On vertex-transitive graphs with a unique hamiltonian circle

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Abstract. It is conjectured that cycles are the only finite regular graphs that have a unique hamiltonian cycle. Mateja Šajna and Andrew Wagner proved in 2014 that the conjecture is true in the special case where the graph is vertex-transitive, which means there is an automorphism that takes any vertex to any other vertex. We will discuss the generalization of the vertex-transitive case to infinite graphs. (Infinite graphs do not have "hamiltonian cycles," but there are natural analogues.) It is not difficult to solve the case where the graph has only finitely many ends, but the case where there are infinitely many ends is not yet understood. Sean Legge recently constructed the first family of infinite examples. Joint work with Bobby Miraftab (and with help from Agelos Georgakopoulos) constructed examples that are Cayley graphs on free groups. All of these examples are outerplanar, but we do not know whether that is always true.

https://deductivepress.ca/dmorris/talks/UniqueHamCircle.pdf

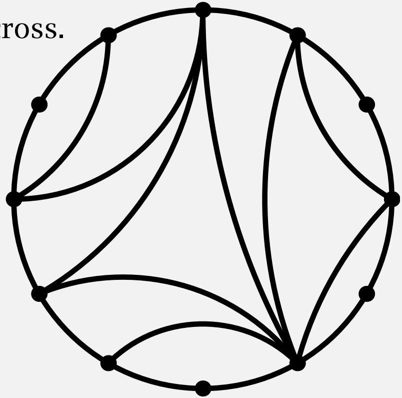
All graphs are simple. (no loops or multiple edges)

Definition

A graph is ***uniquely hamiltonian*** if it has exactly one ham cycle.

Example

Cycle with (inner) chords that do not cross.



Outerplanar graph.

Outerplanar graphs are uniquely hamiltonian.

Exercise

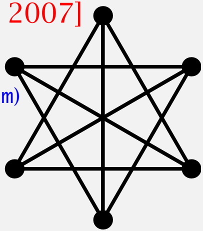
No outerplanar graphs are *regular* (except cycles). (all verts have same valency)

I.e., cycles are the only regular graphs that are outerplanar.

Conjecture (well known)

Cycles are the only regular graphs that are uniquely hamiltonian.

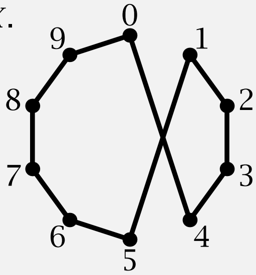
- True for k-regular if k is odd or $k \geq 23$.
  [Thomason 1978, Haxell-Seamone-Verstraete 2007]
- Suffices to show for $k = 4$.
  (cf. https://en.wikipedia.org/wiki/2-factor_theorem)
- True for ***vertex-transitive*** graphs.
  [Šajna-Wagner 2014]
  (All vertices look alike:
  $\forall$ vertices v, w, $\quad \exists$ automorphism φ, $\quad \varphi(v) = w$.



Observation

Cycles are the only vertex-transitive graphs with a unique ham cycle.

Proof. Let C be the unique hamiltonian cycle in X.



For $\varphi \in \operatorname{Aut} X$, $\varphi(C)$ is also a hamiltonian cycle,

so $\varphi(C) = C$.

$\therefore\ \varphi \in \operatorname{Aut} C = D_{2n}$ = symmetries of an n-gon

= rotations + reflections.

Since X is vertex-transitive, $\exists\ \varphi$, $\varphi(0) = 1$.

For simplicity, assume φ is a rotation,

so $\varphi(k) = k + 1$ for all k.

Assume $X \neq C$, so there is an edge $0 \mathbin{—} i$ that is not in C.

Then $1 = \varphi(0) \mathbin{—} \varphi(i) = i + 1$.

So there is another hamiltonian cycle.

Open problem

¿ $\exists$ infinitely many vert-trans graphs that do not have a ham cycle?

(Only 4 are known: Petersen, $\times 3 = 30$, Coxeter, $\times 3 = 84$)

Observation

*Cycles are the only **finite** vert-trans graphs with a unique ham cycle.*

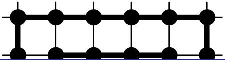
What about infinite graphs? (Assume valency of each vertex is finite.)
Need an appropriate analogue of "hamiltonian cycle."

Definition (1960s)

Two-way-infinite ham path: List all vertices without repeats.

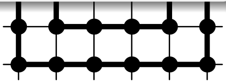
$$\cdots — v_{-1} — v_0 — v_1 — v_2 — \cdots$$

Essentially the same proof as for finite graphs:



Exercise

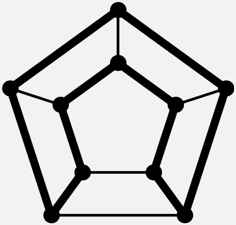
The two-way-infinite path is the only vertex-transitive graph that has a unique two-way-infinite hamiltonian path.



Two-way-infinite hamiltonian paths are very classical.
Recently, a different notion has been introduced from topology.

Note. A ham cycle must contain all of the vertices.

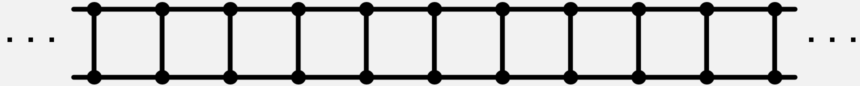
- Finite graph: topologically, a hamiltonian cycle is a copy of the circle S^1 in X.
- Infinite graph: a two-way-infinite hamiltonian path is a copy of S^1 in $X \cup \{\infty\}$: the one-point compactification of X.



Diestel and others ~2004:

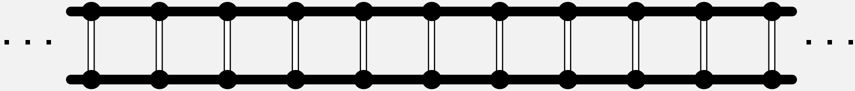
Circles in other compactifications are more interesting.
Specifically, the Freudenthal compactification $|X|$.

Example. The **infinite ladder** has two ends.



Compactify it by adding **two** points at infinity: $|X| = X \cup \{\pm\infty\}$.

Eg. Compactify the infinite ladder by adding 2 pts: $|X| = X \cup \{\pm\infty\}$.



Definition (Bruhn, popularized by Diestel ~2004)

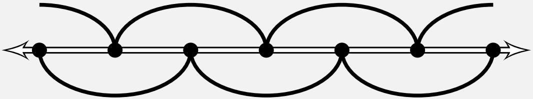
Hamiltonian circle in X is an S^1 in $|X|$ that contains all of the verts.

If X has two ends, then removing $\pm\infty$ from a hamiltonian circle results in a pair of disjoint (two-way-infinite) paths whose union contains all of the vertices, and go from one point at ∞ to the other.

Eg. Infinite ladder has a unique ham circle and is vertex-transitive.

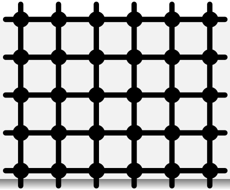
Theorem (Miraftab-Morris 2023^+)

The only other example with two ends is the square of a 2-way-infinite path:



Hamiltonian circle in X is an S^1 in $|X|$ that contains all of the verts.

Some infinite graphs have only one end: Removing a compact [i.e., finite] set of vertices results in only one infinite connected component.



Observation

A ham circle in a graph with one end is a two-way-infinite ham path.

Recall. The two-way-infinite path is the only vertex-transitive graph that has a unique two-way-infinite hamiltonian path.

Corollary

No 1-ended vertex-transitive graph has a unique hamiltonian circle.

Summary so far. Vertex-transitive graphs with a unique ham circle:

- 0 ends (i.e., finite): cycle.
- 1 end: none.
- 2 ends: ladder or square of path.

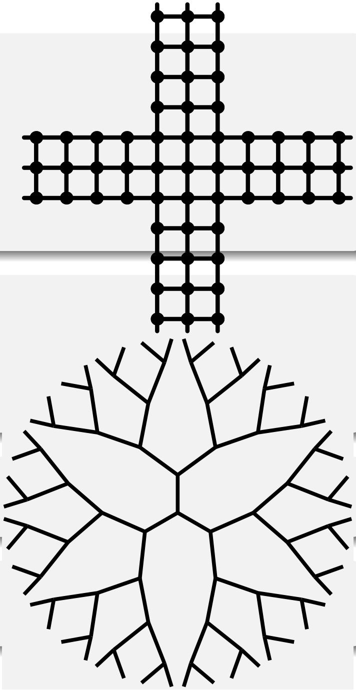
It is easy to construct graphs with any number of ends:

3 ends, 4 ends, ...

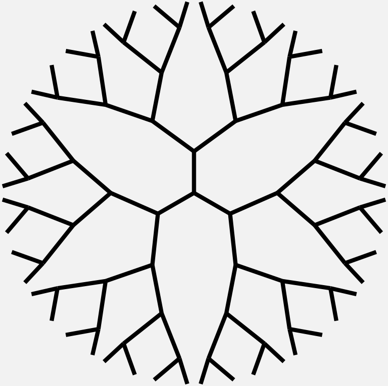
But they will not be vertex-transitive.

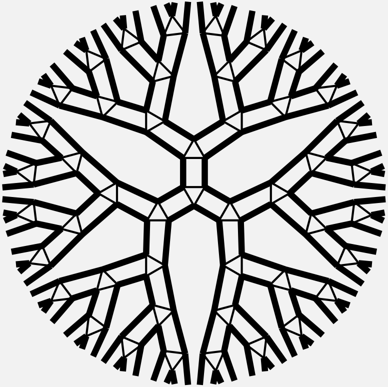
Exercise. A vertex-transitive graph with > 2 ends has ∞ ends.

Example. Regular **tree** T_m of valency m.
Easy: No hamiltonian circle.



Example (S. Legge [personal communication])





Replace each vertex of T_m with an m-cycle, and
replace each edge of T_m with two parallel edges.

The result is vertex-transitive and has infinitely many ends
(because T_m is arc-transitive and has infinitely many ends).

Also, it is **outerplanar** (dark edges form circle, other edges inside),
so it is **uniquely hamiltonian**.

Other examples.
To get a hamiltonian graph,
we can add edges to T_m.

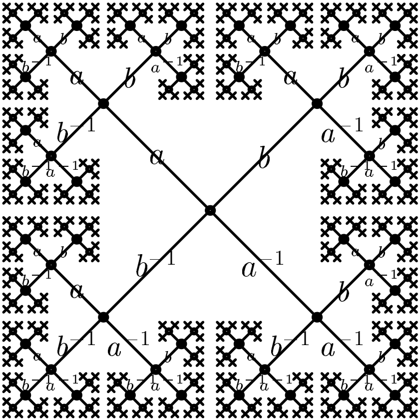
Cayley graph of a free group

$F_2 = \langle a, b \rangle = \{\text{reduced words}\}$
(don't allow xx^{-1})

This tree T_4 is $\mathrm{Cay}(F_2; a, b)$:

- vertices are the elements of F_2,
- g — gs for $g \in F_n$ and $s \in \{a, b\}$.

To add edges, replace $\{a, b\}$
with a bigger set S.



Open problem

Find all $S \subset F_2$, such that $\mathrm{Cay}(F_2; S)$ has a unique hamiltonian circle.

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Find all $S \subset F_2$*, such that* $\mathrm{Cay}(F_2; S)$ *has a unique hamiltonian circle.*

Exercise

Spse C is a unique hamiltonian circle, and let $1 \text{---} s$ be an edge of C.
Show the edge $g \text{---} gs$ is in C, $\forall g \in F_2$, I.e., $C = \mathrm{Cay}(F_2; s)$.

Theorem (Miraftab-Morris 2022^{+})

$\mathrm{Cay}(F_2; s)$ *is a hamiltonian circle in* $\mathrm{Cay}(F_2; S)$
$\Leftrightarrow\ s \in \{\, a^2 b^2,\ a b a^{-1} b^{-1} \,\}$. *(up to an automorphism of* F_2*)*

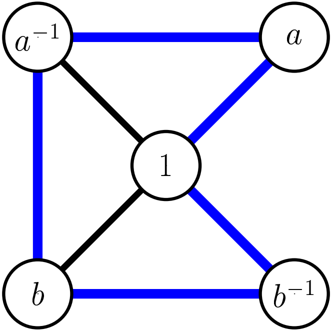
Method of proof ($\Leftarrow$).

Construct quotient graph $\mathrm{Cay}(F_2; s)_r$ by identifying
words that have the same first r letters.
Prove by induction that this is a cycle.
The result follows by passing to the limit. □

Prove by induction that $\mathrm{Cay}(F_2; s)_r$ is a cycle.

Eg. $s = a^2 b^2$.

$r = 1$



$1 \to a^2 b^2 \equiv a$

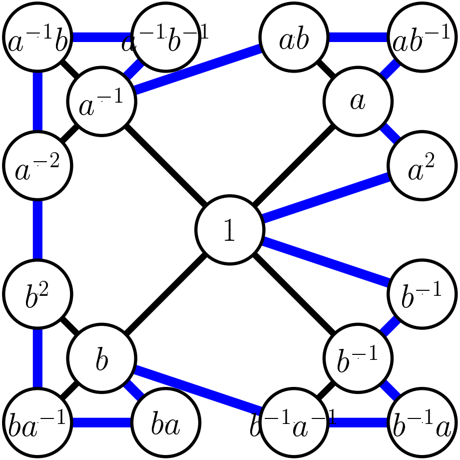
$a \equiv ab^2 \leftarrow a^{-1}$

$a^{-1} \equiv a^{-2} \to b^2 \equiv b$

$b \equiv ba^2 \leftarrow b^{-1}$

$b^{-1} \equiv b^{-2} a^{-2} \to 1$

$r = 2$

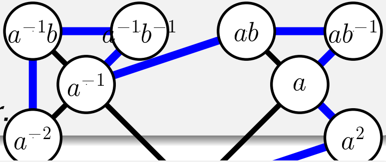


Each vertex of $\mathrm{Cay}(F_2; s)_{r-1}$ expands to a path in $\mathrm{Cay}(F_2; s)_r$.

Thm. $\mathrm{Cay}(F_2; a^2b^2)_r$ *is a cycle.*

In fact:

Thm. $\mathrm{Cay}(F_2;\, a^2b^2,\, a, b)$ *is outerplanar.*

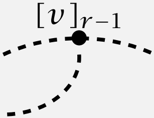


Outline of inductive proof. Each vertex of $\mathrm{Cay}(F_2; a^2b^2)_{r-1}$ is represented by a word of length $\leq r-1$.

{ words of length less than $r-1$ } induces same graph in both $\mathrm{Cay}(F_2; a^2b^2)_{r-1}$ and $\mathrm{Cay}(F_2; a^2b^2)_r$.

Now let v be a word of length $r-1$.

The vertex $[v]_{r-1}$ of $\mathrm{Cay}(F_2; a^2b^2)_{r-1}$ has valency ≤ 3:

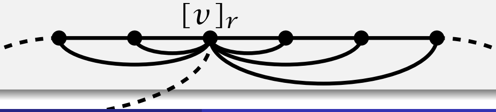


- two edges in the ham cycle $\mathrm{Cay}(F_2; s)$, and
- one edge in $\mathrm{Cay}(F_2; a, b)$ from a word of shorter length.

We also know $[v]_{r-1}$ expands to a path P in $\mathrm{Cay}(F_2; a^2b^2)_r$.

All other edges in the subgraph induced by P are incident with $[v]_r$:

These are chords of the cycle that do not cross each other.



Theorem (Miraftab-Morris 2022^+)

$\mathrm{Cay}(F_2;\ a^2b^2,\ a, b)$ *and* $\mathrm{Cay}(F_2;\ aba^{-1}b^{-1},\ a, b)$ *are outerplanar.*

Corollary

$\mathrm{Cay}(F_2;\ a^2b^2,\ a, b)$ *and* $\mathrm{Cay}(F_2;\ aba^{-1}b^{-1},\ a, b)$ *are uniquely hamiltonian.*

Uniqueness of the hamiltonian cycle was first proved by **A. Georgakopoulos** (personal communication).

Open problem

1. *Find all* S*, such that* $\mathrm{Cay}(F_2; S)$ *is uniquely hamiltonian.* (*or outerplanar or hamiltonian*)
2. *Is there a uniquely hamiltonian vertex-transtive graph that is* ***not*** *outerplanar?* An example would need to have ∞ ends.

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