Entropy solutions to Macroscopic IPM

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IPM

THE FLOW OF
HOMOGENEOUS FLUIDS
THROUGH POROUS MEDIA

BY

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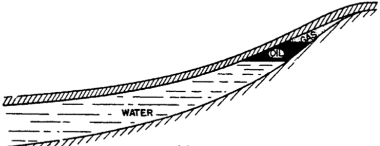


(c)

FIG. 7.—Types of traps that may form oil or gas reservoirs.

The incompressible porous media equation in short (IPM) investigates the movement of a fluid, through a porous medium

Incompressible porous media equation

permeability	viscosity	density	velocity	pressure	gravity
κ	μ	ρ	v	p	ge_2

Conservation of mass: $\partial_t \rho + \mathrm{div}(\rho v) = 0$

Incompressibility: $\mathrm{div}\, v = 0$

Darcy's law: $\frac{\mu}{\kappa} v + \nabla p = \rho g e_n$

(we will assume $\kappa = g = 1$ and $\mu = 1$ when is constant). That is an active scalar equation. Notice that

Velocity= Force !

The Muskat Problem

Issue: Evolution of a two phase fluid with different constant densities ρ_-, ρ_+ (equal viscosity μ for the time being) separated by a curve (an interfase) which we call z°.
As we deal with closed and open curves, it is convenient to fix an orientation for z°. For closed curves we fix the clockwise orientation ($\circlearrowright$) and for open curves the orientation from $x_1 = -\infty$ to $+\infty$.
Then, we denote Ω°_- (Ω°_+) by the domain to the left (right) side of z°.
Thus, the initial density will be written as

$$\rho^\circ(x) := \begin{cases} \rho_-, & x \in \Omega^\circ_-, \\ \rho_+, & x \in \Omega^\circ_+, \end{cases} \tag{1}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$.

The parabolic equation

The classical Muskat Problem leads to a non linear and non local equation. Namely by taking curl in Darcy law, using Bio-Savart in complex notation and the argument principle, one obtains for $x \neq z(t,\beta)$, that

Classical velocity

$$v(t,x) = -\frac{\rho_+ - \rho_-}{2\pi} \int \left(\frac{1}{x - z(t,\beta)} \right)_1 \partial_\alpha z(t,\beta)\, \mathrm{d}\beta, \tag{2}$$

Set

$$B(t,\alpha) := \frac{\rho_+ - \rho_-}{2\pi} \int \left(\frac{1}{z(t,\alpha) - z(t,\beta)} \right)_1 (\partial_\alpha z(t,\alpha) - \partial_\alpha z(t,\beta))\, \mathrm{d}\beta. \tag{3}$$

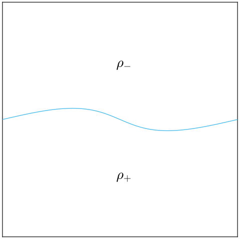
Classical Muskat Equation

$$(\partial_t z - B) = 0. \tag{4}$$

The parabolic analysis

Assuming $z(\alpha) = (\alpha, f(\alpha))$

$$\partial_t f = \frac{\rho_+ - \rho_-}{2\pi} \int \frac{\beta(\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha - \beta))}{\beta^2 + (f(\alpha) - f(\alpha - \beta))^2} \,\mathrm{d}\beta$$

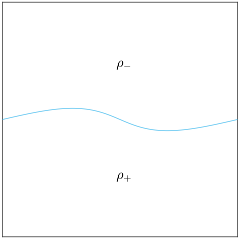


$t = 0$

The parabolic analysis

Assuming $z(\alpha) = (\alpha, f(\alpha))$

$$\partial_t f = \frac{\rho_+ - \rho_-}{2\pi} \int \frac{\beta(\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha - \beta))}{\beta^2 + (f(\alpha) - f(\alpha - \beta))^2} \, \mathrm{d}\beta$$
$$\sim (\rho_+ - \rho_-)(-\frac{1}{2})(-\Delta)^{1/2} f$$

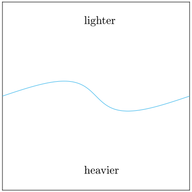


$t = 0$

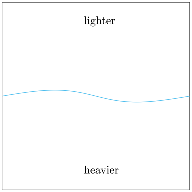
Fully stable regime $\rho^+ > \rho^-$

The interface evolution is well-posed in H^s for $s \geq 3/2$
Non-mixing solutions

Long, long history, Yi '03; Siegel, Caflisch, Howison '04; Ambrose '04; Córdoba, Córdoba, Gancedo '11; Cheng, Granero-Belinchón, Shkoller '16; Constantin, Córdoba, Gancedo, Rodríguez-Piazza, Strain '16; Deng, Lei, Lin '17; Matioc '19; Cameron '19; Córdoba, Lazar '20; Alazard, Lazar '20; Nguyen, Pausader '20; Alazard, Nguyen '21; E.Juarez



$t = 0$



$t = T_1$

Mixing Solutions

A weak solution is a **mixing solution** if, at each $0 < t \leq T$, the space $\mathbb{R}^2$ is split into three complementary open domains, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\rm mix}(t)$, satisfying that (ρ, v) is continuous on the non-mixing zones $\Omega_\pm$:

$$\rho = \pm 1 \quad \text{on} \quad \Omega_\pm, \tag{5}$$

while it behaves wildly inside the mixing zone $\Omega_{\rm mix}$:

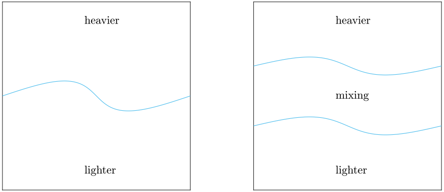
$$\int_\Omega (1-\rho^2)\,\mathrm{d}x = 0 < \int_\Omega (1-\rho)\,\mathrm{d}x \int_\Omega (1+\rho)\,\mathrm{d}x, \tag{6}$$

for every open $\emptyset \neq \Omega \subset \Omega_{\rm mix}(t)$.
Conversely, we say that (ρ, v) is a **non-mixing solution** if $\Omega_{\rm mix} = \emptyset$.

Fully unstable regime $\rho^+ < \rho^-$

Theorem (Castro, Cordoba, F, Inventiones 2022)

Let $f^\circ \in H^5$. There exist (locally in time) infinitely many mixing solutions to IPM starting from the Muskat data given by $(x, f(x))$.



$t = 0$ $\quad$ $t = T_1$

The Mixing zone as an envelop of a pseudointerfase

In all these problems the Mixing Zone is described as an envelop of a curve evolving in time, that we call the pseudointerfase with a certain speed of opening. That is we declare

At each time slice $0 < t \leq T \ll 1$, the mixing zone is the open set in $\mathbb{R}^2$ given by

$$\Omega_{\mathrm{mix}}(t) := \{z_\lambda(t,\alpha) \, : \, c(\alpha) > 0, \, \lambda \in (-1,1)\}, \qquad (7)$$

parametrized by the map

$$z_\lambda(t,\alpha) := z(t,\alpha) + \lambda t c(\alpha) \tau(\alpha)^\perp, \qquad (8)$$

where

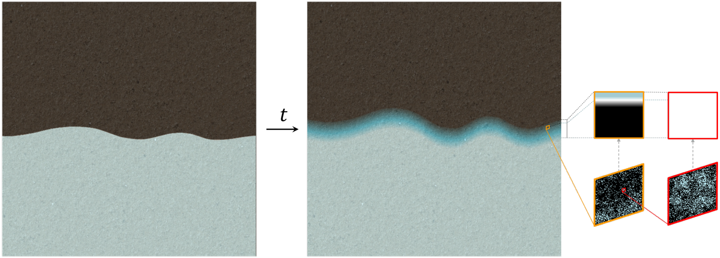
- $\tau^\perp$ is the direction of opening of the mixing zone.
- $c(\alpha)$ is the speed of opening.
- z is a curve evolving in time which a time 0 coincides with the initial interfase.

Macroscopical interpolation between the two fluids

Then, such degraded mixing solutions display a **perfect linearly degraded macroscopic behaviour on contour lines** $\mathrm{x}(\mathbb{R}, \lambda, t)$

$$\lim_{N \to \infty} \fint_{\mathrm{x}(R_N^\delta(\lambda), t)} \rho(x, t)\, \mathrm{d}x = \lambda \tag{9}$$

uniformly in $\lambda \in (-1, 1)$ and $t \in (0, T]$.



There are non linear quantities whose average behaviour can be predicted
$F(u) = v \cdot (v + \rho e_2) \qquad \approx \text{Powerbalance}$

Comments

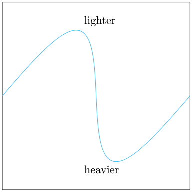
- Córdoba, F. Gancedo' 10. Lack of uniqueness for IPM.
- Székelyhidi '12: Flat interface.
- Other constructions: Förster, Székelyhidi, Noisette Székelyhidi, Arnaiz, Castro, F. '20: Semiclassical analysis.
  - Improvements on the regularity of initial data
  - different proofs.
  - Different Average (macroscopic, coarse grained) behaviour.
- The theorem open the way to model other inestabilities in fluid dynamics
  - Vortex Sheets, Mengual-Székelyhidi CPAM 2022
  - Kelvin Helmholtz (Euler with different densities, Bousinessq (Koluman, Gebhard, Hirsch Székelyhidi).

Partially unstable regime

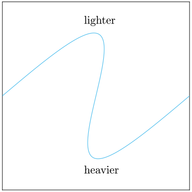
For a general interface z the stability of the Muskat equation depends upon the Rayleigh-Taylor condition

$$\sigma(\alpha) := (\rho_+ - \rho_-)\partial_\alpha z_1(\alpha) > 0$$

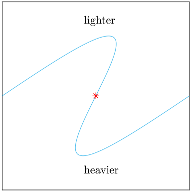
- Rayleigh-Taylor breakdown (Castro, Córdoba, Fefferman, Gancedo, López-Fernández, Ann. of Math. '12)
- Smoothness breakdown (Castro, Córdoba, Fefferman, Gancedo, Arch. Ration. Mech. Anal. '13)



$t = 0$



$t = T_1$



$t = T_2$

Can we continue the solution after T_2?

The answer: Yes, there are solutions which mixes the two fluids sthocastically in a neighborhood of the singularity zone in an (unpredictable manner).

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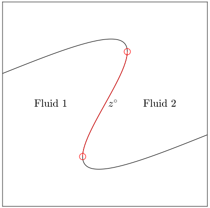
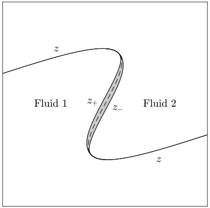
Theorem (Castro, F, Mengual Annals of PDE 2023)

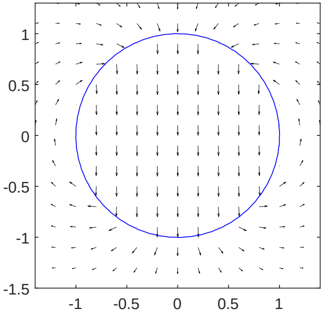
Let $z^\circ \in H^6$ be a chord-arc curve, a turned interface. There exist (locally in time) infinitely many mixing solutions to IPM starting from the Muskat data given by z°.



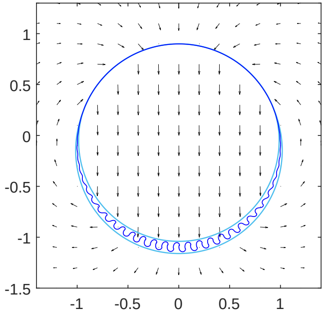


$t = T_2$ $\quad$ $t = T_3$

Numerical experiments



(a) A bubble type initial interface.



(b) The localized mixing zone.

Figure: (a) The initial interface $z^\circ(\alpha)$ separating two fluids with different constant densities. b). At some $t > 0$, the two boundaries of the non-mixing zones (light blue) Inside the mixing zone $\Omega_{\mathrm{mix}}(t)$ we plot the Rayleigh-Taylor curve $z_{\mathrm{per}}(t)$ (dark blue) which starts from a tiny perturbation of z° (via the vortex-blob method). In all the figures we have added the coarse-grained velocity field $\bar{v}(t,x)$ outside Ω_{mix}.

Main Issue in the field

Find a criteria which prescribes uniquely the macroscopical evolution

Lack of uniqueness. of solutions is accepted both in the physics and mathematics literature due to to the sthocasticity (Strong Butterfly effect) and it is consistent both with experiments and numerics.

However, it is to be expected that uniqueness is recovered at the level of subsolutions, as properties such as the size and shape of the mixing zone seems to be a robust observable in the experiments.

All previous results start with an ansatz for the Macroscopical solution, from which you deduce the existence of a microscopical one. Therefore as in the case of Burguers equation for example it is desired that there would be a math/physics criteria which will lead to uniqueness of solutions at the level of subsolutions.

Otto 1:Gradient flow

In the late 90, F.Otto addressed the issue of instabilities in IPM, pionering the use of varitional models in the Wasserstein distance to model parabolic equation. In the context of IPM, we retain two main observations.

IPM is a gradient flow respect to potential energy In Lagragian coordinates Potential energy

$$E[\Phi] = -\int \rho(x,0)^1 \Phi(x) \cdot e_2$$

on the manifold

$$M_0 = \{\Phi \text{ one-to-one and onto, smooth, volume preserving maps}\}.$$

$$\int \partial_t \Phi(\cdot,t) \cdot w = -dE[\Phi(\cdot,t)]w, \quad \forall w \in T_{\Phi(\cdot,t)} M_0, \tag{10}$$

[1] It is better to normalized ρ so that light fluid to be 0

Otto 2: Minimizing Movements schemes

Fast-forwarding a bit, Otto discretizes the problem, declares a relaxation in Lagrangian coordinates and pass back to Eulerian ones. At this point there exists a sequence of functions $\theta^{(k)}$ corresponding to $\theta(\cdot, t)$ at time $t = kh$, but of course potentially on a coarse grained or "locally averaged" level, which is characterized by the following (and first) JKO scheme: $\theta^{(0)} = s(\cdot, 0)$, and given $\theta^{(k)}$, $\theta^{(k+1)}$ is the minimizer in K of

$$\frac{1}{2}\text{dist}^2(\theta^{(k)}, \theta) + \frac{1}{2}\text{dist}^2(1 - \theta^{(k)}, 1 - \theta) - h \int \theta(x) x_2 \tag{11}$$

where the set K consists of measurable θ taking values in $[0, 1]$ and such that $\int \theta = \int \rho(x, 0)$, and $\text{dist}^2(\theta_0, \theta_1)$ is the L^2-Wasserstein distance

$$\text{dist}^2(\theta_0, \theta_1) = \inf_{\Phi \in I(\theta_0, \theta_1)} \int \theta_0(x) |\Phi(x) - x|^2 dx$$

with

$$I(\theta_0, \theta_1) = \{\Phi \, : \, \int \theta_1(y) \zeta(y) dy = \int \theta_0(x) \zeta(\Phi(x)) dx \quad \forall \zeta \in \mathcal{C}_0^0\}.$$

Notice that this indeed is a relaxation of the original problem since the densities are no longer taking values in $\{0, 1\}$ and the transport maps are not necessarily injective. Issue: Convergence as h goes to 0?

Macroscopic IPM(Otto relaxed IPM equation)

Otto conjectured θ_k converges to the entropy solution to the following equation,

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) + \partial_{x_2}(\rho^2) &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2, \end{aligned} \tag{12}$$

Question: are there entropy solutions to this equation with Muskat type initial data?

Theorem (Castro-Gebhard-F September 2023)

-Among all macroscopic solutions, the solution to Macroscopic IPM maximizes potential energy dissipation.
-For a Muskat data (example of SBV data with analytic interphase), there exists entropy solutions to Macroscopic IPM

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Macroscopic behaviour and weak limits (relaxation)

- H principle in PDE, built heavily on Tartar Compensated Compactness theory, among other things a description of Mac behaviour.
- The relaxation is a functional analytic concept which aims to describe all weak limits of the equation, e can interpret weak limits as averages i.e macroscopic quantities. $z_j \rightharpoonup z$. That is, if E is a set of positive measure $|E| > 0$

  $$\int_E z_j dxdt \to \int_E zdxdt$$

  Note: As oppose to coarse grained equations taking by convoluting with ψ_l an scaled bump function , weak limits do not choose a preferable scale, and those perhaps this makes them particularly suitable for turbulent regimes.

Differential inclussions (Tartar framework)

Tartar Compensated Compactness philosophy:
Thas is to say: Rewrite nonlinear P.D.E as a linear system (conservation laws) and a differential inclusion.
We are given a domain $\mathcal{D} \subset \mathbb{R}^d$, a linear differential operator $\mathcal{L}$ and a close set $\mathcal{K} \subset \mathbb{R}^N$.

$$\mathcal{L}(z) = 0, z(y) \in \mathcal{K}$$

Additionally, one can prescribe, boundary conditions, initial conditions if e.g $D = \mathbb{T} \times [0, T]$ or specify the required regularity of the solution.

Tools from Compensated Compactness

- The wave (Λ) cone: 1 dimensional solutions to $\mathcal{L}$ .e states $Z \in \mathcal{R}^d$ such that there exists 1-d solutions. That is for $h : \mathbb{R} \to \mathbb{R}$, exist (ξ, ξ_t) such that,
$$\mathcal{L}(h((x,t) \cdot \xi)Z) = 0$$
- Semiconvex hulls:The Λ (convex) hull. $\mathcal{K}^\Lambda$ and the lamination convex hull (just finite convex combinations along Λ directions.
- $\mathcal{F}$ is a compensated compactness quantity. If $\mathcal{F}(z_j) \rightharpoonup \mathcal{F}(z)$ whenever $z_j \rightharpoonup z$ and $\mathcal{L}(z_j) = 0$.
- $\mathcal{K}^{Cc} = \{(z, \mathcal{F}(z)) \in co(K, \mathcal{F}(K))\}$. The the compensated compactness hull.

Definitions

Definition of Relaxation Using the language of differential inclusions, , weak limits are characterized by finding a set $\mathcal{K} \subset \mathcal{K}^{\mathrm{relaxed}}$ such that weak limits satisfy

$$\mathcal{L}(\bar{U}) = 0, \bar{U} \in \mathcal{K}^{\mathrm{relaxed}}.$$

The problem becomes geometrical as one needs to find $\mathcal{K}^{\mathrm{relaxed}}$. It turns out that, (using the compensated compactness jargon)

$$\mathcal{K}^{\Lambda} \subset \mathcal{K}^{\mathrm{relaxed}} \subset \mathcal{K}^{Cc}$$

Definition of a subsolution: We say that Z is a subsolution if $\mathcal{L}(z) = 0$ and

$$Z(x) \in \mathcal{K}^{\mathrm{relaxed}}$$

Strict subsolutions are defined by the inclusion $Z \in \mathrm{int}(\mathcal{K}^{\mathrm{relaxed}})$.

The H-Principle

The Desired Meta-Theorem: If there is an strict subsolution, there are infinitely many solutions, which share the macroscopical behaviour of the solution.
Macroscopical behaviour $\approx$., initial condition and compensated compactness quantities.

Tartar Framework: The Linear conservation laws

We get rid of the nonlinearities augmenting variables. Rewrite IPM

$$\begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2. \end{aligned} \tag{13}$$

$$m = \rho v, |\rho| = 1$$

A mixing subsolution: The differential inclusions

At each $0 < t \leq T$, the space $\mathbb{R}^2$ is split into three complementary open domains, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\mathrm{mix}}(t)$ satisfying that

$$\bar{\rho} = \pm 1, \quad \bar{m} = \bar{\rho}\bar{v} \quad \text{on} \quad \Omega_\pm, \tag{14a}$$

$$|2(\bar{m} - \bar{\rho}\bar{v}) + (1 - \bar{\rho}^2)i| < (1 - \bar{\rho}^2) \quad \text{on} \quad \Omega_{\mathrm{mix}}. \tag{14b}$$

In addition, it is required that

$$\sup_{0 \leq t \leq T} \|\bar{v}(t)\|_{L^\infty} < \infty. \tag{15}$$

Observe that, outside of the mixing zone a subsolution is a solution.

The relaxation of *IPM*, equal densities (Székelyhidi 12)

The relaxation is actually the 2-Λ hull of the set $K = \{m = \rho u, |\rho| = 1\}$.

$$|2(\bar{m} - \bar{\rho}\bar{v}) + (1 - \bar{\rho}^2)i| < (1 - \bar{\rho}^2)$$

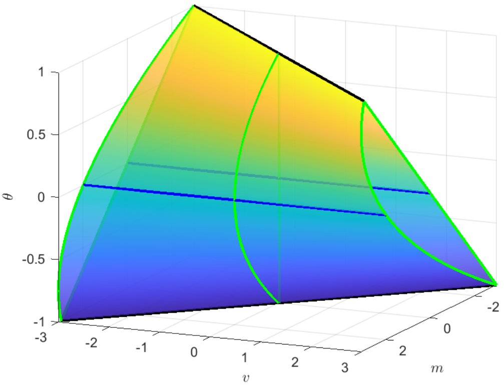


Figure: Intersection of the relaxation with a $3D$ space

Different Viscosities. Mengual 2021

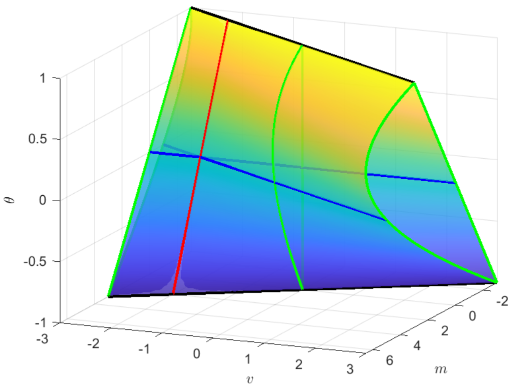


Figure: Intersection of the relaxation with a $3D$ space

Appears a pinch singularity, reflecting the existence of velocities for which no Kelvin-Helmoltz instabilities ara availalbe The proof is very technical and it needs the use of Möbious transformations.

An H-principle for IPM

Theorem (Córdoba-F.Gancedo10,Szekelehidi12,Castro-F-Mengual21)

If there is an strict subsolution, there are infinitely many solutions (comeager in a Baire category sense) solutions with the same initial data and the same average behaviour for all non linear quantities (density, velocity, power balance).

- The first H-principle is Nash isometric emmbedding theorem. If I can find a short map from (M, g) to R^3, then there is a isometry.
- There are constructive proofs, which aim of optimising the regularity (Onsaguer conjecture, non uniqueness for Navier Stokes)
- An elegant use of the fact that the Identity between weak and strong normed space is a Baire-one map yields a comeaguer set of solutions (in a ad-hoc created metric space).

Ansatz subsolutions

The current methods to find subsolutions consist in

- firstly making an ansatz for $\bar{\rho}, \bar{m}$ and the structure of the mixing zone)
- then $\partial_{x_1}\rho$,and hiddenly the mixing zone, determines the velocity v through Bio-Savart. Imposing the continuity equation and being clever in the choice of the mixing zone (i.e chosing the curve z, the speed and direction of opening), such strategy leaves you inside $\mathcal{K}^{\mathrm{relaxed}}$ (at least for a short time).

This typically yields an evolution equation for z.

Depending on our choice of ρ we need to deal with various operators. The building blocks are the interaction operators, which are analogous to B but consider how the various boundaries interact with each others.

Interaction Operators

$$B_{\lambda,\lambda'}(t,\alpha) := \frac{\rho_+ - \rho_-}{4\pi} \int \left(\frac{1}{z_\lambda(t,\alpha) - z_{\lambda'}(t,\beta)} \right)_1 (\partial_\alpha z_\lambda(t,\alpha) - \partial_\alpha z_{\lambda'}(t,\beta)) \, \mathrm{d}\beta. \tag{16}$$

and we replace ± 1 by $\pm$.

Discrete Average Operators

$$Av(B) := \frac{1}{2} \sum_{a=\pm} B_a, \qquad B_a := \sum_{b=\pm} B_{a,b}, \tag{17}$$

Continuous Average Operators For a continuous $\rho = \lambda$ indeed the the relevant operators is a continuous curve

$$\tilde{Av}(B) = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \int B_{\lambda,\lambda'} d\lambda d\lambda'$$

The density is a linear interpolation

In the case of $\rho = \lambda$, it is further assume that γ is scalar, that $z = (1, f)$ is a graph and that we open the mixing zone in the vertical directions in all times. It quickly follows from the conservation of mass, that in order to have continuity at $|\rho| = 1$

$$f_t = \tilde{Av}(B)(f)$$

Once the equation is solved (and for an appropriate ansatz) for m we have a subsolution $\bar{\rho}, \bar{m}, \bar{v}$.

$$f_t = \tilde{Av}(B)(f)$$

which is non linear and nonlocal and degenerates as times goes to 0 After linearization (5 derivatives) one is lead to

$$f_t = \frac{1}{t}\mathrm{Op}(p(x, t|\xi|))f$$

where

$$p(x, \xi) \approx \frac{|\xi|}{1 + c(x)|\xi|}$$

is a non-smooth semiclassical symbol with time playing the role of the Planck-constant. whose theory (Commutators, composition, Gårding inequaltiy)had to be developed Castro-Cordoba-F, Arnaiz-Castro-F.

The toy model

Let us consider the following toy model

$$f_t = \left(\frac{1}{1 + ct|\xi|} \right)^{\check{}} * \Lambda f \quad \text{in } \mathbb{R} \times \mathbb{R}^+ \tag{18}$$

$$f(x,0) = f^0(x),$$

In the Fourier side this equation reads

$$\hat{f}_t(\xi) = \frac{|\xi|}{1 + ct|\xi|} \hat{f}(\xi),$$

which can be solved explicitly. Indeed, the solutions are given by

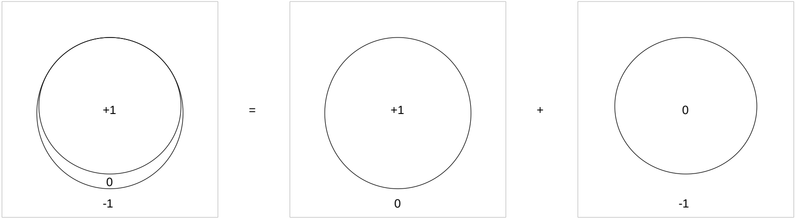
$$\hat{f}(\xi) = (1 + ct|\xi|)^{\frac{1}{c}} \hat{f}_0(\xi). \tag{19}$$

Thus, in this approach the speed of opening, dictates how much regularity is loss respect to initial data. Even the toy model degenerates as c goes to 0.

Unstable situation: The simplest ansatz

The Förster-Székelyhidi ansatz.

$$\bar{\rho}(t,x) := \chi_{\Omega_+(t)}(x) - \chi_{\Omega_-(t)}(x), \tag{20}$$



As before $v = BS(\rho_{x_1}) = Av(B) = \frac{1}{2}(B_+ + B_-)$

A glimplse on the partially unstable case

What to do in the partially unstable case?
The game is to solve the parabolic equation (The Muskat equation) in the stable situation and find a mixing solution in the stable situation.
In all the previous approaches the estimates degenerate when $c = 0$ so the approach seems hopeless.
Idea: treat the interaction between separate boundaries as a perturbation.
Instead of $Av(B)$ deal only with $E = B_{++} + B_{--}$ The corresponding equation can be can be solved energy estimates from the parabolic analysis of the classical Muskat problem.
It remains to show that $B_{+}-$ and the like are a perturbation for which we find some unexpected cancellations.

Dissipating Potential energy leads to Otto equation

Let $(\bar{\rho}, \bar{m}, \bar{v})$ be a relaxed solution. Define its associated relative potential energy

$$E_{rel}(t) := \int_{\mathbb{T}\times\mathbb{R}} (\rho(t,x) - \rho_0(x)) x_2 \, dx \tag{21}$$

Formally (and rigorously wiht more work)

$$\partial_t E_{rel}(t) = -\int_{\mathbb{T}\times\mathbb{R}} x_2 \mathrm{div}\, m(t,x) \, dx = \int_{\mathbb{T}\times\mathbb{R}} m_2(t,x) \, dx. \tag{22}$$

Recall that lying in the hull, is equivalent to:

$$m = \rho v - \frac{1-\rho^2}{2} e_2 + \frac{1-\rho^2}{2} \xi$$

almost everywhere for some $\xi : [0,T) \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}^2$ satisfying $|\xi| < 1$. Plugging this into (22) one deduces

$$\partial_t E_{rel}(t) = \int_{\mathbb{T}\times\mathbb{R}} \rho v_2 - (1-\rho^2)\frac{1-\xi_2}{2} \, dx.$$

which is more negative when $\xi = -e_2$.

Dissipating potential energy selects Otto equation

we deduce that (non-strict) subsolutions that maximize at each time instant the dissipation of potential energy are characterized as solutions of

$$\begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2. \end{aligned} \tag{23}$$

$$\begin{aligned} \partial_t \rho + \operatorname{div} \left(\rho v - (1-\rho^2) e_2\right) &= 0, \\ \operatorname{div} v &= 0, \\ v &= -\nabla p - \rho e_2. \end{aligned} \tag{24}$$

which is nothing but Macroscopic IPM!!
Now the equation can not be solved by using ideas from conservation laws due to the nonlinear relation between v and ρ, thus a new view point is needed.

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Theorem

Let $\gamma_0 : \mathbb{T} \to \mathbb{R}$ be real analytic. Then I.B.V has a solution emanating fom Muskat data with

(i) *at positive times $\rho(t,\cdot)$ is Lipschitz continuous, $v(t,\cdot)$ is log-Lischitz continuous with*

$$\|\nabla\rho(t,\cdot)\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq C_0 t^{-1}, \tag{25}$$

$$|v(t,x) - v(t,x')| \leq C_0 t^{-1} \left|(x-x')\log|x-x'|\right| \tag{26}$$

(ii) *The analytic level sets are given by $\Gamma_t(h) := \{x \in \mathbb{T}\times\mathbb{R} : \rho(t,x) = h\}$, $h \in (-1,1)$ are given by graphs of real analytic functions $\gamma_t(\cdot,h) : \mathbb{T} \to \mathbb{R}$ which moreover*

$$\gamma_t(x_1,h) = \gamma_0(x_1) + t(2h + S_0(\gamma_0')(x_1)) + o(t) \tag{27}$$

(iii)

$$\partial_t(\eta(\rho)) + div\big(\eta(\rho)v + Q(\rho)e_2\big) = 0, \tag{28}$$

for any Lipschitz continuous $\eta : \mathbb{R} \to \mathbb{R}$ there holds the balance with initial data $\eta(\rho)(0,\cdot) = \eta(\rho_0)$ and flux $Q(\rho) := \int_0^\rho 2\eta'(s)s\, ds$.

Step 1: Reparametrization to capture the Burguer´s effect

We look for diffeomorphisms of the type $X_t : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $t \in (0, T)$,

$$X_t(y) = \begin{pmatrix} y_1 \\ ty_2 + f(t, y) \end{pmatrix}$$

where $f(t, y)$ is to be defined.
We now seek to find a solution to MacroIPM with Muskat data on $[0, T)$ having the property that

$$\rho(t, X_t(y)) = \phi_0(y_2) = \begin{cases} +1, & y_2 \geq 2, \\ \frac{1}{2} y_2, & y_2 \in (-2, +2), \\ -1, & y_2 \leq -2. \end{cases} \tag{29}$$

After, recovering the velocity from Darcy´s law Macroscopic IPM translates into an evolution for f.

$$\partial_t f(t,y) = v(t, y_1, ty_2 + f(t,y)) \cdot \begin{pmatrix} -\partial_{y_1} f(t,y) \\ 1 \end{pmatrix} \tag{30}$$

for $(t, y_1, y_2) \in (0, T) \times \mathbb{T} \times (-2, 2)$.
or

$$\partial_t f(t,y) = -\frac{1}{2} \int_{-2}^{2} \int_{\mathbb{T}} K_2(\tilde{\Delta} X_t(y,z))(\partial_{y_1} f(t,y) - \partial_{y_1} f(t,z)) \, dz_1 \, dz_2 \tag{31}$$

The level sets are driven by the normal component of the velocity. We loose y_1 derivative at all times. A priori the whole thing could degenerate at 0.

Removing first order terms

Frontal difficulty

$$\det DX = t + \partial_{y_2} f$$

which might not be positive! this formally prevente X might not be a diffeo, and $\det(X)^{-1}$ appears in the estimates. Solution: We give an ansatz for f up to first order

Declare $f(y,t) = \gamma_0(y_1) + s_0(y_1)t + t^{1+\alpha}\eta$,

$$s_0 = \frac{1}{2}\int_{-2}^{2}\int_{\mathbb{T}} K_2(\Delta X_0(y_1,z_1))\Delta\gamma_0'(y_1,z_1)\, dz_1\, dz_2, \tag{32}$$

The normal velocity at the initial data

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The normal velocity at the initial data

Equation for η,

$$\eta(t,y) = -\frac{1}{t^{1+\alpha}} \int_0^t \int_{-2}^2 \int_{\mathbb{T}} K_2(\Delta X_s(y,z))\Delta\partial_{y_1} f_s(y,z) \\ - K_2(\Delta X_0(y_1,z_1))\Delta\gamma_0'(y_1,z_1)\, dz_1\, dz_2\, ds. \tag{33}$$

$$\eta_0(y) = 0, \quad \eta_t(y) = \frac{1}{t^{1+\alpha}} \int_0^t F_s(\eta_s)(y)\, ds. \tag{34}$$

The equation looses at least one derivative in y_1 as time goes to zero and no derivatives in y_2. A priori it could further degenerates when t goes to 0. However there are some cancellations, e.g the controlled behaviour of $\det X$. In the analytic regime existing can be deduced from the following extension of Nirenberg abstract theorem of Cauchy-Kowaleskaya.

Cauchy-Kovaleskaya(non linear in t)

Theorem (A variant of Nishida)

Let $(B_\rho)_{\rho\in(0,\rho_0)}$, $\rho_0 > 0$ be a scale of Banach spaces with $\|\cdot\|_{\rho'} \leq \|\cdot\|_\rho$ for $0 < \rho' < \rho < \rho_0$ and consider the integral equation

$$u(t) = \frac{1}{a(t)} \int_0^t F(u(s), s)\, ds \tag{35}$$

for a given continuous function $a : [0, \infty) \to \mathbb{R}$ with $a(t) > 0$ for $t > 0$. If F is such that

- (i) *there exists $R > 0$, $T > 0$ such that for every $0 < \rho' < \rho < \rho_0$ the map*

$$\{u \in B_\rho : \|u\|_\rho < R\} \times [0, T) \to B_{\rho'}, \quad (u, t) \mapsto F(u, t)$$

$$\|F(u, t) - F(v, t)\|_{\rho'} \leq \frac{b(t)}{\rho - \rho'} \|u - v\|_\rho ,$$

i.e F behaves as the derivative of an analytic function)More properties.

Then, there is a solution for short time.

$$\int_{\Omega_0} \frac{1}{|\Delta X_t^{\eta}(y,z)|_*}\, dz \le C_0 \, |\log t| \,. \tag{36}$$

$$t\, |y_2 - z_2| \le C_0 \, |\Delta X_t^{\eta}(y,z)|_* \,. \tag{37}$$

$$\left|\partial_{a_2}^{j} K_2(a)\right| \le C\, |a|_*^{-(1+j)} \,. \tag{38}$$

$$\begin{aligned}
&\int_{\Omega_0} |K_2(\Delta X_t^{\eta}) - K_2(\Delta X_t^{\zeta})|\, |\Delta \partial_{y_1} f_t^{\eta}| \; dz \\
&\quad = \int_{\Omega_0} \left| \int_0^1 \partial_{a_2} K_2(\Delta X_t^{\xi_\lambda}) \frac{1}{2} t^{1+\alpha} (\Delta\eta - \Delta\zeta)\, d\lambda \right| |\Delta \partial_{y_1} f_t^{\eta}| \; dz \\
&\quad \le C_0 t^{1+\alpha} \, \|\eta - \zeta\|_{\rho'} \int_{\Omega_0} \int_0^1 \frac{1}{|\Delta X_t^{\xi_\lambda}|_*^2} \frac{|\Delta X_t^{\xi_\lambda}|_*}{\rho - \rho'}\, d\lambda \, dz \le \frac{C_0 t^{1+\alpha} \, |\log t|}{\rho - \rho'} \, \|\eta - \zeta
\end{aligned}$$

Questions

Once we have a solution to Macroscopic IPM, we can run convex integration to have infinitely many solutions. Thus potential energy dissipation seems to solve the selection problem (at least for analytic data in the initial interfase) We also predict the macroscopic behaviour which can be tested experimentally.

1-Can we find a such a well-defined evolution in the partially unstable case?

2-Our mechanism yields a unique evolution, but is there uniqueness of entropy solutions in this setting?

3-We have a solution of the llmit equation but do the JKO scheme really converges?

Gracias !

$$\int_{\Omega_0} \frac{1}{\left|\Delta X_t^{\eta}(y,z)\right|_*}\, dz \le C_0 \left|\log t\right| . \tag{39}$$

$$t\left|y_2 - z_2\right| \le C_0 \left|\Delta X_t^{\eta}(y,z)\right|_* . \tag{40}$$

$$\left|\partial_{a_2}^{j} K_2(a)\right| \le C\left|a\right|_*^{-(1+j)} . \tag{41}$$

$$\begin{aligned} &\int_{\Omega_0} \left|K_2(\Delta X_t^{\eta}) - K_2(\Delta X_t^{\zeta})\right| \left|\Delta \partial_{y_1} f_t^{\eta}\right|\, dz \\ &\quad = \int_{\Omega_0} \left|\int_0^1 \partial_{a_2} K_2(\Delta X_t^{\xi_\lambda}) \frac{1}{2} t^{1+\alpha} (\Delta\eta - \Delta\zeta)\, d\lambda\right| \left|\Delta \partial_{y_1} f_t^{\eta}\right|\, dz \\ &\quad \le C_0 t^{1+\alpha} \left\|\eta - \zeta\right\|_{\rho'} \int_{\Omega_0}\int_0^1 \frac{1}{\left|\Delta X_t^{\xi_\lambda}\right|_*^2} \frac{\left|\Delta X_t^{\xi_\lambda}\right|_*}{\rho - \rho'}\, d\lambda\, dz \le \frac{t^{1+\alpha}\left|\log t\right|}{\rho - \rho'} \left\|\eta - \zeta\right\|_{\rho} \end{aligned}$$