A survey of extremal co-degree problems for hypergraphs

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Alberta-Montana Combinatorics and Algorithms Day

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Erdős-Stone-Simonovits theorem: for F a graph of chromatic number k we have

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- Turán Theorem: $\mathrm{ex}_2(n, K_k) \sim \left(1 - \frac{1}{k-1}\right)\binom{n}{2}$.
- ESS Theorem: $\mathrm{ex}_2(n, F) = \left(1 - \frac{1}{\chi(F)-1}\right)\binom{n}{2} + o(n^2)$.

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- Turán Tetrahedron Conjecture: $\mathrm{ex}_3(n, K_4^3) \sim \frac{5}{9}\binom{n}{3}$.

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Does the limit exist? Yes, the function $\frac{\mathrm{ex}_r(n,F)}{\binom{n}{r}}$ is monotone increasing and bounded above by 1.

Proposition (Katona-Nemetz-Simonovits, 1964)

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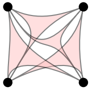
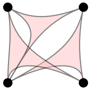
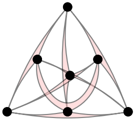
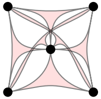
$$\frac{\mathrm{ex}_r(n,F)}{\binom{n}{r}} \leq \frac{n}{n-r}\frac{\mathrm{ex}_r(n-1,F)}{\binom{n}{r}} = \frac{\mathrm{ex}_r(n-1,F)}{\binom{n-1}{r}}. \quad \square$$

F		$\leq \pi(F)$	$\pi(F) \leq$
K_4^3		$5/9$ (Turán)	0.5615 (Baber)
K_4^-		$2/7$ (Frankl-Füredi)	0.28689 (Vaughn)
$\mathbb{F}$		$3/4$ (Sós)	$3/4$ (de Caen-Füredi)
J_4		$1/2$ (Bollobás-Leader-Malvenuto)	0.50409 (Vaughn)

Theorem (Supersaturation)

Fix an r-graph F. For $\varepsilon > 0$, there exists $\delta > 0$ such that if G is an n-vertex r-graph with

$$|E(G)| > (\pi(F) + \varepsilon)\binom{n}{r},$$

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The idea behind the supersaturation theorem is that if we exceed the extremal number of F by enough, then not only must there be a copy of F, but there will be many.

Theorem (Stability)

Fix a 2-graph F. For $\varepsilon > 0$, there exists $\delta > 0$ such that if G is an n-vertex F-free 2-graph with

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- Only for $r = 2$ as the "extremal graph" for F is essentially unique.
- There are many non-isomorphic "extremal 3-graphs" for K_4^3, so a stability theorem seems impossible.
- On the other hand, the extremal graph for the Fano plane $\mathbb{F}$ is unique and has been shown to be stable.

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Theorem (Erdős Degeneracy Theorem)

An r-partite r-graph F has Turán number $\mathrm{ex}(n, F) = o(n^r)$, i.e, $\pi(F) = 0$.

This implies no density in the range $(0, \frac{r!}{r^r})$, e.g., for 3-graphs no value in $\left(0, \frac{2}{9}\right)$.

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Erdős question: Is there a δ such that no 3-graph satisfies $\pi(F) \in \left(\frac{2}{9}, \frac{2}{9} + \delta\right)$?

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Erdős question: Is there a δ such that no 3-graph satisfies $\pi(F) \in \left(\frac{2}{9}, \frac{2}{9} + \delta\right)$? Does density $\frac{2}{9}$ jump?

What densities jump for r-graphs?

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HYPERGRAPHS DO NOT JUMP

P. FRANKL and V. RÖDL

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The number α, $0 \leqq \alpha \leqq 1$, is a jump for r if for any positive ε and any integer m, $m \geqq r$, any r-uniform hypergraph with $n > n_0(\varepsilon, m)$ vertices and at least $(\alpha + \varepsilon)\binom{n}{r}$ edges contains a subgraph with m vertices and at least $(\alpha + c)\binom{m}{r}$ edges, where $c = c(\alpha)$ does not depend on ε and m. It follows from a theorem of Erdős, Stone and Simonovits that for $r = 2$ every α is a jump. Erdős asked whether the same is true for $r \geqq 3$. He offered \$ 1000 for answering this question. In this paper we give a negative answer by showing that $1 - \frac{1}{l^{r-1}}$ is not a jump if $r \geqq 3$, $l > 2r$.

What densities jump for r-graphs?

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Hypergraphs Do Jump

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We say that $\alpha \in [0, 1)$ is a *jump* for an integer $r \geqslant 2$ if there exists $c(\alpha) > 0$ such that for all $\epsilon > 0$ and all $t \geqslant 1$, any r-graph with $n \geqslant n_0(\alpha, \epsilon, t)$ vertices and density at least $\alpha + \epsilon$ contains a subgraph on t vertices of density at least $\alpha + c$.

The Erdős–Stone–Simonovits theorem [4, 5] implies that for $r = 2$, every $\alpha \in [0, 1)$ is a jump. Erdős [3] showed that for all $r \geqslant 3$, every $\alpha \in [0, r!/r^r)$ is a jump. Moreover he made his famous 'jumping constant conjecture', that for all $r \geqslant 3$, every $\alpha \in [0, 1)$ is a jump. Frankl and Rödl [7] disproved this conjecture by giving a sequence of values of non-jumps for all $r \geqslant 3$.

We use Razborov's flag algebra method [9] to show that jumps exist for $r = 3$ in the interval $[2/9, 1)$. These are the first examples of jumps for any $r \geqslant 3$ in the interval $[r!/r^r, 1)$. To be precise, we show that for $r = 3$ every $\alpha \in [0.2299, 0.2316)$ is a jump.

We also give an improved upper bound for the Turán density of $K_4^- = \{123, 124, 134\}$: $\pi(K_4^-) \leqslant 0.2871$. This in turn implies that for $r = 3$ every $\alpha \in [0.2871, 8/27)$ is a jump.

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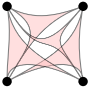
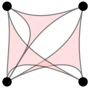
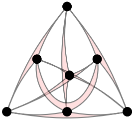
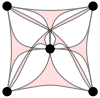
Does the limit exist? Yes, due to a nice probablistic argument of Mubayi and Zhao.

F		$\leq \gamma(F)$	$\gamma(F) \leq$
K_4^3		$1/2$ (Nagle-Czygrinow)	0.529 (Balogh-Clemens-Lidický)
K_4^-		$1/4$ (Nagle)	$1/4$ (Falgas-Ravry-Pikhurko-Vaughan-Volec)
$\mathbb{F}$		$1/2$ (Mubayi)	$1/2$ (Mubayi)
J_4		$1/4$ (Balogh-Clemens-Lidický)	0.473 (Balogh-Clemens-Lidický)

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- Claim: H has co-degree $\frac{n}{2} - o(n)$ with positive probability.

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- Claim: H is K_4^3-free.
- Claim: H has co-degree $\frac{n}{2} - o(n)$ with positive probability.

If we instead make a 3-edge if ij, jk and ki are arcs, then H will be K_4^--free with co-degree $\frac{n}{4} - o(n)$.

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Jumps (Mubayi-Zhao): The values of $\gamma(F)$ are dense in $[0, 1]$. They conjecture that all values in $[0, 1]$ are possible.

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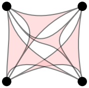
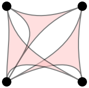
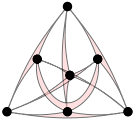
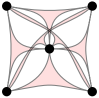
Does the limit exist? Supersaturation?

Define the **positive co-degree density** of r-graph F as

$$\gamma(F) := \lim_{n\to\infty} \frac{\mathrm{co}^{+}\mathrm{ex}(n, F)}{n}.$$

This is the maximum possible minimum positive co-degree of an n-vertex F-free r-graph normalized by n.

Does the limit exist? Supersaturation? Yes, but you'll have to wait for the next talk!

F		$\leq \gamma^+(F)$	$\gamma^+(F) \leq$
K_4^3		$1/2$ (Halfpap-Palmer-Lemons)	0.54296 (Volec)
K_4^-		$1/3$ (Halfpap-Palmer-Lemons)	$1/3$ (Halfpap-Palmer-Lemons)
$\mathbb{F}$		$2/3$ (Halfpap-Palmer-Lemons)	$2/3$ (Halfpap-Palmer-Lemons)
J_4		$1/2$ (Halfpap-Palmer-Lemons)	0.58 (Balogh-Lidický)

Stability:

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Jumps: Still open in general.

- No value of $\gamma^{+}(F)$ in $(0, \frac{1}{3})$ (Halfpap-Lemons-Palmer, '23+).
- No value in $(\frac{1}{3}, \frac{2}{5})$ (Balogh-Halfpap-Lidický-Palmer, '23+).

F	$\leq \pi(F)$	$\pi(F) \leq$	$\leq \gamma(F)$	$\gamma(F) \leq$	$\leq \gamma^+(F)$	$\gamma^+(F) \leq$
K_4	$5/9$	0.5615	$1/2$	0.529	$1/2$	0.54296
K_4^-	$2/7$	0.28689	$1/4$	$1/4$	$1/3$	$1/3$
$\mathbb{F}$	$3/4$	$3/4$	$1/2$	$1/2$	$2/3$	$2/3$
F_5	$2/9$	$2/9$	0	0	$1/3$	$1/3$
$F_{3,2}$	$4/9$	$4/9$	$1/3$	$1/3$	$1/2$	$1/2$
J_4	$1/2$	0.50409	$1/4$	0.473	$1/2$	0.58
$F_{3,3}$	$3/4$	$3/4$	$1/2$	0.604	$3/5$	0.616
C_5	$2\sqrt{3}-3$	0.46829	$1/3$	0.3993	$1/2$	$1/2$
C_5^-	$1/4$	0.25074	0	0.136	$1/3$	$1/3$

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	Turán $\pi(F)$	Co-degree $\gamma(F)$	Pos. co-degree $\gamma^+(F)$
Supersat.	✓	✓	✓
Stability	$r=2$	K_4^-	K_4^-
Jumps	strange	dense in $[0,1]$	No $(0,\frac{1}{3})\cup(\frac{1}{3},\frac{2}{5})$

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Stability	$r = 2$	K_4^-	K_4^-
Jumps	strange	dense in $[0, 1]$	No $(0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{5})$

Thank you for your attention!