Bypassing Hölder super-criticality barriers in viscous, incompressible fluids

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New Trends in Fluids and Collective Dynamics

BIRS

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- $\partial_t u(t,x) - \nabla \cdot [C(t,x)\nabla u(t,x)] = 0$, $u(0,x) = u_0(x)$, whole space
- C symmetric matrix, uniform ellipticity

$$\lambda|z|^2 \leq \sum_{i,j} C_{i,j}(t,x) z_i z_j \leq \Lambda |z|^2, \quad \forall z \in \mathbb{R}^d.$$

- Nash [1958]: there exists $\alpha = \alpha(\lambda, \Lambda, d) \in (0,1)$, $A = A(\lambda, \Lambda, d)$ such that

$$|u(t,x) - u(t_0,y)| \leq A \|u_0\|_{L^\infty} \left[\frac{|x-y|^\alpha}{t^{\alpha/2}} + \frac{(t-t_0)^{\frac{\alpha}{2(\alpha+1)}}}{t^{\frac{\alpha}{2(\alpha+1)}}} \right].$$

- De Giorgi [1957] (elliptic case), Moser [1960] (another proof for elliptic).
- No regularity assumption on ∇C whatsoever!!! In one dimension:

$$\partial_t u - c\partial_x^2 u - \partial_x c \partial_x u = 0$$

- What about $\partial_t u - \Delta u + (b \cdot \nabla) u = 0$? Should we expect analogous result? Regularity of u without making any assumptions on b?
- No! Counter-example by Silvestre, Vicol, and Zlatoš [2013]: let $p \in [1, 2)$. Then there exists $T > 0$ and smooth u_0, such that for any modulus of continuity ρ, there exist $b \in L_t^\infty L_x^p$ such that $u(t, \cdot)$ violates ρ before T.
- Scale invariance: $u_\epsilon(t, x) := \epsilon u(\epsilon^2 t, \epsilon x)$, $b_\epsilon(t, x) := \epsilon b(\epsilon^2 t, \epsilon x)$
- $\|u_\epsilon\|_X = \epsilon^\kappa \|u\|_X$, subcritical if $\kappa > 0$, critical $\kappa = 0$, supercritical $\kappa < 0$.
- Nonlinear models: good LWP in critical/subcritical spaces
- So solution is regular if critical norm under control
- "Obvious" arguments fail in supercritical regime
- "Expect" regularity for u if b is in critical space

- Consider the incompressible NSE

$$\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \nabla \cdot u = 0$$

- Scales like drift-diffusion
- Ladyzenskaja-Proddi-Serrin: $u \in L_t^q L_x^p$, $d/p + 2/q < 1$ then u is regular (subcritical)
- Critical case (when $d = 3$) is $L_t^\infty L_x^3$, proven by Escauriaza, Seregin, and Sverak [2003]
- A-priori bounds: $L_t^\infty L_x^2$, $L_t^2 \dot{H}_x^1$, $L_t^1 L_x^\infty$, $L_t^{\gamma_m} \dot{H}_x^m$, and others, all at the supercritical level.
- No known regularity criteria below critical level, no known a-priori bound above supercritical: scaling gap

Drift-Diffusion

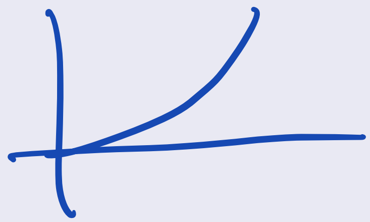
Main Result

$\partial_t u - \Delta u + b \cdot \nabla u = 0$, the space $L^p_t C^{0,\beta}_x$ is critical when $p = 2/(1+\beta)$.

Theorem (I. 2022 (JMFM))

Assume $[b(t,\cdot)]_{C^{0,\beta}_x} \le g(t)$ *and* $g'(t) \ge 0$. *Then there exists a constant* $M_0 = M_0(\|u_0\|_{W^{1,\infty}})$ *(independent of* $\beta \in [0,1]$*) such that*

$$\int_0^T \|\nabla u(t,\cdot)\|_{L^\infty} dt \le M_0 \int_0^T g^{1/(1+\beta)}(t) dt.$$



Theorem (I. 2023 (preprint))

Drop $g' \ge 0$. *There exists a constant* $M_1 = M_1(\|\nabla u_0\|_{L^\infty}, \beta)$ *(blows up as* $\beta \to 0^+$*) such that*

$$\int_0^T \|\nabla u(t,\cdot)\|_{L^\infty} dt \le M_1 e^{\|g\|_{L^1}} \int_0^T g(t) dt.$$

Incompressible Navier-Stokes

Main Result

Theorem (I. 2023 (preprint))

Let $u_0 : \mathbb{R}^d \to \mathbb{R}^d$ be smooth, divergence-free. Let $T_* > 0$ be the maximal time of existence of classical solution. Then $T_* < \infty$ if and only if for any $\beta \in (0,1)$,

$$\int_0^{T_*} [u(t,\cdot)]_{C_x^{0,\beta}} dt = +\infty.$$

Moreover, given any $T \in (0, T_*)$, $q \in [1,\infty)$ and any $\beta \in (0,1)$, there exists a constant $M = M(T_*, d, \beta, \|u\|_{L_t^1 \dot{C}_x^{0,\beta}}, q)$ such that

$$\int_0^T \|u(t,\cdot)\|_{L^\infty}^q dt \le \|u_0\|_{L^\infty}^q M$$

$u \in L_t^1 C_x^{0,\beta}$+smooth initial data $\Rightarrow$ no blow up. $L_t^p C_x^{0,\beta}$ is critical $p = p^* = 2/(1+\beta)$, supercritical $p < p^*$. $p = p^*$ Silvestre and Vicol [2012], different proof by I. [2022]

Reducing to one-dimensional problem

Transferring evolution to moduli of continuity

- Use elegant ideas introduced by Kiselev, Nazarov, and Volberg 2007 (critical SQG) (also Kiselev, Navarov, and Shterenberg 2008 critical Burgers).
- For $\partial_t u - \Delta u + b \cdot \nabla u = 0$ with $[b(t,\cdot)]_{C_x^{0,\beta}} \le g(t)$, construct Ω such that $\Omega(t,0) = 0$, $\Omega(t,\cdot)$ non-decreasing, concave (typically):

$$\begin{cases} \partial_t \Omega - 4\partial_\xi^2 \Omega \ge g(t)\xi^\beta \partial_\xi \Omega, & (t,\xi) \in (0,T] \times (0,\infty), \\ |u_0(x) - u_0(y)| < \Omega(0,|x-y|), & x \ne y. \end{cases}$$

- Then $|u(t,x) - u(t,y)| \le \Omega(t,|x-y|)$ $\Rightarrow \|\nabla u(t,\cdot)\|_{L^\infty} \le \partial_\xi \Omega(t,0)$
  $\Rightarrow \|u(t,\cdot)\|_{L^\infty} \le \|\Omega(t,\cdot)\|_{L^\infty}$
- For incompressible NSE with $[u(t,\cdot)]_{C_x^{0,\beta}} \le g(t)$:

$$\partial_t \Omega - 4\partial_\xi^2 \Omega \ge g(t)\xi^\beta \partial_\xi \Omega + C_{d,\beta} g(t) \int_0^\xi \eta^{\beta-1} \partial_\eta \Omega(t,\eta) d\eta.$$

$u \in C_x^{0,\alpha} \Rightarrow P \in C^{0,2\alpha}$

$\nabla P \in C^{0,2\alpha-1}$

- Advantage: one dimension, extra degree of freedom

Drift-Diffusion

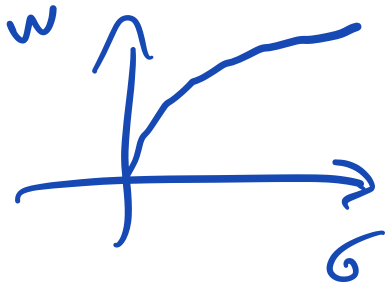
Supercritical regularity

$$\partial_t \Omega - 4\partial_\xi^2 \Omega - g(t)\xi^\beta \partial_\xi \Omega \geq 0$$

- Dynamic rescaling: $\Omega(t,\xi) := \omega(\mu\xi)$, $\mu = \mu(t)$, $\sigma := \mu\xi$:

$$\mu^{-1}\mu'\sigma\omega'(\sigma) - 4\mu^2\omega''(\sigma) - g\mu^{1-\beta}\sigma^\beta\omega'(\sigma) \geq 0$$

- Balance dissipation with transport: $-\mu^2[4\omega'' + g\mu^{-\beta-1}\sigma^\beta\omega'] \geq 0$
- $\mu \approx g^{1/(1+\beta)}$, leads to $4\omega'' + \sigma^\beta\omega' \leq 0$
- Assume g non-decreasing $\|\nabla u(t,\cdot)\|_{L^\infty} \leq M_0[g(t)]^{\frac{1}{1+\beta}}$



$$\int_0^T \|\nabla u(t,\cdot)\|_{L^\infty} dt \leq M_0 \int_0^T [g(t)]^{\frac{1}{1+\beta}} dt$$

- $L_t^p C_x^{0,\beta}$ critical when $p = 2/(1+\beta)$!! Silvestre and Vicol [2012] showed $u \in L_t^\infty C_x^{0,\alpha}$, any $\alpha \in (0,1)$, if b critical.

Incompressible NSE

A-priori Bounds

- For drift diffusion:

$$\int_0^T \|\nabla u(t,\cdot)\|_{L^\infty} dt \leq M_0 \int_0^T [g(t)]^{\frac{1}{1+\beta}} dt$$

- Foiaș, Guillopé, and Temam [1981]: 3D periodic NSE, $m \geq 1$, $T > 0$,

$$\int_0^T \|u(t,\cdot)\|_{\dot{H}^m}^{\gamma_m} dt < \infty, \ \gamma_m := \frac{2}{2m-1}$$

$\gamma_2 = \frac{2}{4-1} = 2/3$

- From $[u(t,\cdot)]_{C_x^{0,1/2}} \lesssim \|u(t,\cdot)\|_{\dot{H}^2}$

$$\int_0^T [u(t,\cdot)]_{C_x^{0,1/2}}^{2/3} dt < \infty, \quad \frac{1}{1+1/2} = 2/3.$$

- We also have $u \in L_t^1 L_x^\infty$ Constantin [1990, 2014], Tao [2013]

Evolution of Ω

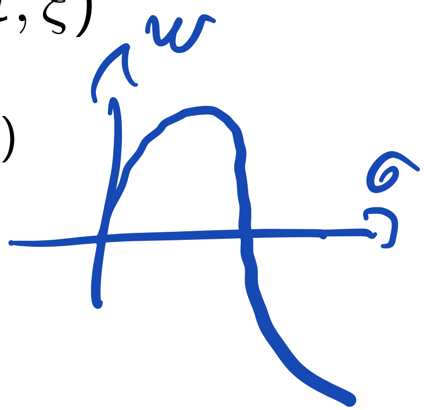
Incompressible NSE

- For NSE, the inequality (roughly)

$$\partial_t\Omega - 4\partial_\xi^2\Omega \geq g(t)\xi^\beta\partial_\xi\Omega + C_{d,\beta}g(t)\xi^{\beta-1}\Omega(t,\xi)$$

- Dynamic rescaling: $\Omega(t,\xi) := \omega(\mu\xi)$, $\mu(t) = g(t)^{1/(1+\beta)}$

$$4\omega'' + \sigma^\beta\omega'(\sigma) + C_{d,\beta}\sigma^{\beta-1}\omega(\sigma) \leq 0$$



- Issue: no non-decreasing solutions, no longer modulus of continuity
- Take advantage of time-derivative: $\Omega(t,\xi) := \lambda(t)\omega(\mu\xi)$

$$\lambda'(t) \approx g^{2/(1+\beta)}\lambda(t) \to \lambda(t) \approx e^{\|g\|_{L^{2/(1+\beta)}}}$$

- $\|\nabla u(t,\cdot)\|_{L^\infty} \lesssim \lambda(t)g^{1/(1+\beta)}(t)$

Drift-Diffusion: Enhanced Parabolic Regularity

- Focus on solutions to $\partial_t \Omega(t,\xi) - 4\partial_\xi^2 \Omega(t,\xi) - g\xi^\beta \partial_\xi \Omega(t,\xi) = 0$ on $(0,T] \times (0,\infty)$, with concave, non-decreasing initial data
- Let h be odd extension to ξ^β, mollify, $\mathcal{V} := \partial_\xi \Omega$, *even* solution to

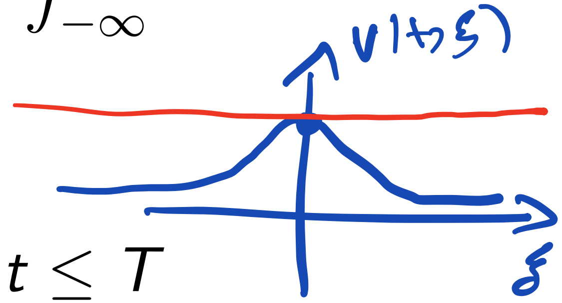
$$\partial_t \mathcal{V}(t,\xi) - \partial_\xi^2 \mathcal{V}(t,\xi) - g(t)\partial_\xi[h(\xi)\mathcal{V}(t,\xi)] = 0, \quad (t,\xi) \in [0,T] \times \mathbb{R}$$

- $\mathcal{V} \geq 0$ on $\mathbb{R}$, even, and decreasing on $[0,\infty)$, so maximum at $\xi = 0$. Heart of the matter: h' has singularity at 0, "obvious" bound is not allowed. By symmetry, $g(t)h(\xi)\partial_\xi V(t,\xi) \leq 0$ always. Enhanced regularity.

solution variables

$$\mathcal{V}(t,\xi) = \int_{-\infty}^{\infty} Z(t,\xi;0,\sigma)\mathcal{V}_0(\sigma)d\sigma \leq \|\mathcal{V}_0\|_{L^\infty} \int_{-\infty}^{\infty} Z(t,\xi;0,\sigma)d\sigma$$

- Max at $\xi = 0$, so need to control



$$\int_{-\infty}^{\infty} Z(t,0;s,\sigma)d\sigma, \quad 0 \leq s < t \leq T$$

evolution variables.

Enhanced Parabolic Regularity

The fundamental solution

$$\mathcal{D}_{t,\xi} := \partial_t - \partial_\xi^2 - g(t)\partial_\xi[h(\xi)\cdot]$$

- Need $\|Z(t,0;s,\cdot)\|_{L^1}$. Adjoint equation: fix $(t,\xi) \in (s,T] \times \mathbb{R}$

$$-\partial_r Z(t,\xi;r,\sigma) - \partial_\sigma^2 Z(t,\xi;r,\sigma) + g(r)h(\sigma)\partial_\sigma Z(t,\xi;r,\sigma) = 0$$

- Set $f(r) = \|Z(t,0;r,\cdot)\|_{L^1}$, evaluate at $\xi = 0$ and integrate:

$$f'(r) = \frac{d}{dr}\int_{-\infty}^{\infty} Z(t,0;r,\sigma)d\sigma = -g(r)\int_{-\infty}^{\infty} h'(\sigma)Z(t,0;r,\sigma)d\sigma$$

- $h' \in L^1(-1,1) \cap L^\infty(\mathbb{R}\backslash[-1,1])$ as $h'(\sigma) \approx |\sigma|^{\beta-1}$:

$$f'(r) \geq -g(r)\|Z(t,0;r,\cdot)\|_{L^\infty} - g(r)f(r)$$

- Transport diffusion, no h', maximum principle. Issue: singular initial data

Enhanced Parabolic Regularity

Controlling the volume potential

- $Z(t,\xi;s,\sigma) = \Psi(t-s,\xi-\sigma) + \tilde{Z}(t,\xi;s,\sigma)$

$$\tilde{Z}(t,\xi;s,\sigma) = \int_s^t \int_{-\infty}^{\infty} \Psi(t-r,\xi-\mu)Q(r,\mu;s,\sigma)d\mu dr$$

$\frac{|\sigma|^{\beta+1}}{t-r} \frac{e^{-\sigma^2/(t-r)}}{\sqrt{t-r}} \leq (t-r)^{\frac{\beta+1}{2}-\frac{1}{2}-1} = (t-r)^{\frac{\beta-1}{2}}.$

- Set $\mathcal{W}(r,\sigma) := \tilde{Z}(t,0;r,\sigma)$, $r \in [s,t)$

$$-\partial_r \mathcal{W} - \nu \partial_\sigma^2 \mathcal{W} + g(r)h(\sigma)\partial_\sigma \mathcal{W} = -g(r)h(\sigma)\partial_\sigma \Psi(t-r,\sigma),$$

$$\lim_{r\to t^-} \|\mathcal{W}(r,\cdot)\|_{L^p} = 0$$

$\forall\, p \in [1,\infty)$ (1 dimension)

- We have $|h(\xi)| \leq |\xi|^\beta$, reduces singularity in forcing term

$$\|\tilde{Z}(t,0;s,\cdot)\|_{L^\infty} \leq C_\beta \int_s^t (t-r)^{(\beta-2)/2} g(r)dr.$$

- Bootstrap L^∞ to L^1

Enhanced Parabolic Regularity

Drift-Difffusion

Theorem (I. 2023 (preprint))

Let Z be the fundamental solution to the operator $\mathcal{D}_{t,\xi} := \partial_t - \partial_\xi^2 - g(t)\partial_\xi[h(\xi)\cdot]$, set $\Gamma(s,t) := \int_s^t g(r)dr$. It follows that

$$\int_{-\infty}^{\infty} Z(t,0;s,\sigma)d\sigma$$

$$\leq 1 + C_\beta e^{\Gamma(s,t)} \left[\int_s^t (t-r)^{\frac{\beta-1}{2}} g(r)dr + \Gamma(s,t) \int_s^t (t-r)^{\frac{\beta-2}{2}} g(r)dr \right].$$

There exists a function $G : [0,T] \to [1,\infty)$ s.t. $\|G\|_{L^1} \leq M(\beta, \|g\|_{L^1})$ for which the solution to $\mathcal{D}_{t,\xi}\mathcal{V} = 0$ satisfies $\|\mathcal{V}(t,\cdot)\|_{L^\infty} \leq \|\mathcal{V}_0\|_{L^\infty} G(t)$
No non-decreasing assumption on g, requires L^1 instead of $L^{1/(1+\beta)}$.

Enhanced Parabolic Regularity

General initial data

- Result is false without drift or switch sign of the drift
- Quantify enhanced regularity: consider $\mathcal{D}_{t,\xi;\lambda} := \partial_t - \nu\partial_\xi^2 - b(t,\xi)\partial_\xi - \lambda\partial_\xi b(t,\xi)$, some $\lambda > 0$
- Ignore diffusion: $\partial_t \mathcal{V} - b\partial_\xi \mathcal{V} - \lambda\partial_\xi b\mathcal{V} = 0$
- Flow map: $\partial_t \phi = -b(t,\phi)$, $\mathcal{A}(t,\xi) := \phi^{-1}(t,\xi)$ solves $\partial_t \mathcal{A}(t,\xi) - b(t,\xi)\partial_\xi \mathcal{A}(t,\xi) = 0$, $\mathcal{A}(0,\xi) = \xi$
- Characteristics:

$$\mathcal{V}(t,\xi) = [\partial_\xi \mathcal{A}(t,\xi)]^\lambda \mathcal{V}_0(\mathcal{A}(t,\xi))$$

- Diffusion: stochastic flow $\partial_t \phi = -b(t,\phi) + \sqrt{2\nu}\dot{W}$, Feynman-Kac

$$\mathcal{V}(t,\xi) = \mathbb{E}\left[[\partial_\xi \mathcal{A}(t,\xi)]^\lambda \mathcal{V}_0(\mathcal{A}(t,\xi))\right]$$

- Constantin and Iyer [2008] $\mathcal{A} = \phi^{-1}$ solves

$$\partial_t \mathcal{A} - \nu\partial_\xi^2 \mathcal{A} - b\partial_\xi \mathcal{A} + \sqrt{2\nu}\dot{W}\partial_\xi \mathcal{A} = 0$$

Enhanced Parabolic Regularity

Stochastic Flow

$\lambda = 1$

$$\mathcal{D}_{t,\xi;\lambda} := \partial_t - \nu\partial_\xi^2 - b(t,\xi)\partial_\xi - \lambda\partial_\xi b(t,\xi) \qquad B = \partial_\xi A$$

$|V(t,\xi)| \leq \|V_0\|_{L^\infty} E[B^\lambda(t,\xi)]$ $\quad \geq 1$

- $\mathcal{V}(t,\xi) = \mathbb{E}\left[[\partial_\xi \mathcal{A}(t,\xi)]^\lambda \mathcal{V}_0(\mathcal{A}(t,\xi))\right]$ solves $\mathcal{D}_{t,\xi;\lambda}\mathcal{V} = 0$.
- $\mathcal{A}$ solves $\partial_t \mathcal{A} - \nu\partial_\xi^2\mathcal{A} - b\partial_\xi\mathcal{A} + \sqrt{2\nu}\dot{W}\partial_\xi\mathcal{A} = 0$, $\mathcal{A}(0,\xi) = \xi$ a.s.
- $\bar{\mathcal{B}} := \mathbb{E}[\partial_\xi \mathcal{A}]$ symmetric solution to $\quad \|\bar{B}(t,\cdot)\|_{L^\infty} \leq G(t)$

$$\partial_t\bar{\mathcal{B}} - \nu\partial_\xi^2\bar{\mathcal{B}} - \partial_\xi[b\bar{\mathcal{B}}] = 0, \quad \bar{\mathcal{B}}(0,\xi) = 1$$

- If $\lambda \leq 1$, then $\mathbb{E}\left[[\partial_\xi\mathcal{A}(t,\xi)]^\lambda\right] \leq [\bar{\mathcal{B}}(t,\xi)]^\lambda \leq G^\lambda(t)$
- Thus, $\|\mathcal{V}(t,\cdot)\|_{L^\infty} \leq \|\mathcal{V}_0\|_{L^\infty} G^\lambda(t)$ any initial data. Enhanced regularity is property of the parabolic operator
- If $\lambda > 1$, very hard to control $\quad \sup_\xi E[B^\lambda(t,\xi)] \leq G^\lambda(t)$

Enhanced Parabolic Regularity

$\lambda \in (1,2]$: L^p bounds

$$\mathcal{D}_{t,\xi;\lambda} := \partial_t - \nu\partial_\xi^2 - b(t,\xi)\partial_\xi - \lambda\partial_\xi b(t,\xi)$$

- $\mathcal{V}(t,\xi) = \mathbb{E}\left[[\partial_\xi \mathcal{A}(t,\xi)]^\lambda \mathcal{V}_0(\mathcal{A}(t,\xi))\right]$ solves $\mathcal{D}_{t,\xi;\lambda}\mathcal{V} = 0$.
- $\mathcal{A}$ is C^2 diffeomorphism almost surely ($\mathcal{B} := \partial_\xi \mathcal{A} \geq 1$):

$$\begin{aligned}\|\mathcal{V}(t,\cdot)\|_{L^1} \leq& \mathbb{E}\left[\int_{-\infty}^{\infty} [\partial_\xi \mathcal{A}(t,\xi)]^\lambda |\mathcal{V}_0(\mathcal{A}(t,\xi))| d\xi\right] \\ =& \int_{-\infty}^{\infty} \mathbb{E}[\mathcal{B}^{\lambda-1}(t,\Phi(t,\sigma))] |\mathcal{V}_0(\sigma)| d\sigma\end{aligned}$$

- If $\lambda - 1 \in [0,1]$, $\mathbb{E}[\mathcal{B}^{\lambda-1}(t,\Phi(t,\sigma))] \leq \|\bar{\mathcal{B}}(t,\cdot)\|_{L^\infty}^{\lambda-1}$
- Thus,

$$\|\mathcal{V}(t,\cdot)\|_{L^1} \leq \|\mathcal{V}_0\|_{L^1} G^{\lambda-1}(t)$$

- Can get L^p bounds provided $p\lambda \in [1,2]$.

Incompressible Navier-Stokes

The pressure as a perturbation

Assume $|u(t,x) - u(t,y)| \leq g(t)|x-y|^{\beta}$

$$\partial_t \Omega - 4\partial_\xi^2 \Omega - g\xi^\beta \partial_\xi \Omega \geq C_{d,\beta} g(t) \int_0^\xi \eta^{\beta-1} \partial_\eta \Omega(t,\eta) d\eta$$

- $\Omega(t,\cdot)$ non-decreasing, inequality, multiply drift by any $\mu \geq 1$:

$$\partial_t \Omega - 4\partial_\xi^2 \Omega = \mu g \xi^\beta \partial_\xi \Omega + C_{d,\beta} g(t) \int_0^\xi \eta^{\beta-1} \partial_\eta \Omega(t,\eta) d\eta$$

- $\mathcal{V} := \partial_\xi \Omega$ solves

$$\partial_t \mathcal{V} - 4\partial_\xi^2 \mathcal{V} = (\mu + C_{d,\beta}) g(t) h'(\xi) \mathcal{V} + \underbrace{\mu g(t) h(\xi) \partial_\xi \mathcal{V}}_{\leq 0}$$

- Set $\lambda := (\mu + C_{d,\beta})/\mu = 1 + C_{d,\beta}\mu^{-1}$, then if μ is large, $\lambda \in [1,2]$:

$$\|\mathcal{V}(t,\cdot)\|_{L^1} \leq \|\mathcal{V}_0\|_{L^1} G_\mu^{\lambda-1}(t)$$

$$G_\mu(t) \approx 1+\mu e^{\mu\|g\|_{L^1}}\left[\int_0^t (t-r)^{\frac{\beta-1}{2}} g(r)dr + \mu\|g\|_{L^1}\int_0^t (t-r)^{\frac{\beta-2}{2}} g(r)dr\right]$$

- Recall $|u(t,x)-u(t,y)| \le \mathcal{V}(t, |x-y|) = \int_0^{|x-y|} \mathcal{V}(t,\xi)d\xi.$

$$|u(t,x)-u(t,y)| \le \int_0^{|x-y|} \mathcal{V}(t,\xi)d\xi \le \|\mathcal{V}(t,\cdot)\|_{L^1}$$

- Thus,

$$\|u(t,\cdot)\|_{L^\infty} \le \|\mathcal{V}_0\|_{L^1} G_\mu^{\lambda-1}(t)$$

- Recall $\lambda = 1 + C_{d,\beta}\mu^{-1}$, given $q \ge 1$ choose $\mu = \mu(q,d,\beta) \ge 1$ such that $\lambda - 1 = 1/q$:

$$\int_0^T \|u(t,\cdot)\|_{L^\infty}^q dt \le \|\mathcal{V}_0\|_{L^1}^q \int_0^T G_\mu(t)dt = \|\mathcal{V}_0\|_{L^1}^q M(\beta,d,q,\|g\|_{L^1})$$

- Regularity follows by LPS when $q > 2$.

Theorem (I. 2023 (in preparation))

Let $d \in \mathbb{Z}^+$, $q, q_b \geq 1$, $\beta \in [0,1)$ *be such that*

$$\gamma := \frac{d}{2}\left(\frac{1}{q_b} + \frac{1}{q}\right) + \frac{\beta+1}{2} < 1,$$

let u *solve* $\partial_t u - \Delta u + b \cdot \nabla u = 0$, $u(0,x) = u_0(x)$. *It follows that there exists a constant* $M = M(\gamma, d)$ *such that*

$$[u(t,\cdot)]_{C_x^{0,\beta}} \leq M \|u_0\|_{L^q}\left[t^{-\frac{d}{2q}-\frac{\beta}{2}} + \int_0^t (t-r)^{-\gamma}\|b(r,\cdot)\|_{L^{q_b}}\,dr\right].$$

Thank you!

Moduli of Continuity

$$\begin{cases} \partial_t \theta - \Delta\theta = F(\theta, \nabla\theta), \\ \theta(0,x) = \theta_0(x). \end{cases}$$

- Assume $|\theta_0(x) - \theta_0(y)| < \Omega(0, |x-y|),\ x \neq y$, suppose θ is smooth

$\tau := \sup\left\{t \in [0,T] : |\theta(s,x) - \theta(s,y)| < \Omega\left(s, |x-y|\right),\ s \in [0,t],\ x \neq y\right\}$

- Goal: construct Ω such that $\tau = T$, whence

$$\|\nabla\theta(t,\cdot)\|_{L^\infty} \leq \partial_\xi \Omega(t,0), \quad \forall t \in [0,T]$$

- Can show $\tau \in (0,T]$ and if $\tau < T$ then $\theta(\tau,x_0) - \theta(\tau,y_0) - \Omega(\tau, |x_0 - y_0|) = 0$ for some $x_0 \neq y_0$.
- Study $\gamma(t) := \theta(t,x_0) - \theta(t,y_0) - \Omega(t, |x_0 - y_0|)$
- Notice $\gamma(t) < 0$ on $[0,\tau)$. Construct Ω such that $\gamma'(\tau) < 0$, done.
- Choice of Ω depends on model.

Local Dissipation/Max Principle

$$\gamma'(\tau) = \Delta\theta(\tau, x_0) - \Delta\theta(\tau, y_0) - \partial_t\Omega(\tau, |x_0 - y_0|) + F(\theta(\tau, x_0), \nabla\theta(\tau, x_0)) - F(\theta(\tau, y_0), \nabla\theta(\tau, y_0))$$

Lemma

Suppose θ *is* C^2 *and* $|\theta(x) - \theta(y)| \leq \omega(|x - y|)$. *If* $\theta(x_0) - \theta(y_0) = \omega(|x_0 - y_0|)$ *for some* $x_0 - y_0 = \xi e_1$, $\xi > 0$ *then*

$$\partial_j\theta(x_0) = \partial_j\theta(y_0) = \begin{cases} \omega'(\xi), & j = 1 \\ 0, & j \neq 1 \end{cases}$$

and $\Delta\theta(x_0) - \Delta\theta(y_0) \leq 4\omega''(\xi)$.

$F(\theta, \nabla\theta) = b \cdot \nabla\theta$, assume $|b(t, x) - b(t, y)| \leq g(t)|x - y|^\beta$

$$b \cdot \nabla\theta(x_0) - b \cdot \nabla\theta(y_0) = (b_1(x_0) - b_1(y_0))\partial_\xi\Omega(\xi) \leq g(\tau)\xi^\beta\partial_\xi\Omega(\tau, \xi)$$
$$\Rightarrow \gamma'(\tau) \leq 4\partial_\xi^2\Omega(\tau, \xi) - \partial_t\Omega(\tau, \xi) + g(\tau)\xi^\beta\partial_\xi\Omega(\tau, \xi)$$

Incompressible Navier-Stokes

The pressure term

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0$$

- Nonlinear system of equations, still ok:

$$\gamma'(\tau) \leq 4\partial_\xi^2 \Omega - \partial_t \Omega + \Omega \partial_\xi \Omega + |\nabla p(\tau, x_0) - \nabla p(\tau, y_0)|$$

- $-\Delta p = \mathrm{div}\,[(u \cdot \nabla)u] = \sum_{i,j} \partial_i \partial_j (u_i u_j) \Rightarrow p = \sum_{i,j} R_i R_j (u_i u_j)$
- $u \in C^{0,\alpha} \cap L^\infty$ then $p \in C^{0,\alpha}$ (nonhomgenous)
- Silvestre [2010] (unpublished) showed that $p \in C^{0,2\alpha}$. If $\alpha \in (1/2, 1)$ then $\nabla p \in C^{0,2\alpha-1}$, homogenous estimate. Also, Constantin [2014], Isett [2013], De Lellis and Székelyhidi [2014]

$$\sum_{i,j} \partial_{z_i} \partial_{z_j} [u_i(z) u_j(z)] = \sum_{i,j} \partial_{z_i} \partial_{z_j} [(u_i(z) - u_i(x))(u_j(z) - u_j(x))]$$

Incompressible Navier-Stokes

Nonlinear regularity criterion

$$-\Delta p(z) = \sum_{i,j} \partial_{z_i}\partial_{z_j} \left[(u_i(z) - u_i(x))(u_j(z) - u_j(x))\right]$$

- This means if $\Phi(z) = C_d|z|^{2-d}$ fundamental sol to Laplace then

$$\nabla p(x) = \int_{\mathbb{R}^d} \nabla\partial_i\partial_j\Phi(y)(u_i(x-y) - u_i(x))(u_j(x-y) - u_j(x))\, dy$$

- $|u(x) - u(y)| \leq \omega(|x-y|)$

$$|\nabla p(x) - \nabla p(y)| \lesssim_d \int_0^{|x-y|} \frac{\omega^2(\eta)}{\eta^2}\, d\eta + \omega(|x-y|)\int_{|x-y|}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta$$

- Construct *any* global in time solution to

$$\partial_t\Omega - 4\partial_\xi^2\Omega - \Omega\partial_\xi\Omega - C_d\left[\int_0^{\xi} \frac{\Omega^2(t,\eta)}{\eta^2}\, d\eta + \Omega(t,\xi)\int_\xi^{\infty} \frac{\Omega(t,\eta)}{\eta^2} d\eta\right] \geq 0$$

then no blowup for NSE in finite time I. [2022].

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