Decorated discrete conformal maps and convex polyhedral cusps

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A Little Bit of History

the initial spark:

- Koebe–Andreev–Thurston *circle-packing theorem* (early 1980s)

circle patterns/packings:

- Rodin–Sullivan's *approx. Riemann map* (1987)
- Bower–Stephenson's *inversive distance circle packings* (2004)

hyperbolic polyhedra:

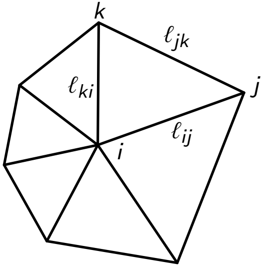
- polyhedral realization: Rivin, Schlenker, Fillastre (1990s to 2000s)
- complete solution: Fillastre (2008)

discrete conformal equivalence: ...

Discrete Conformal Equivalence (DCE)

Given: triangulated marked surface $(\mathcal{S}_g, V, \mathcal{T})$:

- genus g surface,
- finite $V \subset \mathcal{S}_g$,
- surface triangulation $\mathcal{T} = (V, E, F)$.



Definition

A *discrete metric* on $(\mathcal{S}_g, V, \mathcal{T})$ is a function

$$\ell : E \to \mathbb{R}_{>0}, \quad ij \mapsto \ell_{ij}$$

satisfying all triangle inequalities:

$$\forall\, ijk \in F: \qquad \begin{aligned} \ell_{ij} &< \ell_{jk} + \ell_{ki} \\ \ell_{jk} &< \ell_{ki} + \ell_{ij} \\ \ell_{ki} &< \ell_{ij} + \ell_{jk} \end{aligned}$$

Discrete Conformal Equivalence (DCE)

Definition ([Luo '04])

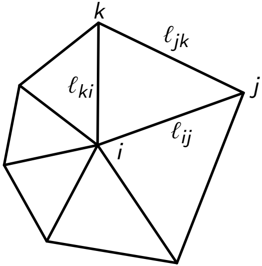
Two PE-metrics ℓ, $\tilde{\ell}$ are *discrete conformally equivalent* if there is $u\colon V \to \mathbb{R}$ with

$$\tilde{\ell}_{ij} = e^{(u_i+u_j)/2}\,\ell_{ij}.$$

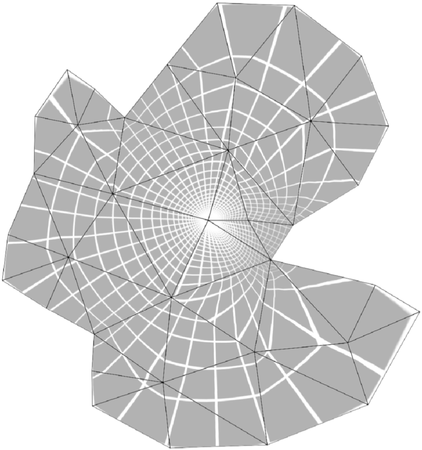
- *Mapping Problem*: find $u\colon V \to \mathbb{R}$ for desired angle sums Θ_i
- *discrete Ricci-flow*:

$$\frac{d}{dt}u_i(t) = -(\Theta_i - \theta_i(t))$$

- Variational principle, uniqueness results, computations
  CG: [Springborn–Pinkall–Schröder '08]
  DG: [Bobenko–Pinkall–Springborn '15]



Discrete Conformal Equivalence (DCE): Example



[Bobenko–Pinkall–Springborn '15]

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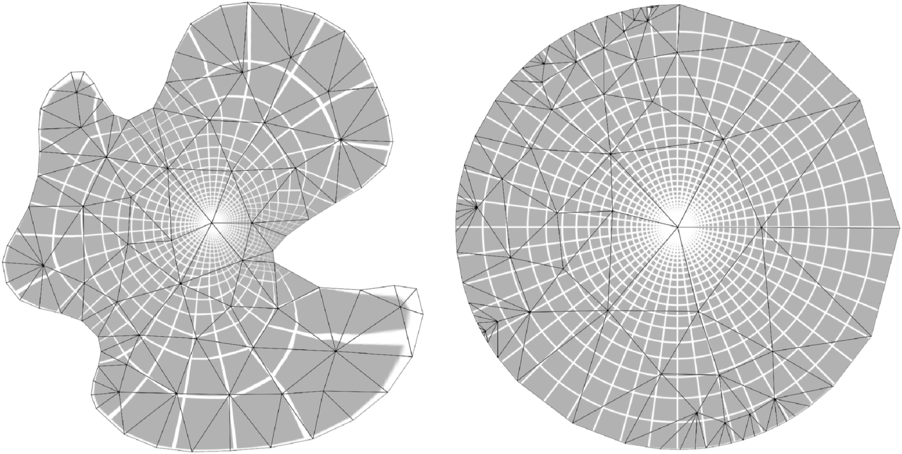

[Bobenko–Pinkall–Springborn '15]

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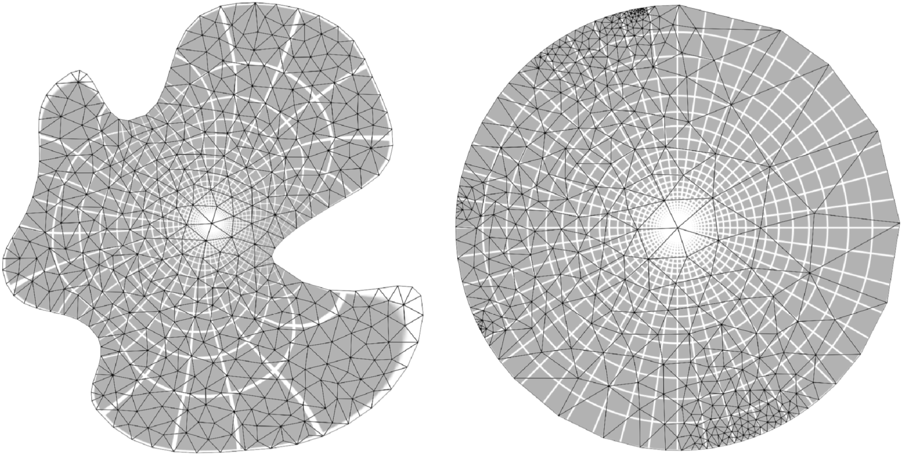


[Bobenko–Pinkall–Springborn '15]

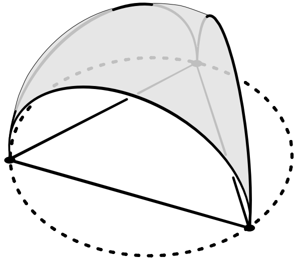
Origin of Rich Theory: Hyperbolic Geometric Interpretation

Observations:

- euclidean triangles $\rightarrow$ ideal hyperbolic triangles
- turns $\mathcal{S}_g$ into hyperbolic cusp surface Σ_g

Theorem ([Bobenko–Pinkall–Springborn '15])

$(\mathcal{T}, \ell)$ and $(\mathcal{T}, \tilde{\ell})$ DCE $\Leftrightarrow$ same Σ_g.



Idea for Existence:

- $(\mathcal{T}, \ell)$ induces piecewise euclidean (PE) metric $\mathrm{dist}_{\mathcal{S}_g}$ on $\mathcal{S}_g$
- consider *Delaunay triangulations* of $(\mathcal{S}_g, \mathrm{dist}_{\mathcal{S}_g})$
- DCE with *different triangulations* $\Leftrightarrow$ same hyperbolic metric.

Discrete Uniformization

Theorem ([Gu–Luo–Sun–Wu '18])

For any PE-metric $\mathrm{dist}_{\mathcal{S}_g}$ *on a closed marked genus* g *surface* $(\mathcal{S}_g, V)$ *and for any* Θ_i *satisfying the Gauß-Bonnet condition*

$$\frac{1}{2\pi}\sum \Theta_i = 2g - 2 + |V|$$

there exists a DCE metric with the desired angle sums Θ_i*. It is unique up to scale.*

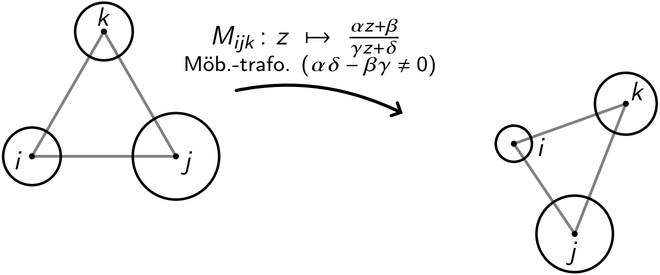
- **DG** [Gu–Luo–Sun–Wu '18] sequence of Delaunay triangulations
- **CG** [Gillespie–Springborn–Crane '21] efficient numerical realization

What to Learn for Generalizations?

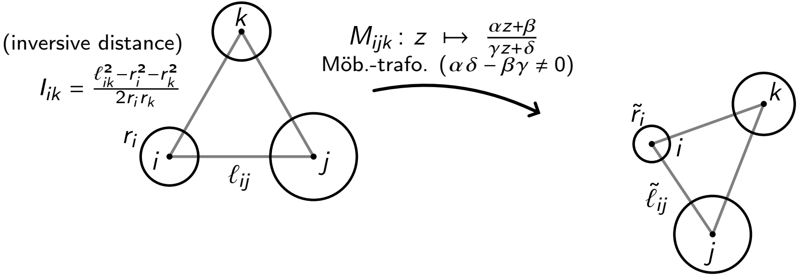
Two geometric ideas:

- Use intrinsic triangulations determined by the metric of the PE-surface.
- Use hyperbolic geometry to determine invariants.

Discrete Conformal Equivalence (DCE) with Decorations



Discrete Conformal Equivalence (DCE) with Decorations



Proposition [Bobenko–L. '23]

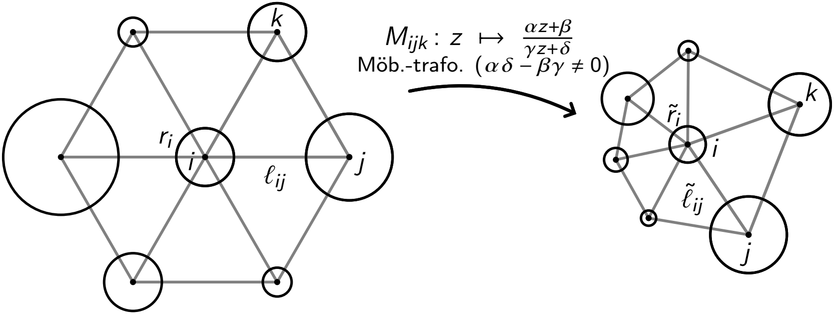
Given two decorated triangles. The following statements are equivalent:

- they are Möbius equivalent,
- the inversive distances of their edges coincide,
- there are $u_i, u_j, u_k \in \mathbb{R}$ such that

$$\tilde{r}_i = e^{u_i}\, r_i$$

$$\tilde{\ell}_{ij}^2 = (e^{2u_i} - e^{(u_i+u_j)})\, r_i^2 + e^{(u_i+u_j)}\, \ell_{ij}^2 + (e^{2u_j} - e^{(u_i+u_j)})\, r_j^2.$$

Discrete Conformal Equivalence (DCE) with Decorations



Definition

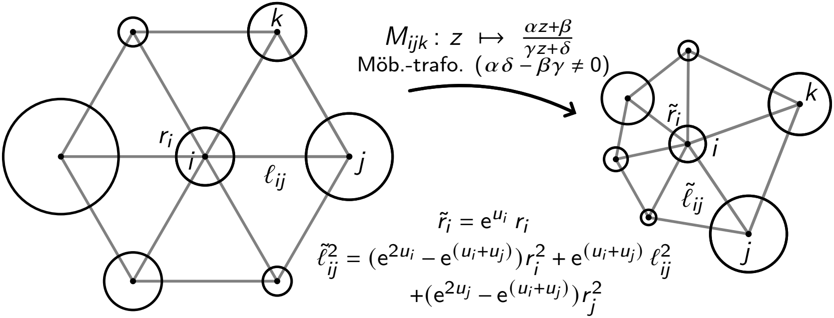
Two decorated PE-metrics (ℓ, r) and $(\tilde{\ell}, \tilde{r})$ are *(decorated) DCE* if one of the following is true:

- corresponding decorated triangles $ijk \in F$ are Möbius equivalent,
- there is $u\colon V \to \mathbb{R}$ such that for all edges $ij \in E$

$$\tilde{r}_i = \mathrm{e}^{u_i}\, r_i$$

$$\tilde{\ell}_{ij}^2 = (\mathrm{e}^{2u_i} - \mathrm{e}^{(u_i+u_j)})r_i^2 + \mathrm{e}^{(u_i+u_j)}\,\ell_{ij}^2 + (\mathrm{e}^{2u_j} - \mathrm{e}^{(u_i+u_j)})r_j^2 .$$

Relationship to “Classical” DCE



$$\tilde{r}_i = \mathrm{e}^{u_i}\, r_i$$

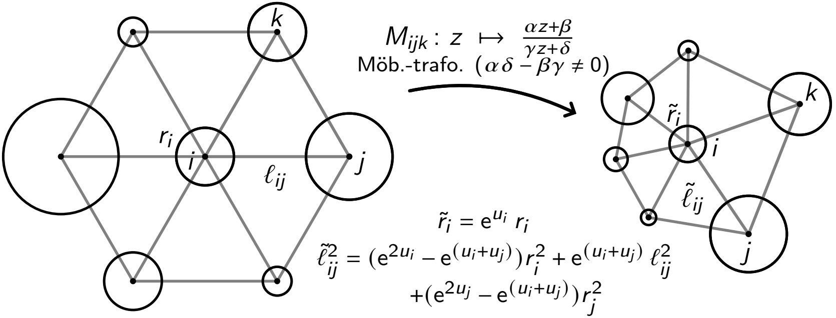
$$\tilde{\ell}_{ij}^2 = (\mathrm{e}^{2u_i} - \mathrm{e}^{(u_i+u_j)})r_i^2 + \mathrm{e}^{(u_i+u_j)}\,\ell_{ij}^2 + (\mathrm{e}^{2u_j} - \mathrm{e}^{(u_i+u_j)})r_j^2$$

Problem: any two undecorated triangles are Möbius equivalent (no inversive distance for $r_i = 0$!)

Idea: consider *infinitesimal circles* $r_i \longrightarrow 0 \rightsquigarrow \tilde{\ell}_{ij} = \mathrm{e}^{(u_i+u_j)/2}\,\ell_{ij}$

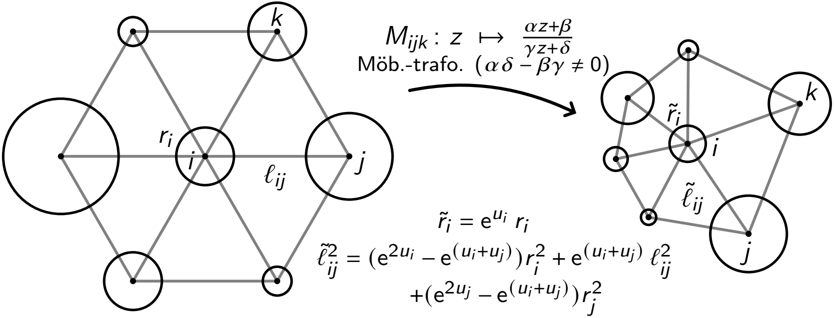
$\Longrightarrow$ *New definition* for $r_i = 0$ via $|M'_{ijk}(p_i)| = |M'_{ijl}(p_i)| = e^{u_i}$.

Previous Work



- if all $r_i > 0$: *inversive distance circle packing* [Bowers–Stephenson '04; Bowers–Hurdal '03]
- *discrete conformal structures via duality structures* [Glickenstein '11]
- *Unified Ricci Flow* [Zhang et al. '14]

Decorated DCE Mapping Problem



DCE Mapping Problem

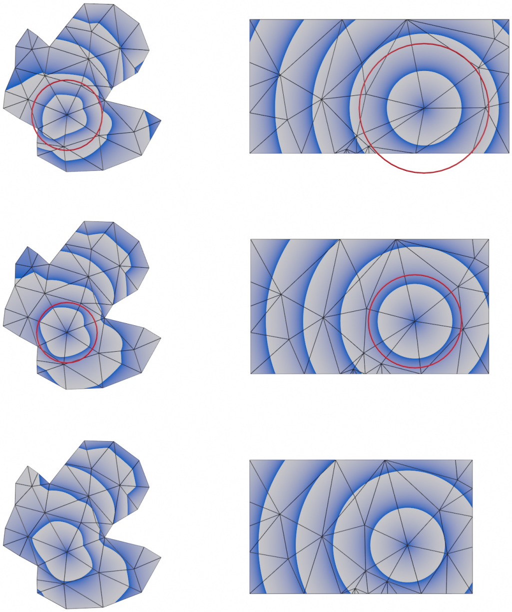
Given

- a triangulation $\mathcal{T}$ of the surface $\mathcal{S}_g$,
- a decorated PE-metric (ℓ, r),
- and a desired angel sum Θ_i for each vertex $i \in V$.

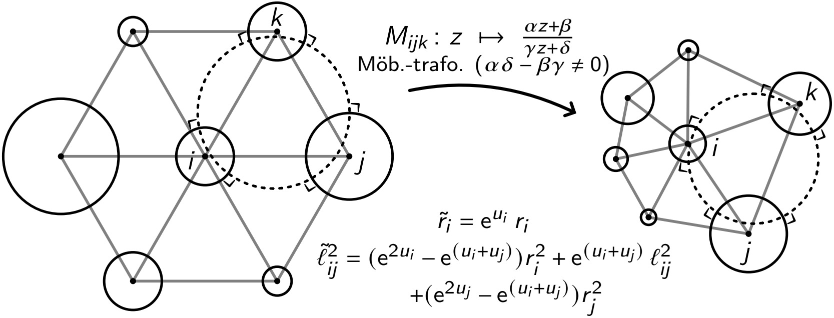
Find

u_i such that the DCE-changed metric *w.r.t.* u_i has angle sum Θ_i about each vertex $i \in V$.

Decorated DCE Mapping Problem: Example



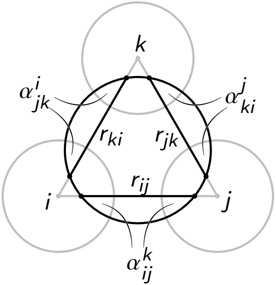
Decorated DCE Mapping Problem

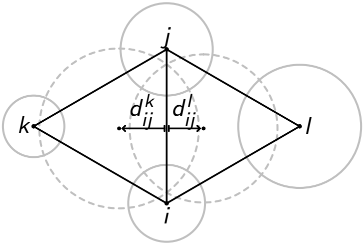


Varying Combinatorics:

- Consider PE-metrics $(\mathcal{T}, \ell) \leftrightarrow \mathrm{dist}_{\mathcal{S}_g}$
- if circles non-intersecting ("hyperideal"), there exist weighted Delaunay triangulations (wDt), "empty disk property" [Bobenko–Springborn '07; L. '23]
- sequences of wDts (similar to [Gu–Luo–Sun–Wu '18; Springborn '19])

Properties of the Curvature Flow





$$d^k_{ij} = r_{ij} \cot \alpha^k_{ij}$$

The quadratic form associated to the Jacobian $\left(\frac{\partial \theta_i}{\partial u_j}\right)_{i,j \in V}$ is the *discrete Dirichlet energy:*

$$\sum d\theta_i du_i = -\sum_{ij} w_{ij} \left(du_j - du_i\right)^2$$

with *decorated cotan-weights* $w_{ij} = {(d^k_{ij} + d^l_{ij})}/{\ell_{ij}} = (\cot \alpha^k_{ij} + \cot \alpha^l_{ij}) \frac{r_{ij}}{\ell_{ij}}$.

Our Main Result (Bobenko–L. 2023)

Theorem

Given a hyperideally decorated PE-metric $(\mathrm{dist}_{\mathcal{S}_g}, r)$ *on the closed marked genus* g *surface* $(\mathcal{S}_g, V)$. *Then*

- *(existence) a decorated PE-metric DCE to* $(\mathrm{dist}_{\mathcal{S}_g}, r)$ *realizing* $\Theta \in \mathbb{R}^V_{>0}$ *exists iff* Θ *satisfies the Gauß–Bonnet condition*
$$\frac{1}{2\pi} \sum \Theta_i \;=\; 2g - 2 + |V|.$$
- *(uniqueness) there exists at most one decorated PE-metric DCE to* $(\mathrm{dist}_{\mathcal{S}_g}, r)$ *realizing* $\Theta \in \mathbb{R}^V_{>0}$*, up to scale.*
- *(variational principle)* $u \in \mathbb{R}^V$ *giving the change of metric is a maximum point of the discrete Hilbert–Einstein functional* $\mathcal{H}_{\Sigma_g,\Theta}$.

Remark: In particular we can realize the uniform angle distribution
$$\Theta_i \;\equiv\; \frac{2\pi\,(2g-2+|V|)}{|V|}.$$

Hyperideal Polyhedral Cusps

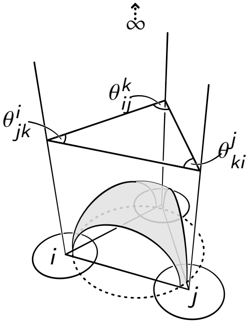
Observations:

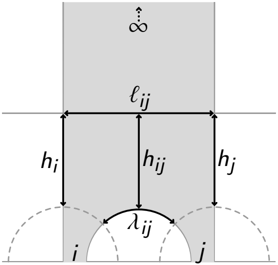
- decorated euclidean triangles
  $\longrightarrow$ hyperideal hyperbolic triangles
- turns $\mathcal{S}_g$ into complete hyperbolic surface Σ_g

Proposition ([Bobenko–L. '23])

$(\mathcal{T}, \ell, r)$ and $(\mathcal{T}, \tilde{\ell}, \tilde{r})$ DCE $\Leftrightarrow$ same Σ_g.

If $\mathcal{T}$ is wDt of $(\mathrm{dist}_{\mathcal{S}_g}, r) \rightsquigarrow \Sigma_g$ fundamental discrete conformal invariant



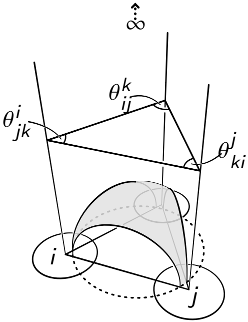


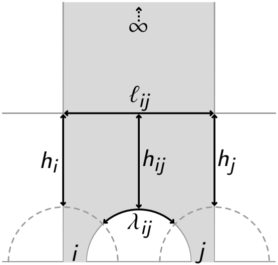
Hyperideal Polyhedral Cusps

- Can construct a *hyperideal horoprism* over each triangle.
- Relationship between parameters:

$$r_i = e^{-h_i}$$
$$r_{ij} = e^{-h_{ij}}$$
$$u_i = h_i - \tilde{h}_i$$
$$l_{ij} = \cosh(\lambda_{ij})$$

- The collection of hyperideal horoprisms is called a *polyhedral cusp* $P_{\mathcal{T},h}$.





The Variational Principle

Consider: hyperbolic surface Σ_g, $\Theta \in \mathbb{R}^V_{>0}$, $h \in \mathbb{R}^V$.

$$\mathcal{H}_{\Sigma_g,\Theta}(h) := -2\,\mathrm{Vol}(P_{\mathcal{T},h}) + \sum_{i \in V}(\Theta_i - \theta_v)h_i + \sum_{ij \in E_{\mathcal{T}}}(\pi - \alpha_{ij})\lambda_{ij}$$

is called the *discrete Hilbert–Einstein functional*.

Properties

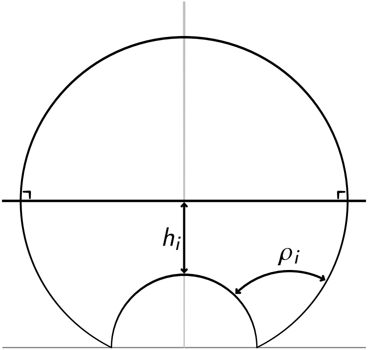
- *first derivative*: $d\,\mathcal{H}_{\Sigma_g,\Theta} = \sum(\Theta_i - \theta_i)dh_i$
- *second derivative*: $D^2\,\mathcal{H}_{\Sigma_g,\Theta} = -\sum_{ij} w_{ij}(dh_i - dh_j)^2$
- *shift-invariance*:

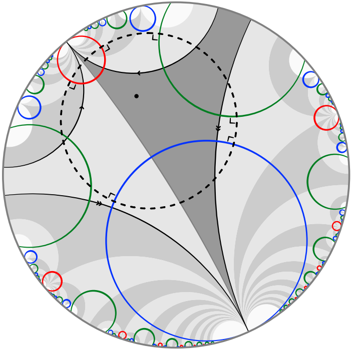
$$\mathcal{H}_{\Sigma_g,\Theta}(h + c1_V) = \mathcal{H}_{\Sigma_g,\Theta}(h) \Leftrightarrow \Theta \text{ satisfies Gauß–Bonnet condition}$$

- *coercive* on $\{h \in \mathbb{R}^V : \sum h_i = 0\}$, *i.e.*,

$$\lim_{\|h\| \to \infty} \mathcal{H}_{\Sigma_g,\Theta}(h) = -\infty$$

Canonical Tessellations of Hyperbolic Surfaces

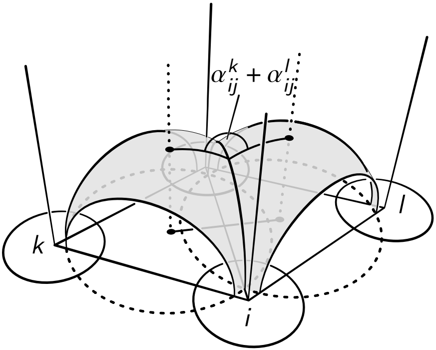


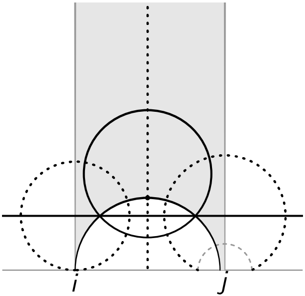


- The h_i induce a decoration of Σ_g
- The space of canonical tessellations is well understood [Penner '87; Epstein–Penner '88; L. '23]

$\Rightarrow$ only finite number of canonical tessellations for fixed Σ_g.

Canonical Tessellations of Hyperbolic Surfaces





Proposition (Bobenko-L. 2023)

$\mathcal{T}$ wDt of $(\mathrm{dist}_{\mathcal{S}_g}, r)$ $\Leftrightarrow$ $\mathcal{T}$ canonical tessellation of Σ_g $\Leftrightarrow$ $P_{\mathcal{T},h}$ is convex

Concluding Remarks and Open Problems

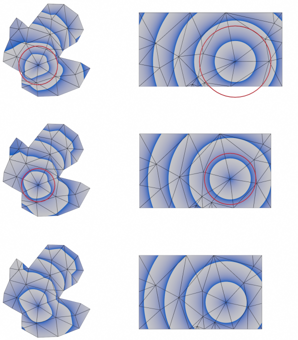
More information in papers:
Bobenko, L.: Decorated discrete conformal maps and convex polyhedral cusps, `arXiv:2305.10988 [math.GT]`

L.: Canonical tessellations of decorated hyperbolic surfaces, Geom. Dedicata, 2023

Upcoming paper for non-euclidean geometries:
Bobenko, L.: Decorated discrete conformal equivalence in non-euclidean geometries

Open problems:

- Convergence to smooth limit.
- Analysis of more general $r^2 \in \mathbb{R}$.



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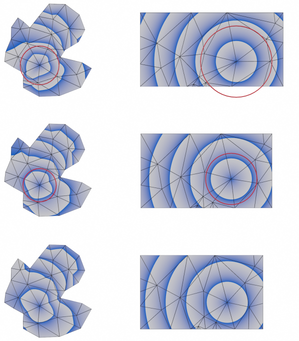

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Thank You!