The $(3+1)$-free conjecture of chromatic symmetric functions

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Chromatic polynomial: Birkhoff 1912

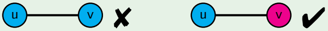
Given G with vertices $V(G)$ a proper colouring κ of G in k colours is

$$\kappa : V(G) \to \{1, 2, 3, \dots, k\}$$

so if $u, v \in V(G)$ are joined by an edge then

$$\kappa(u) \neq \kappa(v).$$

Example



Chromatic polynomial: Birkhoff 1912

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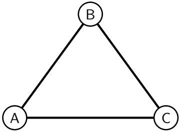
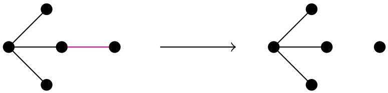


has $\chi_G(k) = k(k-1)(k-2)$.

Deletion-contraction

Delete ϵ: remove edge ϵ to get $G - \epsilon$.



Contract ϵ: shrink edge ϵ + identify vertices to get G/ϵ.

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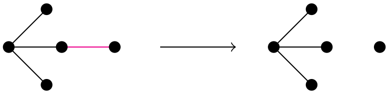

Contract ϵ: shrink edge ϵ + identify vertices to get G/ϵ.

Theorem (Deletion-contraction)

$$\chi_G(k) - \chi_{G-\epsilon}(k) + \chi_{G/\epsilon}(k) = 0$$

Chromatic symmetric function: Stanley 1995

Given G with vertices $V(G)$ a proper colouring κ of G is

$$\kappa : V(G) \to \{1, 2, 3, \dots\}$$

so if $u, v \in V(G)$ are joined by an edge then

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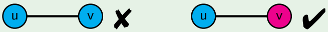
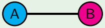


Chromatic symmetric function: Stanley 1995

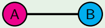
Given a proper colouring κ of vertices $v_1, v_2, \ldots, v_N$ associate a monomial in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}.$$

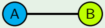
Example



gives $x_1 x_2$.



gives $x_2 x_1 = x_1 x_2$.



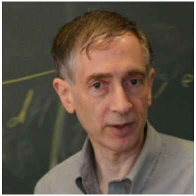
gives $x_1 x_3$.

Chromatic symmetric function: Stanley 1995

Given G with vertices $v_1, v_2, \ldots, v_N$ the chromatic symmetric function is

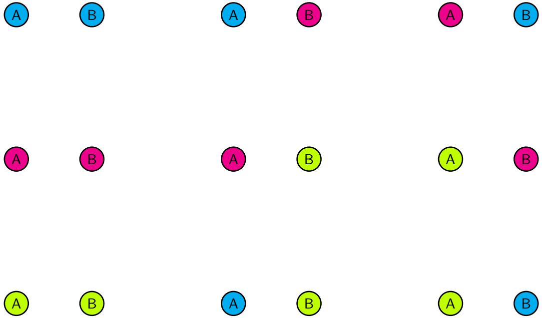
$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

where the sum over all proper colourings κ.



Chromatic symmetric function: Stanley 1995

has $X_G(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$.



Multi-deletion

Theorem (Triple-deletion: Orellana-Scott 2014)

Let G be such that $\epsilon_1, \epsilon_2, \epsilon_3$ *form a triangle. Then*

$$X_G - X_{G-\{\epsilon_1\}} - X_{G-\{\epsilon_2\}} + X_{G-\{\epsilon_1,\epsilon_2\}} = 0.$$

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Theorem (k-deletion: Dahlberg-vW 2018)

Let G be such that $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ form a k-cycle for $k \geq 3$. Then

$$\sum_{S \subseteq [k-1]} (-1)^{|S|} X_{G - \cup_{i \in S}\{\epsilon_i\}} = 0.$$

Deletion-contraction weighted X_G: Crew-Spirkl 2020.

Symmetric functions

A symmetric function is a formal power series f in commuting variables $x_1, x_2, \ldots$ such that for all permutations π

$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

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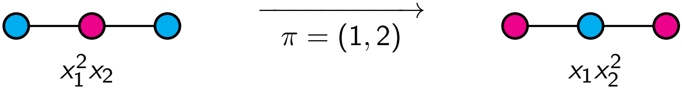
X_G is a symmetric function.



Let

$$\Lambda = \bigoplus_{N \geq 0} \Lambda^N \subset \mathbb{Q}[[x_1, x_2, \ldots]]$$

be the algebra of symmetric functions with Λ^N spanned by ...

Classical basis: power sum

A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a list of positive integers whose sum is N: $3221 \vdash 8$.

The i-th power sum symmetric function is

$$p_i = x_1^i + x_2^i + x_3^i + \cdots$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

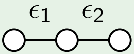
Example

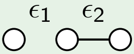
$p_{21} = p_2 p_1 = (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots)$

Classical basis: power sum

Given $S \subseteq E(G)$, $\lambda(S)$ is the partition determined by the connected components of G restricted to S.

Example

$G =$ 

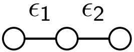
G restricted to $S = \{\epsilon_2\}$ is 

and $\lambda(S) = 21$.

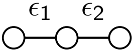
Theorem (Stanley 1995)

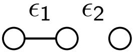
$$X_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(S)}$$

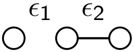
Classical basis: power sum

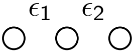
$G =$ 

G restricted to

- $S = \{\epsilon_1, \epsilon_2\}$ is 

 and $\lambda(S) = 3$
- $S = \{\epsilon_1\}$ is 

 and $\lambda(S) = 21$
- $S = \{\epsilon_2\}$ is 

 and $\lambda(S) = 21$
- $S = \emptyset$ is 

 and $\lambda(S) = 111$.

$$X_G = p_3 - 2p_{21} + p_{111}$$

Classical basis: elementary

The *i-th elementary symmetric function* is

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i}$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}.$$

Example

$$e_{21} = e_2 e_1 = (x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$

$G =$  $\quad X_G = 3e_3 + e_{21}$

Classical basis: elementary

Theorem (Stanley 1995)

If

$$X_G = \sum_{\lambda} c_\lambda e_\lambda$$

then

$$\sum_{\lambda \text{ with } k \text{ parts}} c_\lambda = \text{ number of acyclic orientations with } k \text{ sinks.}$$

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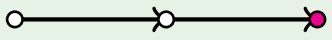
$G =$ 

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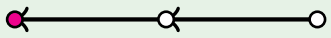
$G =$  $\qquad X_G = 3e_3 + e_{21}$

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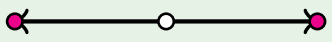
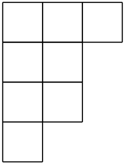
$G =$ 

$$X_G = 3e_3 + e_{21}$$

Partitions and diagrams

A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a list of positive integers whose sum is N: $3221 \vdash 8$.

The diagram $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ is the array of boxes with λ_i boxes in row i from the top.



3221

Semi-standard Young tableaux

A semi-standard Young tableau (SSYT) T of shape λ is a filling with $1, 2, 3, \dots$ so rows weakly increase and columns increase.

1	1	1
2	4	
4	5	
6		

Given an SSYT T we have

$$x^T = x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \cdots .$$

$$x_1^3 x_2 x_4^2 x_5 x_6$$

Classical basis: Schur

The Schur function is

$$s_\lambda = \sum_{T \text{ SSYT of shape } \lambda} x^T.$$

Example

$$s_{21} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3 + \cdots$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

$G =$  $\qquad X_G = s_{21} + 4 s_{111}$

(Wang-Wang 2020) Intricate formula for X_G.

e-positivity and Schur-positivity

G is *e*-positive if X_G is a positive linear combination of e_λ.

G is Schur-positive if X_G is a positive linear combination of s_λ.

e-positivity and Schur-positivity

G is *e-positive* if X_G is a positive linear combination of e_λ.

G is *Schur-positive* if X_G is a positive linear combination of s_λ.

 has

$$X_G = e_{21} + 3e_3 \checkmark$$
$$X_G = 4s_{111} + s_{21} \checkmark$$

e-positivity and Schur-positivity

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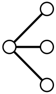
 has $X_G = e_{21} + 3e_3$ ✔ $X_G = 4s_{111} + s_{21}$ ✔

 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘ $X_G = 8s_{1111} + 5s_{211} - s_{22} + s_{31}$ ✘

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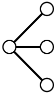

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$X_G = 8s_{1111} + 5s_{211} - s_{22} + s_{31}$ ✘

K_{13} : Smallest graph that is not e-positive. Smallest graph that is not Schur-positive.

e-positivity and Schur-positivity

For $\lambda = \lambda_1 \cdots \lambda_\ell$

$$e_\lambda = \sum_\mu K_{\mu\lambda} s_{\mu^t}$$

where $K_{\mu\lambda} = \#$ SSYTs of shape μ filled with λ_1 1s, …, λ_ℓ ℓs, and μ^t is the transpose of μ along the downward diagonal.

Hence $K_{\mu\lambda} \geq 0$ and

e-positivity implies Schur-positivity.

Example

$$e_{21} = s_{21} + s_{111}$$

1	1
2	

1	1	2

Why e-positivity and Schur-positivity?

- If e-positive, then it is related to permutation representations.
- We have e-positivity implies Schur-positivity.
- If Schur-positive, then it arises as the Frobenius image of some representation of a symmetric group.
- If Schur-positive, then it arises as the character of a polynomial representation of a general linear group.
- The Stanley-Stembridge conjecture.

e-positivity and Schur-positivity

Conjecture (Stanley-Stembridge 1993)

If G is an incomparability graph of a $(3+1)$-free poset then X_G is e-positive.

CONJECTURE (STANLEY-STEMBRIDGE 1993)

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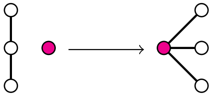

e-POSITIVITY AND SCHUR-POSITIVITY

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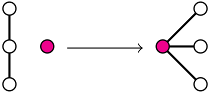
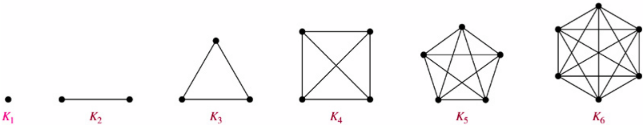


THEOREM (GASHAROV 1996)

If G is an incomparability graph of a $(3+1)$-free poset then X_G is Schur-positive.

e-POSITIVITY AND SCHUR-POSITIVITY

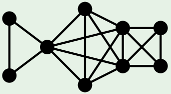
Guay-Paquet showed enough to prove it for unit interval graphs, namely a connected intersection of complete graphs in a row.



CONJECTURE (STANLEY-STEMBRIDGE 1993)

If G is a connected intersection of complete graphs then G is e-positive.

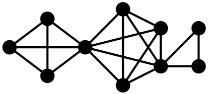
EXAMPLE



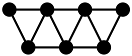
Known cases of e-positive graphs

- 1993 Stanley-Stembridge: two complete graphs intersecting.
- 1995 Stanley: complete graphs K_2 intersecting at vertices

  making a path.
- 2001 Gebhard-Sagan: complete graphs intersecting only at vertices.

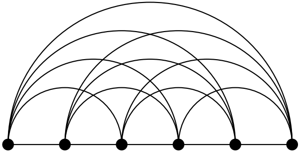


- 2018 Dahlberg: complete graphs K_3 intersecting only at edges.



Results: Aliniaeifard, Wang, vW 2021

Note: We can draw the complete graph as follows.



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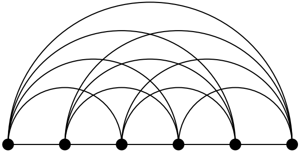
We have *e*-positivity for ice cream scoops: Take the complete graph and melt edges away from the right (or left).



Note: This is an intersection of two complete graphs.

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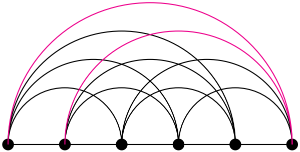

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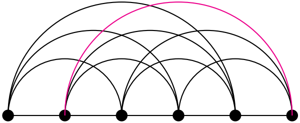

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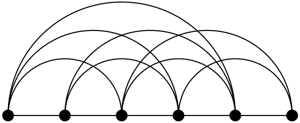

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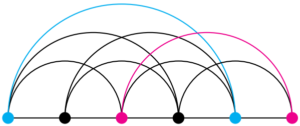
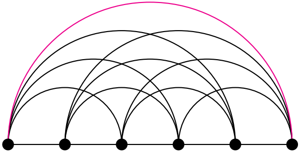


Note: This is an intersection of two complete graphs.

Results: Aliniaeifard, Wang, vW 2021

We have *e*-positivity for snowy peaks: Take the complete graph and melt one edge away and add dribbles from the right (or left).



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Results: Aliniaeifard, Wang, vW 2021

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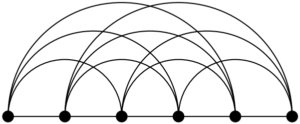

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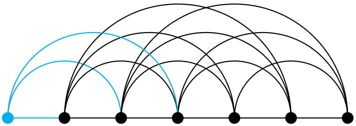

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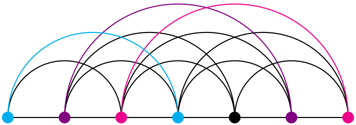
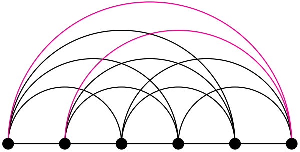


Note: This is an intersection of complete graphs.

Results: Aliniaeifard, Wang, vW 2021

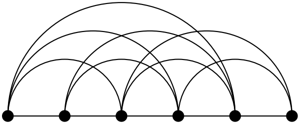
We have e-positivity for peaky snows: Take the complete graph and melt edges away and add one dribble from the right (or left).



Note: This is an intersection of complete graphs.

Results: Aliniaeifard, Wang, vW 2021

We have *e*-positivity for peaky snows: Take the complete graph and melt edges away and add one dribble from the right (or left).



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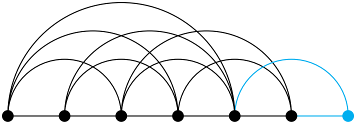

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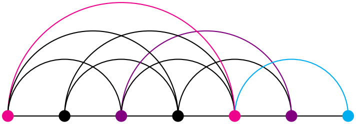
Results: Aliniaeifard, Wang, vW 2021

We have *e*-positivity for peaky snows: Take the complete graph and melt edges away and add one dribble from the right (or left).



Note: This is an intersection of complete graphs.

New cases of Stanley-Stembridge

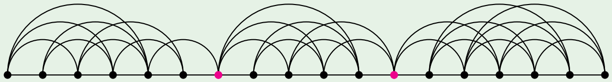
Theorem (Aliniaeifard-Wang-vW 2021)

If G is a connected intersection at the rightmost and leftmost vertex of any combination of

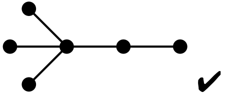
- *ice cream scoops*
- *snowy peaks*
- *peaky snows*
- *complete graphs*
- *triangular ladders*

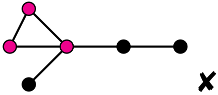
then G is e-positive.

Example



Widen the conjecture - Part 1
e-positivity of trees: Dahlberg, She, vW 2020





N	1	2	3	4	5	6	7	8	9	10	11	12	13
trees	1	1	1	2	3	6	11	23	47	106	235	551	1301
e-pos	1	1	1	1	2	1	3	1	2	2	5	1	4

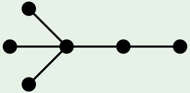
Theorem (Dahlberg-She-vW 2020)

Any tree with N vertices and a vertex of degree

$$d \geq \log_2 N + 1$$

is not e-positive.

Example



is not e-positive.

e-positivity of trees

Conjecture (Dahlberg-She-vW 2020)

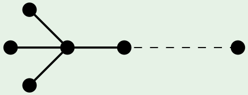
Any tree with N vertices and a vertex of degree

$$d \geq 4$$

is not e-positive.

(Zheng 2020) True for $d \geq 6$.

Example



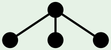
is not e-positive.

e-positivity test of Wolfgang III 1997

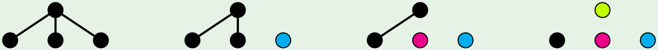
A graph has a connected partition of type $\lambda = \lambda_1 \cdots \lambda_\ell$ if we can find disjoint subsets of vertices $V_1, \ldots, V_\ell \in V(G)$ so

- $V_1 \cup \cdots \cup V_\ell = V(G)$
- restricting edges to each V_i gives connected components with λ_i vertices.

Example



has connected partitions of type $4, 31, 211$ and 1111



but is missing a connected partition of type 22.

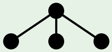
e-positivity test of Wolfgang III 1997

Theorem (Wolfgang III 1997)

If a connected graph G with N vertices is e-positive, then G has a connected partition of type λ for every partition $\lambda \vdash N$.

Test: If G does not have a connected partition of some type then G is not e-positive.

Example



does not have a connected partition of type 22. Hence it is not e-positive.

Schur-positivity of trees

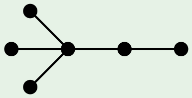
Theorem (Dahlberg-She-vW 2020)

Any tree with N vertices and a vertex of degree

$$d > \left\lceil \frac{N}{2} \right\rceil$$

is not *Schur-positive.*

Example



is not Schur-positive.

Conjectures

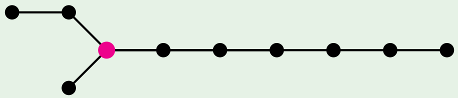
A spider

$$S(i, j, k, \ldots)$$

consists of disjoint paths $P_i, P_j, P_k, \ldots$ and a central vertex joined to a leaf in each path.

Example

$S(6, 2, 1)$



1. Any tree with a vertex of degree 4 or 5 is not e-positive.
2. The family of spiders $S(2(2m+1), 2m, 1)$ is e-positive. More generally, $S(n(n!m+1), n!m, 1)$ is e-positive.
3. If a spider is e-positive, then its line graph is as well.

Widen the conjecture - Part 2
Stanley's widening: Dahlberg, Foley, vW JEMS 2020

Stanley 1995:

> We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive.

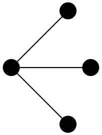
Widen the conjecture - Part 2
Stanley's widening: Dahlberg, Foley, vW JEMS 2020

Stanley 1995:

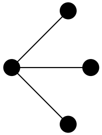
> We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive.

We do.

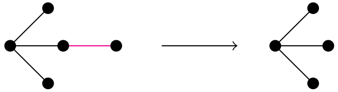
The claw aka K_{13}



The claw aka K_{13}



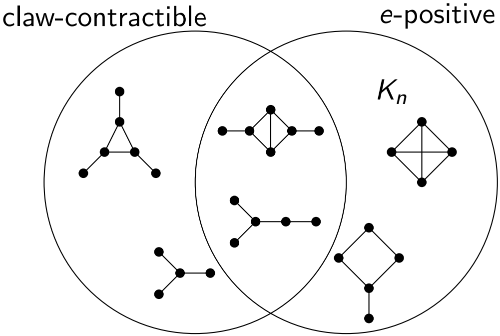
Contracts to the claw: shrinking edges + identifying vertices + removing multiple edges = claw.



A picture speaks 1000 words

Stanley 1995:

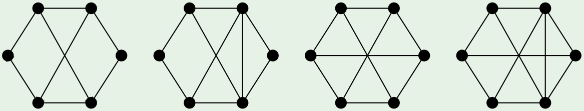
We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive.



Claw-contractible-free: Brouwer-Veldman 1987

G is claw-contractible-free if and only if deleting all sets of 3 non-adjacent vertices gives disconnection.

Example



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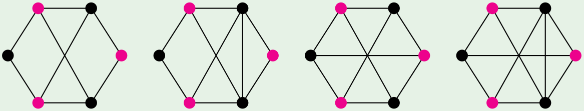
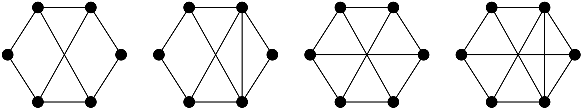
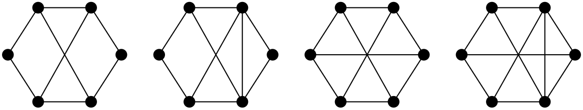


…WITH CHROMATIC SYMMETRIC FUNCTION



$$
\begin{array}{ccccccccc}
2e_{222} & - & 6e_{33} & + & 26e_{42} & + & 28e_{51} & + & 102e_{6} \\
2e_{321} & - & 6e_{33} & + & 24e_{42} & + & 40e_{51} & + & 120e_{6} \\
2e_{222} & - & 12e_{33} & + & 30e_{42} & + & 24e_{51} & + & 186e_{6} \\
2e_{321} & - & 6e_{33} & + & 20e_{42} & + & 32e_{51} & + & 228e_{6}
\end{array}
$$

...WITH CHROMATIC SYMMETRIC FUNCTION

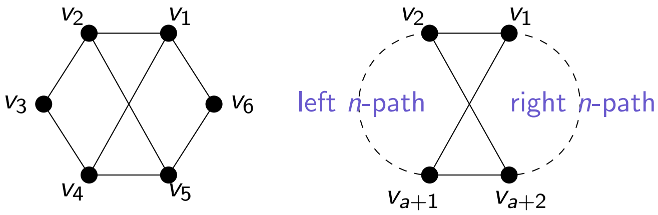


$$2e_{222} - 6e_{33} + 26e_{42} + 28e_{51} + 102e_{6}$$
$$2e_{321} - 6e_{33} + 24e_{42} + 40e_{51} + 120e_{6}$$
$$2e_{222} - 12e_{33} + 30e_{42} + 24e_{51} + 186e_{6}$$
$$2e_{321} - 6e_{33} + 20e_{42} + 32e_{51} + 228e_{6}$$

Smallest counterexamples to Stanley's statement.

Infinite family: saltire graphs

The saltire graph $SA_{n,n}$ for $n \geq 3$ is given by



with $SA_{3,3}$ on the left.

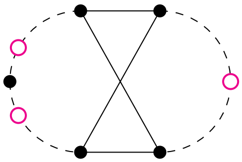
Infinite family: saltire graphs

Theorem (Dahlberg-Foley-vW 2020)

$SA_{n,n}$ for $n \geq 3$ is claw-contractible-free and

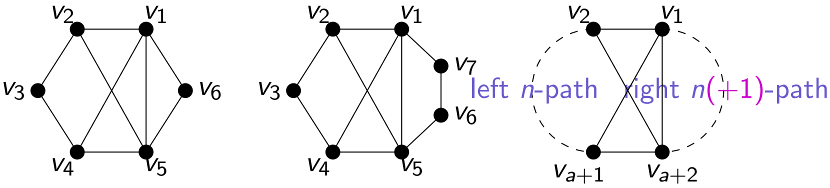
$$[e_{nn}]X_{SA_{n,n}} = -n(n-1)(n-2).$$

CCF:



For any n: augmented saltire graphs

The augmented saltire graphs $AS_{n,n}$, $AS_{n,n+1}$ for $n \geq 3$.



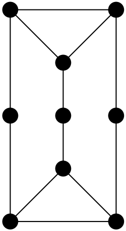
Theorem (Dahlberg-Foley-vW 2020)

$AS_{n,n}$ and $AS_{n,n+1}$ for $n \geq 3$ are claw-contractible-free and

$$[e_{nn}]X_{AS_{n,n}} = [e_{(n+1)n}]X_{AS_{n,n+1}} = -n(n-1)(n-2).$$

Claw-free: Beineke 1970

Claw-free: does not contain the claw as an induced subgraph of the graph.



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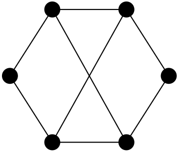

✘

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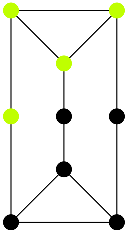

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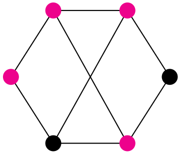

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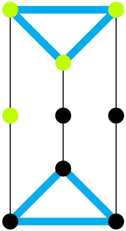
Claw-free: Beineke 1970

G is claw-free if there exists an edge partition giving complete graphs, every vertex in at most two.



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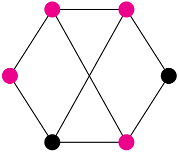
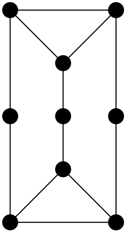
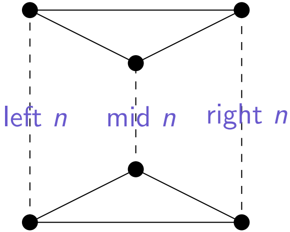


✘

And claw-free: triangular tower graphs

The triangular tower graph $TT_{n,n,n}$ for $n \geq 3$ is given by





with $TT_{3,3,3}$ on the left.

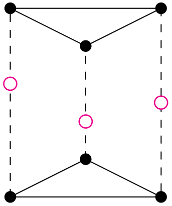
And claw-free: triangular tower graphs

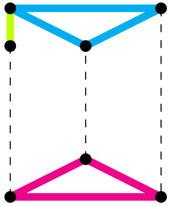
Theorem (Dahlberg-Foley-vW 2020)

$TT_{n,n,n}$ for $n \geq 3$ is claw-contractible-free, claw-free and

$$[e_{nnn}]X_{TT_{n,n,n}} = -n(n-1)^2(n-2).$$

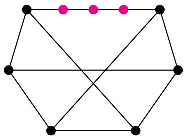
CCF+CF:





Conjectures

1. Bloated $K_{3,3}$:



with $3n$ vertices has

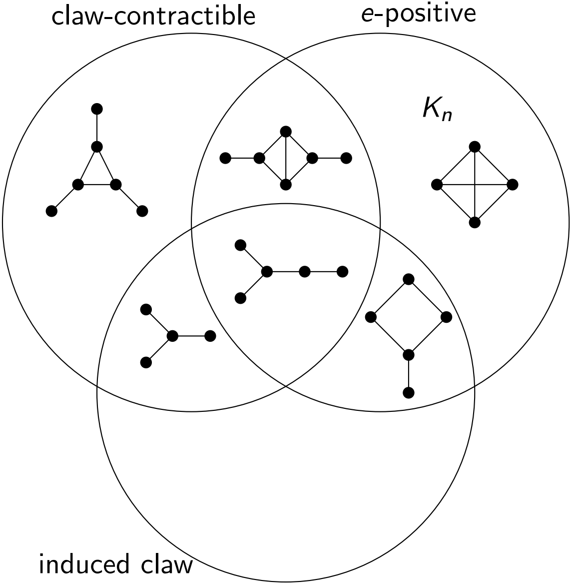
$$-(3 \times 2^n)e_{3^n}.$$

2. No G exists that is connected, claw-contractible-free, claw-free and not e-positive with 10, 11 vertices.

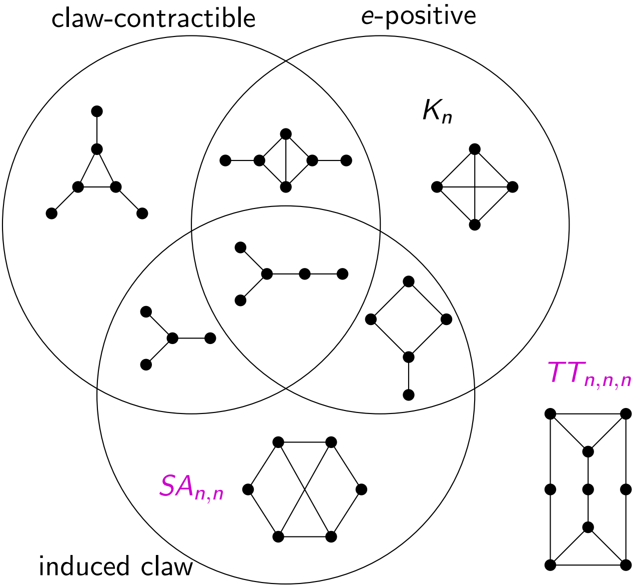
Scarcity

- $N = 6$: 4 of 112 connected graphs ccf and not e-positive.
- $N = 7$: 7 of 853 connected graphs ccf and not e-positive.
- $N = 8$: 27 of 11117 connected graphs ccf and not e-positive.
- Of 293 terms in $TT_{7,7,7}$ only $-$ve at e_{777}.
- Of 564 terms in $TT_{8,8,8}$ only $-$ves at e_{888} and $-1944e_{444444}$.
- Of 1042 terms in $TT_{9,9,9}$ only $-$ves at e_{999}, $-768e_{333333333}$.

A picture speaks 1000 words



A PICTURE SPEAKS 1000 WORDS



In general, *e*-positivity has nothing to do with the claw.

Thank you very much!