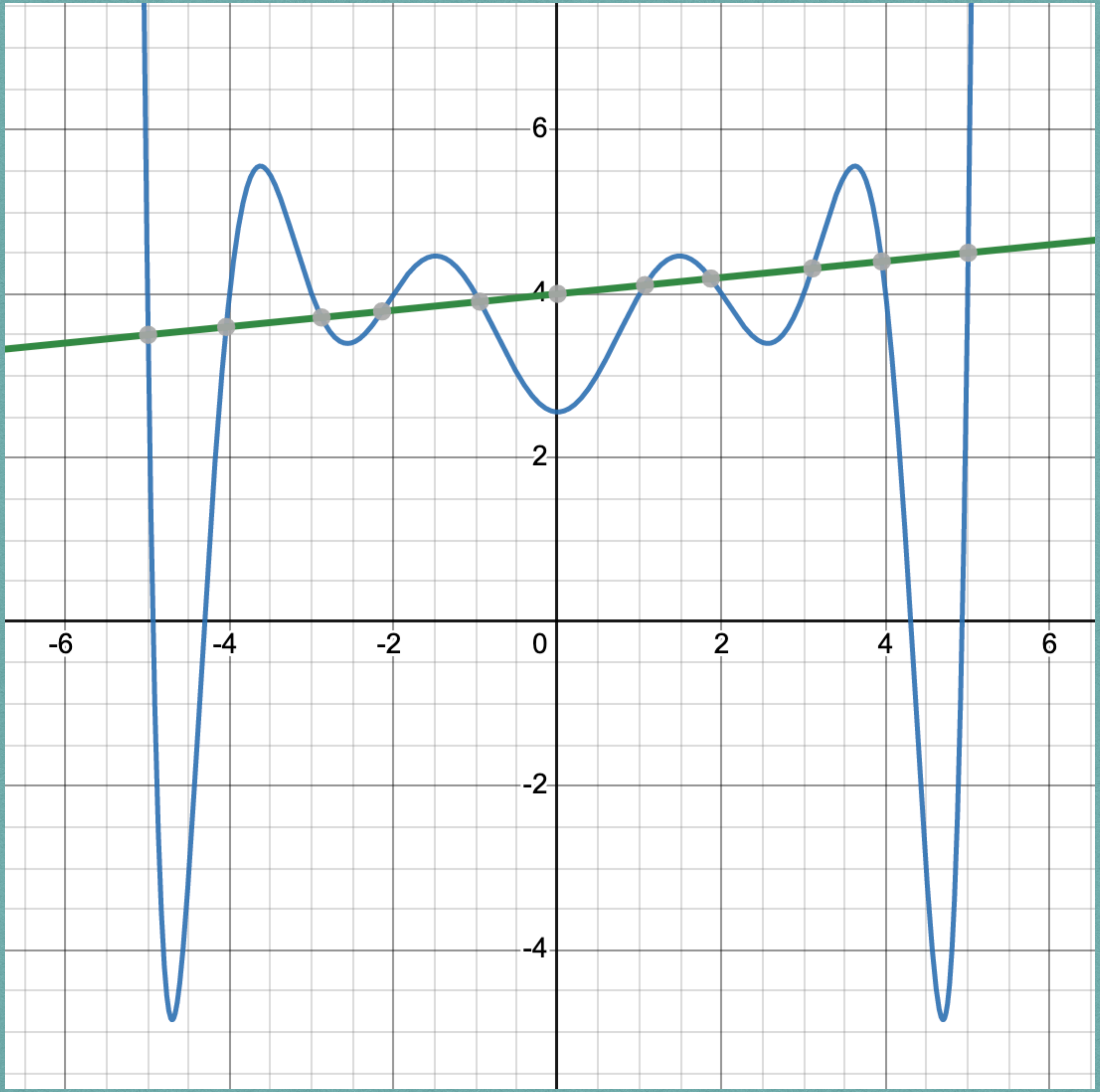
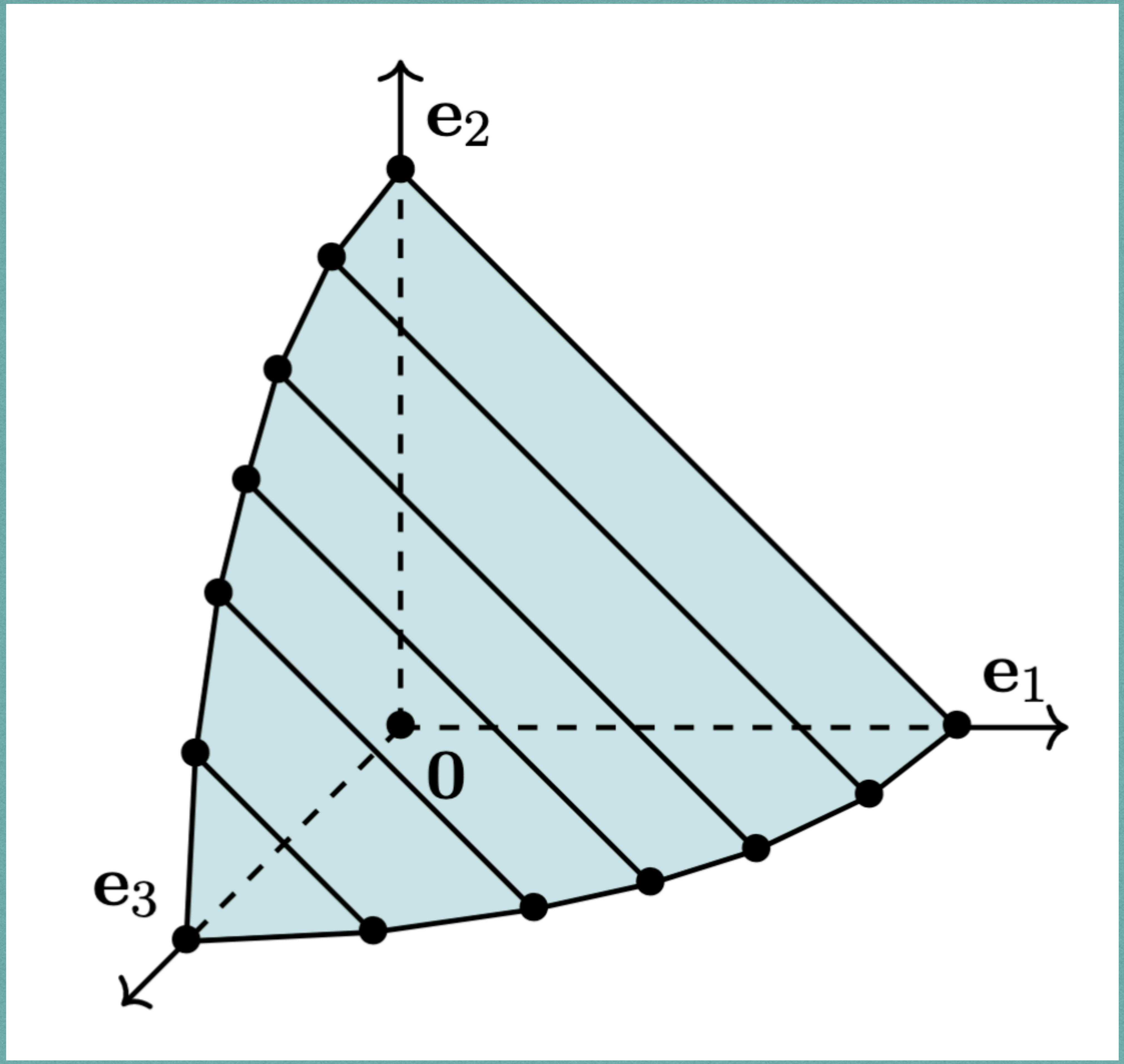
AVERAGE DEGREE OF THE ESSENTIAL VARIETY





Random Algebraic Geometry at BIRS

Samantha Fairchild, MPI MiS

AVERAGE DEGREE OF THE ESSENTIAL VARIETY

ARXIV 2212.01596

Paul Breiding

University of Osnabrück

Pierpaola Santarsiero

University of Osnabrück

Elima Shehu

University of Osnabrück + MPI MiS

AVERAGE DEGREE OF THE ESSENTIAL VARIETY

- $\mathscr{E}$ = Essential Variety
- Degree is 10: $\#\mathscr{E} \cap L \leq 10$ for random linear space L

Theorem [Breiding—F.—Santarsiero—Shehu '22]

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Theorem [Breiding—F.—Santarsiero—Shehu '22]

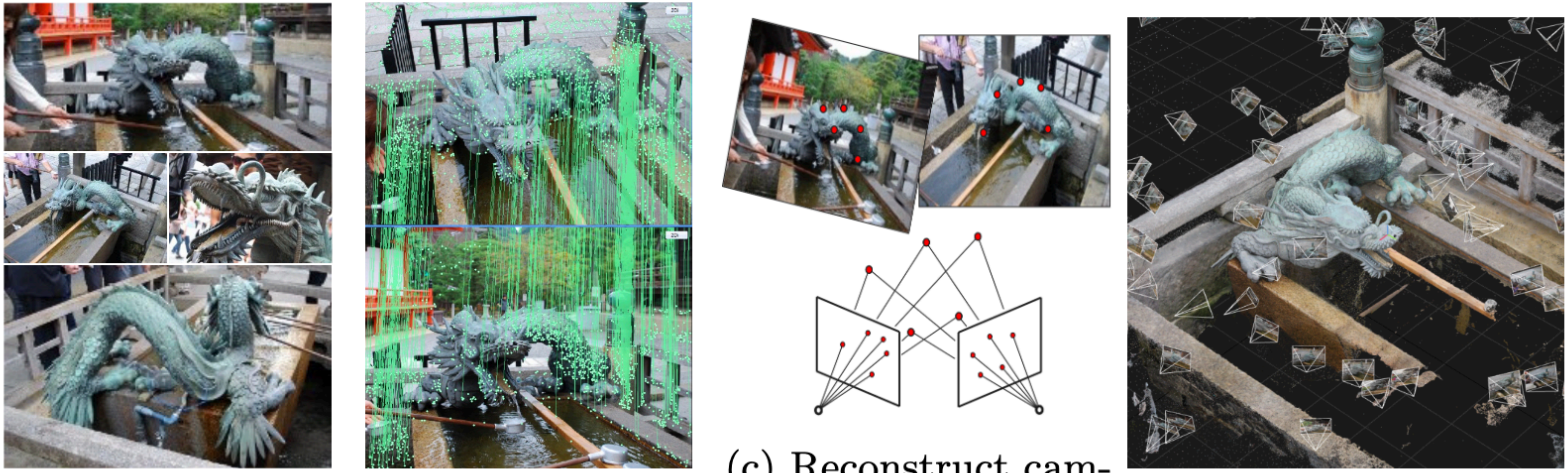
$$\mathbb{E}_{L \sim \psi} \#(\mathscr{E} \cap L) = 30\pi^2 \mathrm{vol}(K)$$

K = essential zonoid

THE PLAN

- What is the Essential Variety?
- Using the Co-area formula to see $\mathbb{E}_{L \sim O(9)} \#(\mathscr{E} \cap L) = 4$
- Experiments and bounds for the essential zonoid
- Further directions

WHAT IS ALGEBRAIC VISION?



(a) Input images (b) Image matching (c) Reconstruct cameras and 3D points (d) Output

Figure 1: 3D reconstruction pipeline (courtesy of Tomas Pajdla).

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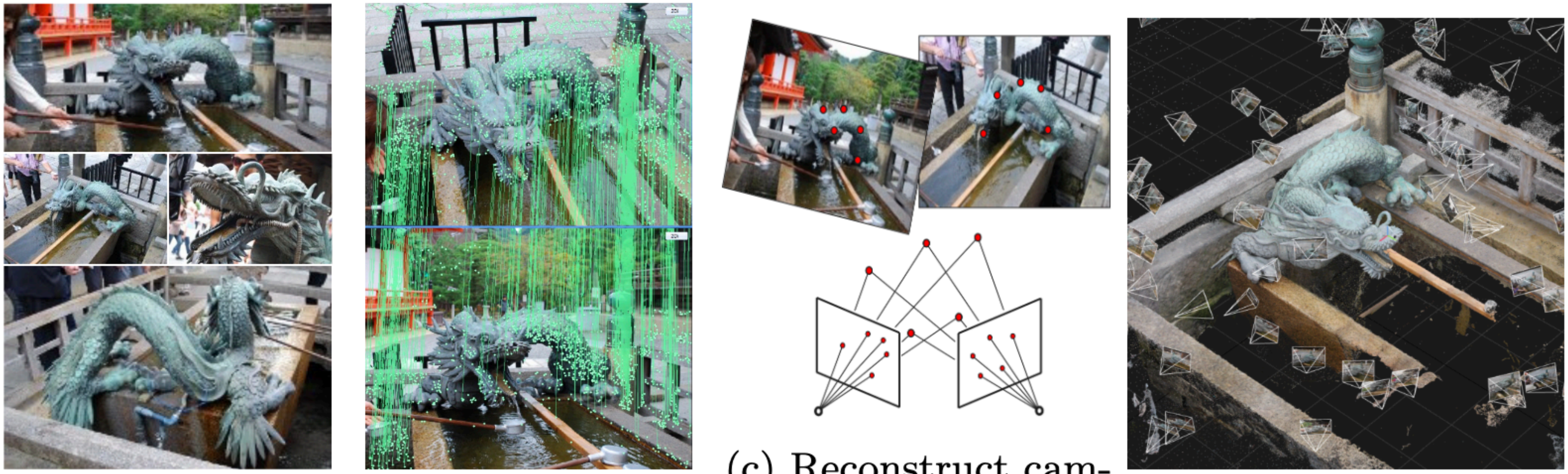

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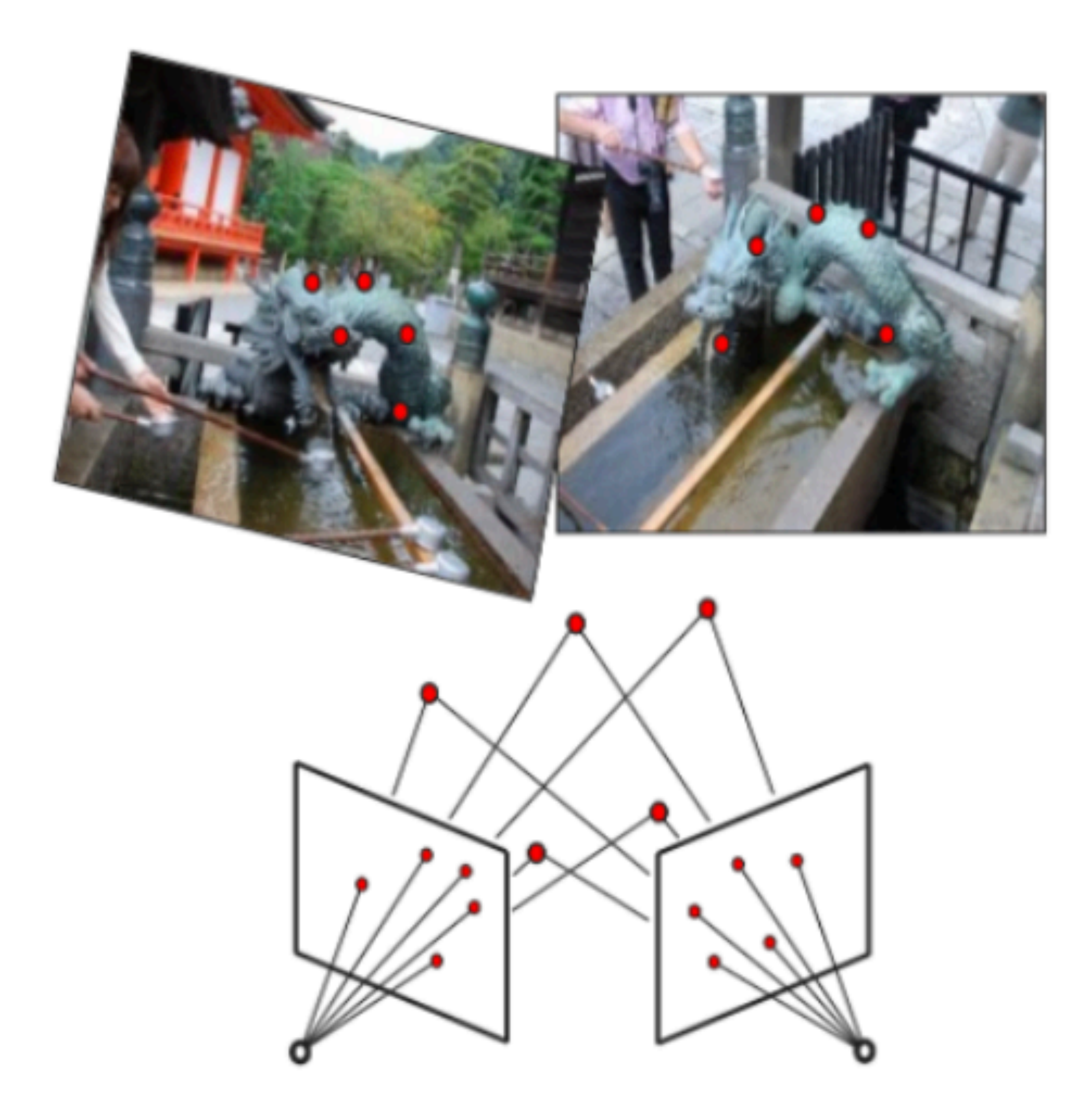
Photo credit and more information see Kileel and Kohn *Snapshot of Algebraic Vision*

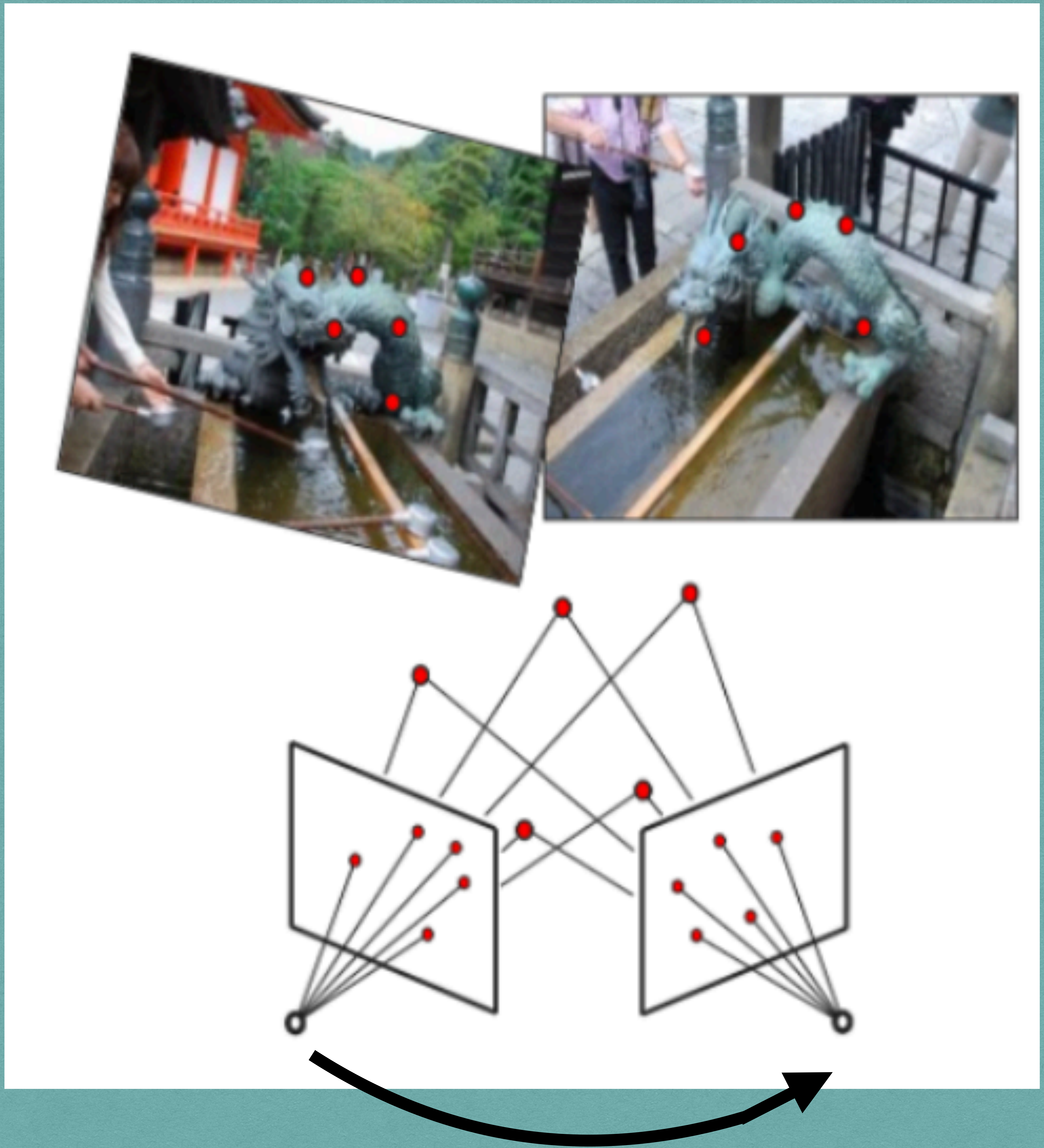
THE 5 POINT RELATIVE POSE PROBLEM

- Two pinhole cameras
- 5 point-correspondences



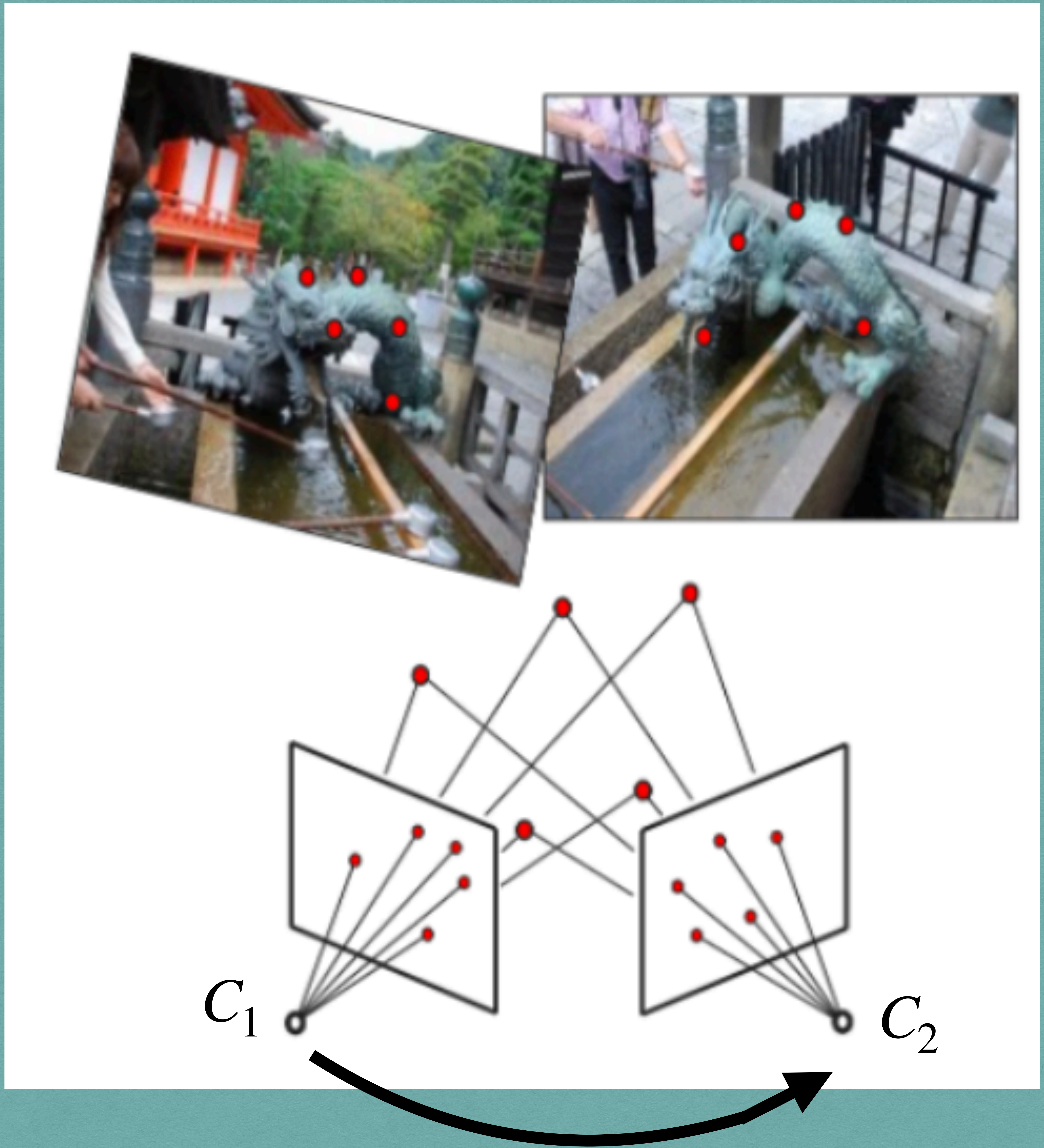
THE 5 POINT RELATIVE POSE PROBLEM

- Two pinhole cameras
- 5 point-correspondences
- **Goal**: reconstruct the relative position between the two cameras



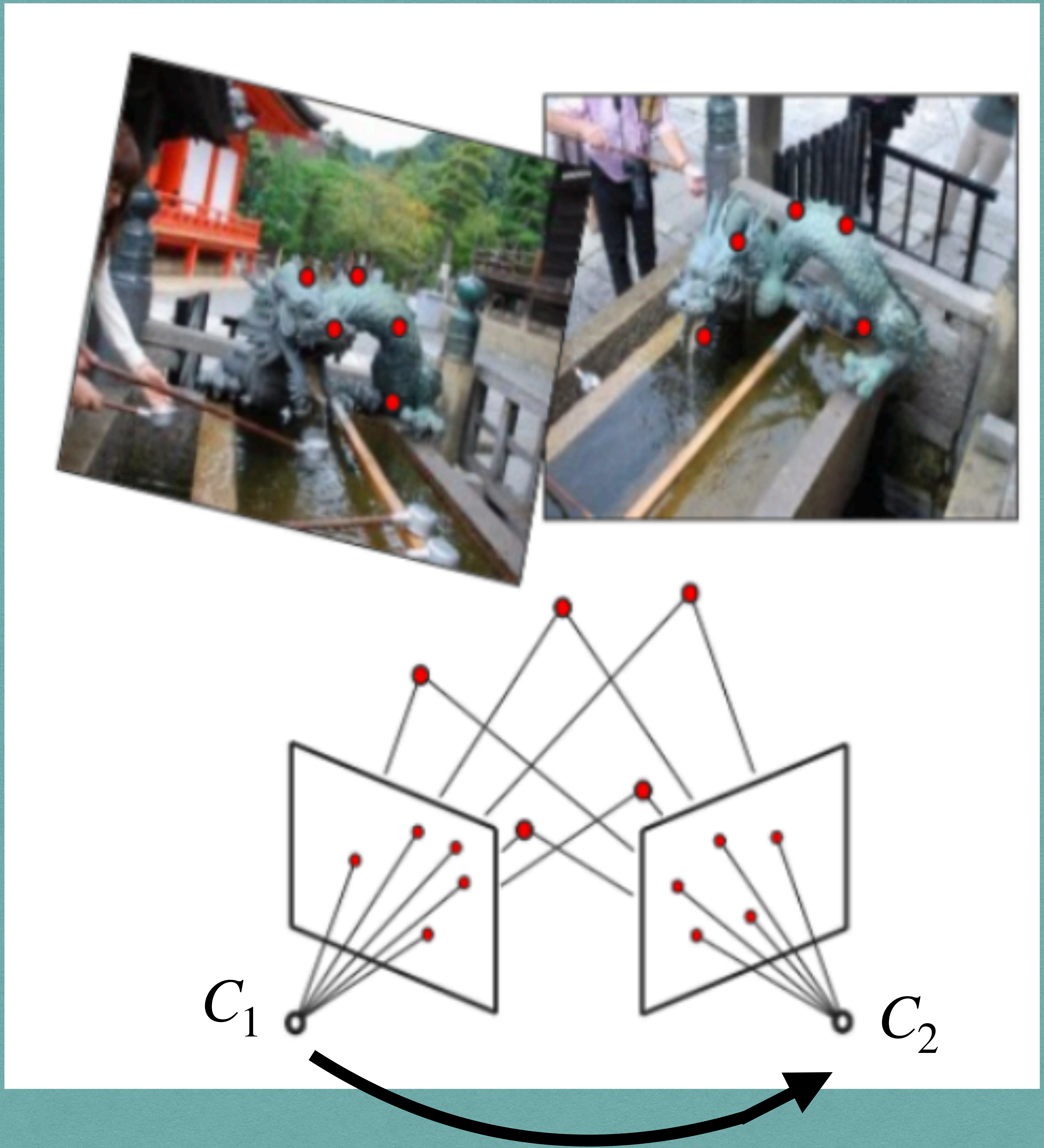
THE 5 POINT RELATIVE POSE PROBLEM

- Two pinhole cameras
  - $C_1, C_2 : \mathbb{P}^3 \to \mathbb{P}^2$
  - $C_j \in \mathbb{R}^{3\times 4}$, rank 3
  - Calibrated cameras:
    $C_j = [R, \mathbf{t}]$ where $R \in SO(3), \mathbf{t} \in \mathbb{R}^3$



THE 5 POINT RELATIVE POSE PROBLEM

- Two pinhole cameras
  - Since we are interested in the *relative* position: $C_1 = [I_3, \mathbf{0}] \quad C_2 = [R, \mathbf{t}]$



THE 5 POINT RELATIVE POSE PROBLEM

- $C_1 = [I_3, \mathbf{0}] \quad C_2 = [R, \mathbf{t}]$
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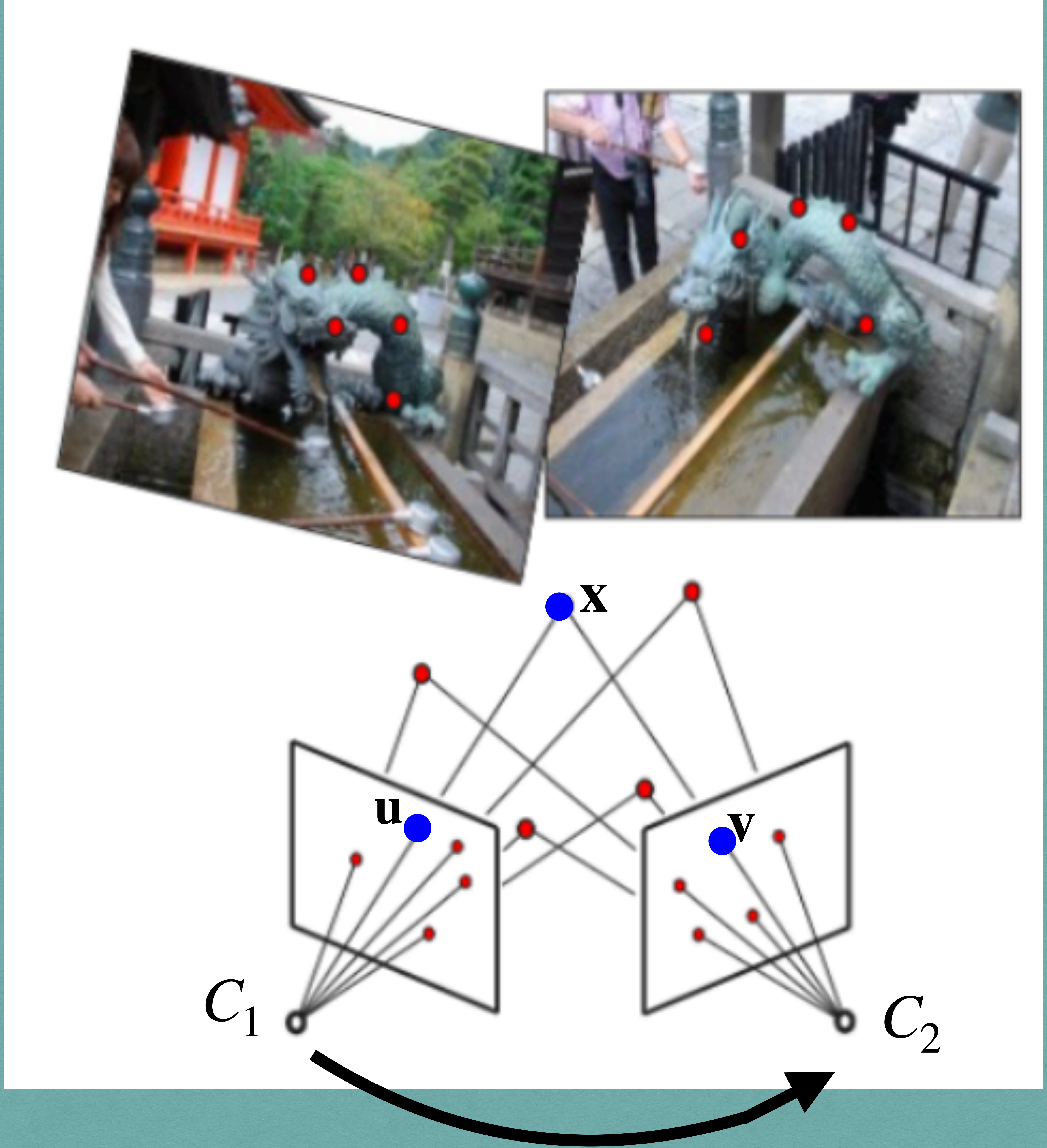


THE ESSENTIAL VARIETY

- For a point correspondence $C_1\mathbf{x} = \mathbf{u} \quad C_2\mathbf{x} = \mathbf{v}$
  - We can write $\mathbf{u}^T E\mathbf{v} = 0$
  - Where Essential matrices are of the form $E = [\mathbf{t}]_\times R \quad \mathbf{t} \in \mathbb{R}^3, R \in SO(3)$

THE ESSENTIAL VARIETY

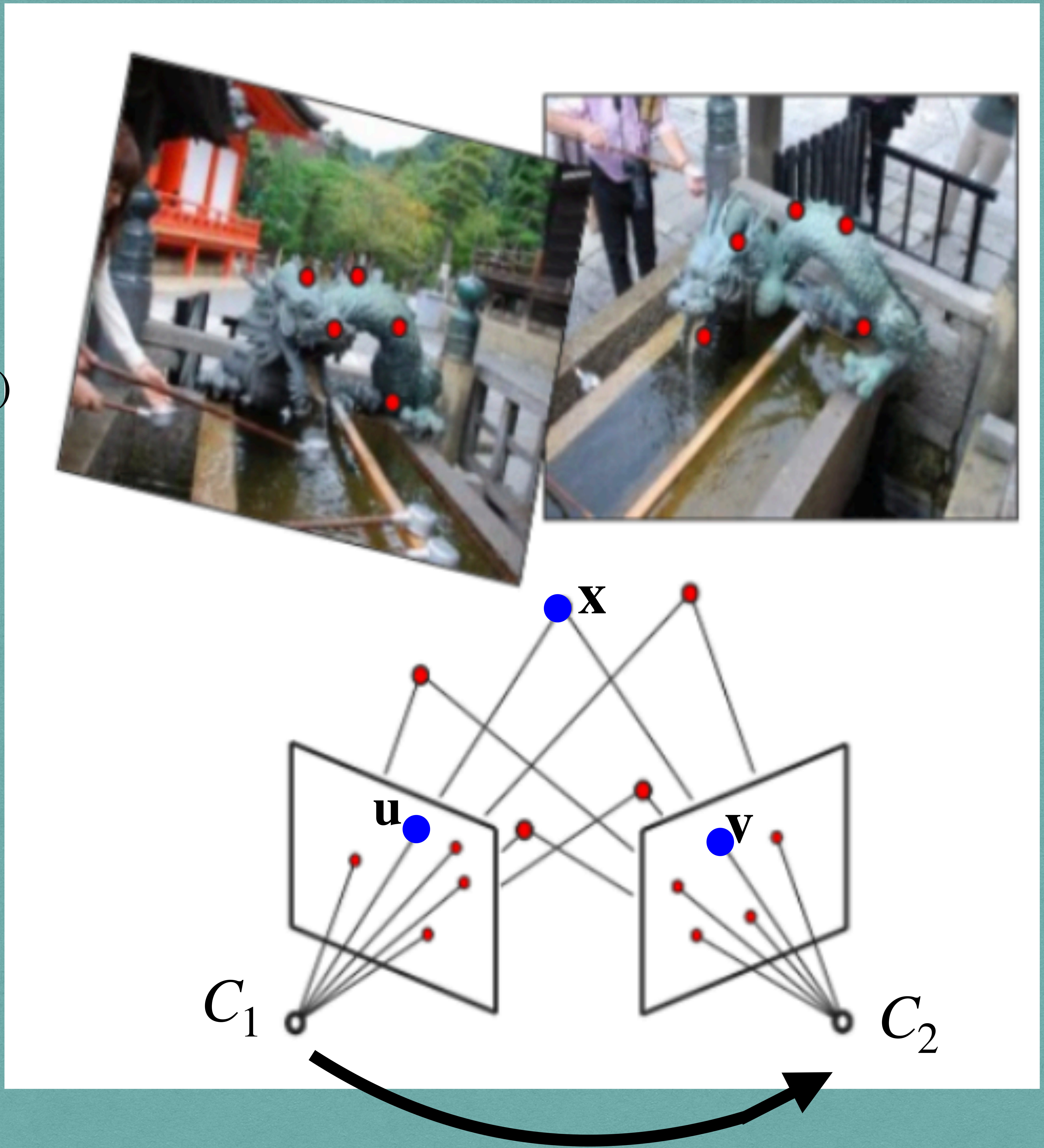
- $\mathscr{E} = \pi\left(\left\{E \in \mathbb{R}^{3\times 3} \,|\, E = [\mathbf{t}]_\times R \textit{ and } R \in SO(3) \textit{ and } \mathbf{t} \in \mathbb{R}^3\right\}\right) \subset \mathbb{P}^8$

THE ESSENTIAL VARIETY

- $\mathscr{E} = \left\{ E = [\mathbf{t}]_\times R \right\} \subset \mathbb{P}^8$
  - *[Demazure '88] Dimension 5, degree 10*
  - *Cut out by 10 cubic equations:*
    $\det(E) = 0, \quad 2EE^TE - \mathrm{tr}(EE^T)E = 0$

THE 5 POINT RELATIVE POSE PROBLEM

- $C_1 = [I_3, \mathbf{0}] \quad C_2 = [R, \mathbf{t}]$
- *5 point correspondences* $(\mathbf{u_j}, \mathbf{v_j})$
  - $L = \{E \in \mathbb{P}^8 \mid \mathbf{u_1}^T E \mathbf{v_1} = \cdots = \mathbf{u_5}^T E \mathbf{v_5} = 0\} \in G(3, \mathbb{P}^8)$
- *The number of real solutions is*
  - $\#(\mathscr{E} \cap L) \in \{0,2,4,6,8,10\}$



PROOF TECHNIQUES FOR O(9)

- $\mathscr{E}$ = Essential Variety
- $L_0 \in G(3,\mathbb{P}^8)$ and $L \sim O(9)$ means that $L = U \cdot L_0$ for U uniform in $O(9)$

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PROOF TECHNIQUES FOR O(9)

Proof (2) [Coarea formula]

suffices to show $\mathrm{vol}(\mathscr{E}) = 4\mathrm{vol}(\mathbb{P}^5)$

$\mathscr{E}$ = image$\{(R, \mathbf{t}) \mapsto E\}$

$$\mathrm{vol}(\mathscr{E}) = \int_{SO(3)\times\mathbb{S}^2} \sqrt{\det(JJ^T)}\, dR\, d\mathbf{t},$$ where J is Jacobian of $(R, \mathbf{t}) \mapsto E$

PROOF TECHNIQUES FOR O(9)

Proof (3) Key components

(1) **Need J independent of $R, \mathbf{t}$**

(2)
$$\text{vol}(\mathscr{E}) = \int_{SO(3)\times\mathbb{S}^2} \sqrt{\det(JJ^T)}\, dR\, d\mathbf{t} = \text{vol}(SO(3))\text{vol}(\mathbb{S}^2)\sqrt{\det JJ^T} = 32\pi^3\sqrt{\det JJ^T}$$

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(2) **Compute explicit basis elements $T_{I_3}SO(3) \times T_{\mathbf{e_1}}\mathbb{S}^2$**

(3)

$$\{(1_3, \mathbf{e}_2), (1_3, \mathbf{e}_3), (F_{1,2}, \mathbf{e}_1), (F_{1,3}, \mathbf{e}_1), (F_{2,3}, \mathbf{e}_1)\}$$

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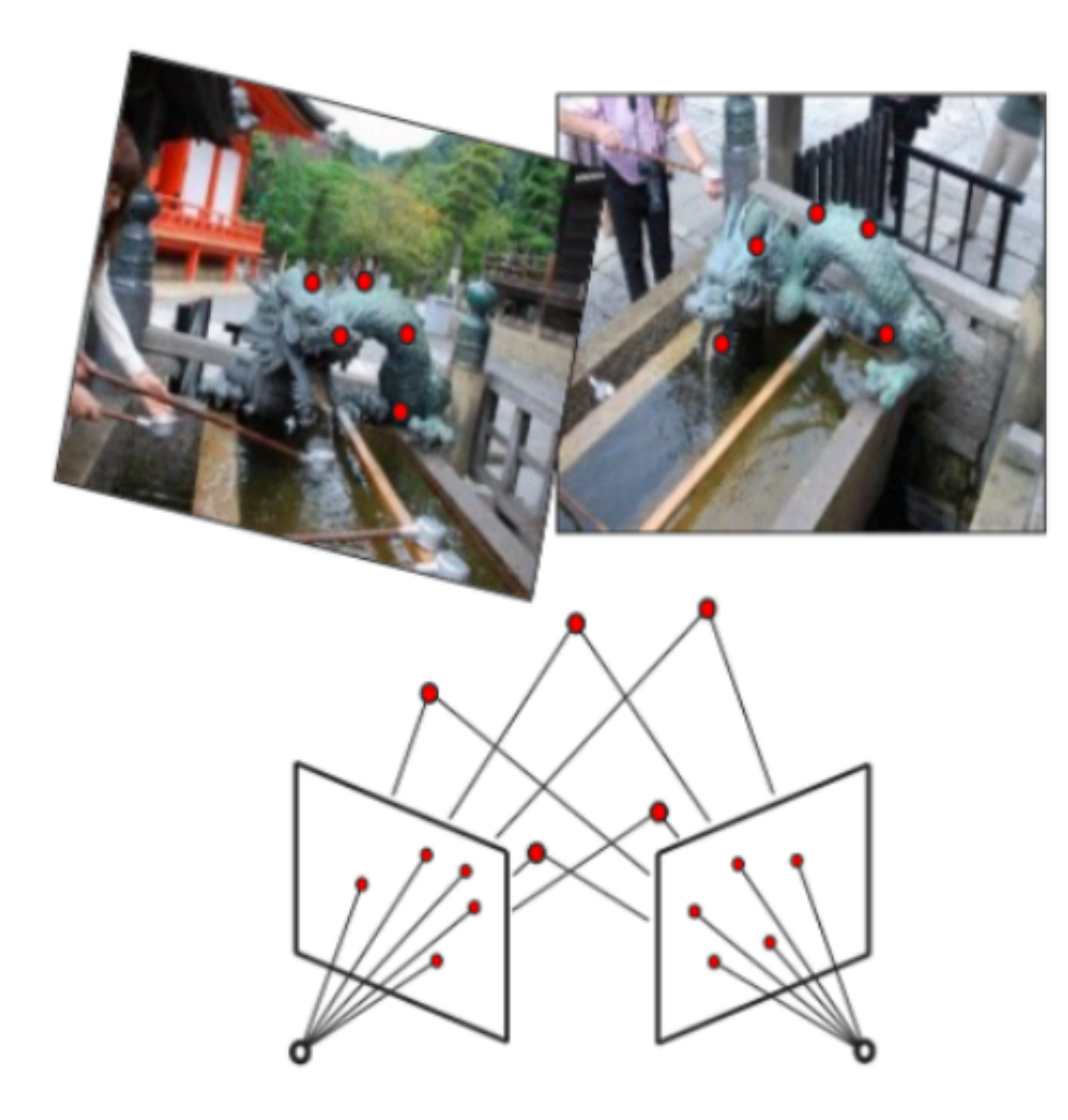
(3) Compute directly derivative with respect to this basis

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

WHAT HAPPENED TO THE 5 POINT PROBLEM?

- *5 point correspondences* $(\mathbf{u_j}, \mathbf{v_j})$
  - $L = \{E \in \mathbb{P}^8 \,|\, \mathbf{u_1}^T E \mathbf{v_1} = \cdots = \mathbf{u_5}^T E \mathbf{v_5} = 0\} \in G(3, \mathbb{P}^8)$
- *Now want to sample* $L \sim \psi$ *where* ψ *samples* $\mathbf{u_1}, \mathbf{v_1}, \ldots, \mathbf{u_5}, \mathbf{v_5}$ *uniformly i.i.d in* $\mathbb{P}^2$



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Proof key components

(1) Same use of Coarea formula, but now $J \in \mathbb{R}^{5\times 30}$, more complicated change of basis to get expected value of determinant of a random matrix

(2) Expected value of determinant of a random matrix is the volume of a convex body K called a *zonoid* with support function $h_K(x) = \frac{1}{2}\mathbb{E}\,|\,\langle x, \mathbf{z}\rangle\,|$

$$\mathbf{z} = \begin{bmatrix} b \cdot r \cdot \sin\theta \\ b \cdot r \cdot \cos\theta \\ a \cdot s \cdot \sin\theta \\ a \cdot s \cdot \cos\theta \\ rs \end{bmatrix}, \quad a, b, r, s \sim N(0,1), \quad \theta \sim \mathrm{Unif}([0, 2\pi)), \quad \text{all independent.}$$

EXPERIMENTS AND BOUNDS

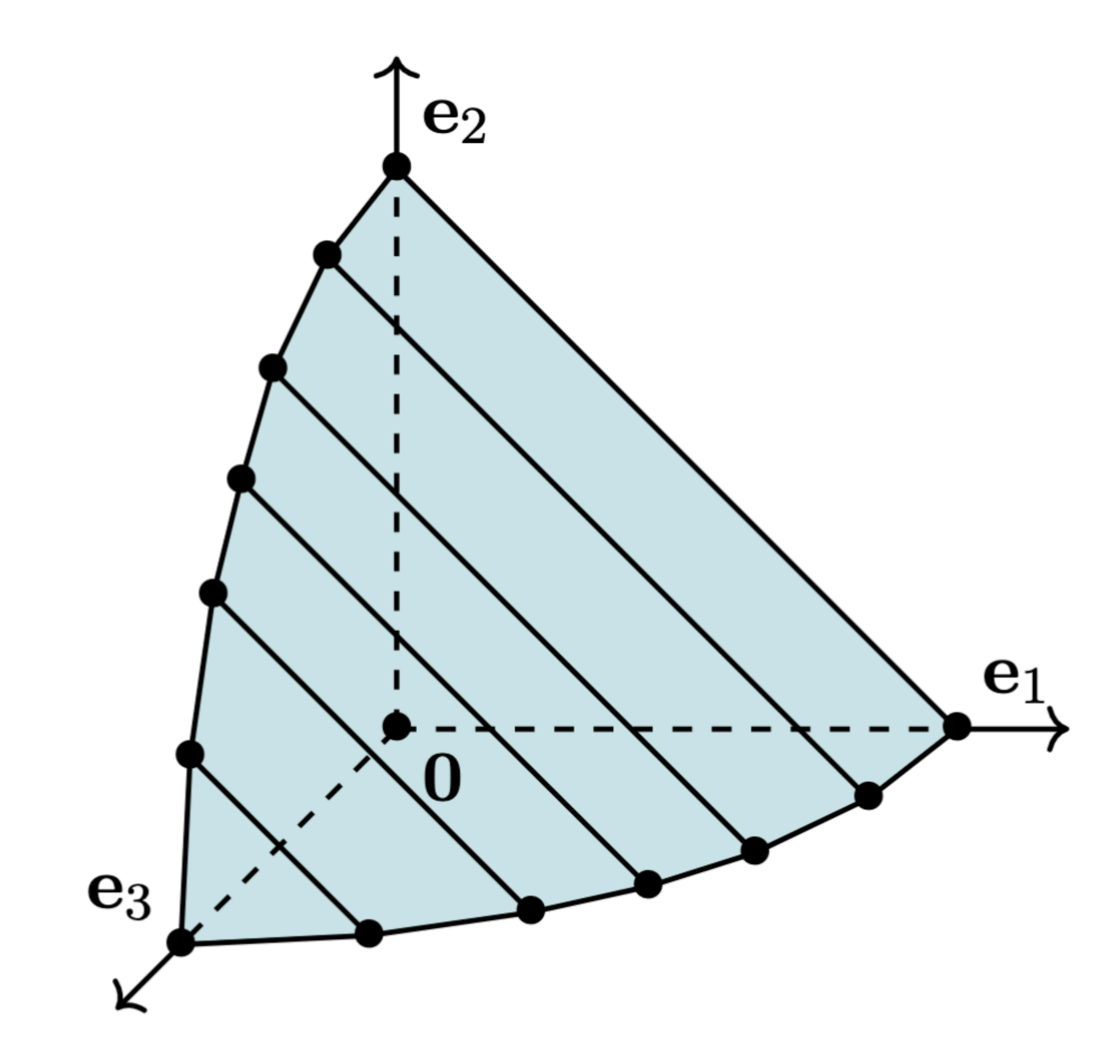
$$\mathbb{E}_{L\sim\psi}\#(\mathscr{E}\cap L) = 30\pi^2\mathrm{vol}(K)$$

K is the essential zonoid with support function

$$h_K(x) = \frac{1}{2}\mathbb{E}\,|\,\langle x, \mathbf{z}\rangle\,|$$

A lower bound: $\mathbb{E}_{L\sim\psi}\#(\mathscr{E}\cap L) \geq .93$

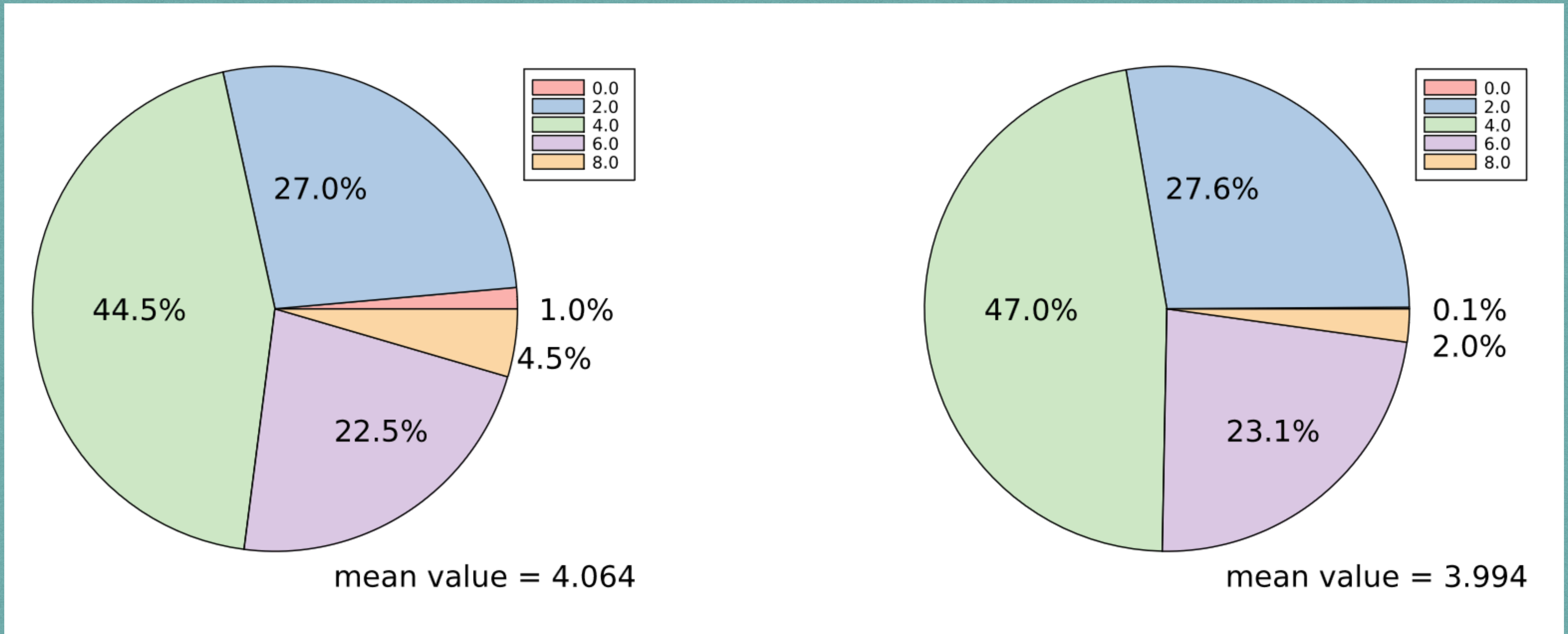
Computed polytope contained in essential zonoid, evaluated with Mathematica



EXPERIMENTS AND BOUNDS

With high probability

$$3.90 < \mathbb{E}_{L\sim\psi}\#(\mathscr{E}\cap L) < 4$$



$O(9)$ ψ

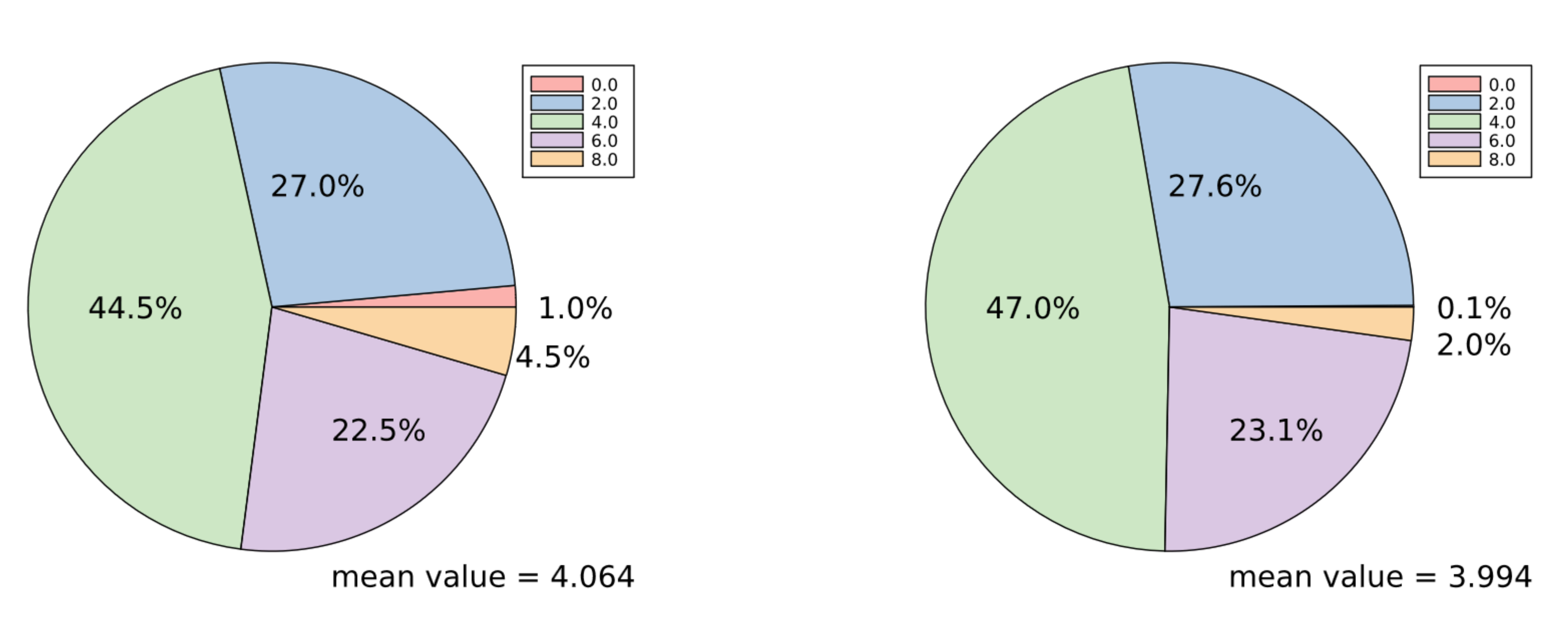
FURTHER DIRECTIONS

- Other Minimal problems
  - *Minimal problem* has the minimum amount of data so the solution is uniquely determined up to finitely many solutions.

to be reconstructed	minimal data	degree
essential matrix	5 point pairs	10
fundamental matrix	7 point pairs	3
relative pose of 2 calibrated cameras with unknown common focal length	6 point pairs	15
absolute pose of 1 calibrated camera (P3P, image registration)	3 world-image point pairs	4
planar homography	4 point pairs	1
trifocal tensor	9 line triples	36
calibrated trifocal tensor	3 point triples +1 line triple	216
relative pose of 2 projective cameras with unkown radial lens distortion	8 point pairs	16
world point under noise (triangulation, reprojection error)	known cameras with: • 1 point pair • 1 point triple	 • 6 • 47

FURTHER DIRECTIONS

- When are there zero solutions?
- Other probability distributions
- Exact values for essential zonoid



THANK YOU!