Robust hedging of American options in continuous time

Ivan Guo

School of Mathematics
Centre for Quantitative Finance and Investment Strategies
Monash University

BIRS Workshop — Stochastic Mass Transports
March 2022

Joint work with Gregoire Loeper and Jan Obłój

Section 1

Path-dependent optimal transport

Semimartingale optimal transport in continuous time

(Tan & Touzi (2013); Huesmann & Trevisan (2017); Backhoff-Veraguas, Beiglböck, Huesmann & Källblad (2017), etc.) Consider probability measures $\mathbb{P}$ such that X is a semimartingale,

$$X_t = X_0 + \int_0^t \alpha_s^{\mathbb{P}}\, ds + M_t, \quad \langle X\rangle_t = \langle M\rangle_t = \int_0^t \beta_s^{\mathbb{P}}\, ds, \quad \mathbb{P}\text{-a.s.},$$

We say $\mathbb{P}$ has *characterstics* $(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$.

Semimartingale optimal transport problem

We want to minimise

$$V(\mu_0, \mu_1) = \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt,$$

where $\mathcal{P}(\mu_0, \mu_1)$ contains probability measures satisfying

$$\mathbb{P}\circ X_0^{-1} = \mu_0, \quad \mathbb{P}\circ X_1^{-1} = \mu_1.$$

The cost function H is convex in (α, β) and may depend on (t, X) as well.

Path-dependent constraints

Instead of the marginal constraint $\mathbb{P} \circ X_1^{-1} = \mu_1$, how about other types of constraints? For example:

$$\mathbb{E}^{\mathbb{P}} X_1 = c, \quad \mathbb{E}^{\mathbb{P}} G(X) = c, \quad \mathbb{P} \circ G^{-1} = \rho, \quad \mathbb{P}(G(X) \leq 0) \leq c, \quad \text{etc.}$$

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General abstract constraints

Let $\mathcal{N} \subseteq \mathcal{P}$ be a convex subset that is closed with respect to the weak topology and define $F : C_b(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by $F(\psi) := \sup_{\mu \in \mathcal{N}} \int_\Omega \psi \, d\mu$.

$$F^*(\mu) = \sup_{\psi \in C_b(\Omega)} \int_\Omega \psi \, d\mu - F(\psi) = \begin{cases} 0, & \mu \in \mathcal{N}, \\ +\infty, & \mu \notin \mathcal{N}. \end{cases}$$

This function penalises measures outside $\mathcal{N}$. Some examples:

$$\mathbb{E}^{\mathbb{P}} G(X) = c \quad \Longrightarrow \quad F^*(\mu) = \sup_{\lambda \in \mathbb{R}^m} \lambda \cdot (c - \mathbb{E}^{\mu}(G(X))),$$

$$\mathbb{P} \circ G^{-1} = \rho \quad \Longrightarrow \quad F^*(\mu) = \sup_{\lambda \in C_b(\mathbb{R}^m)} \int_{\mathbb{R}^m} \lambda (d\rho - d\mu).$$

Path-derivatives

Space of all paths: $\Omega = C([0,1], \mathbb{R}^d)$, X is the canonical process.
Space of all stopped paths: $\Lambda = \{(t, \omega_{\cdot \wedge t}) : t \in [0,1], \omega \in \Omega\}$.

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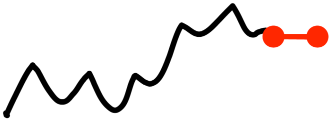
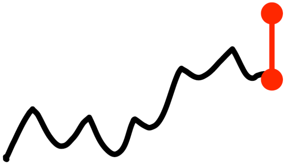
Dupire (2009) introduced non-anticipative *path-derivatives* operating on functions $C^{1,2}(\Lambda)$. Also see Cont & Fournié (2013); Ekren, Touzi & Zhang. (2016).

- $\mathcal{D}_t$: a time derivative where we extend forward in time by dt while remaining constant in space.



- ∇_x, ∇_x^2: space derivatives where we perturb the end point by dx.



Functional Itô formula

Formally, we define the path-derivatives using the functional Itô formula.

Definition

We say $\phi \in C^{1,2}(\Lambda)$ if there exist functions $(\mathcal{D}_t\phi, \nabla_x\phi, \nabla_x^2\phi) \in C(\Lambda; \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d)$ such that, for any semimartingale measure $\mathbb{P}$, the following *functional Itô formula* holds:

$$\phi(t, X) - \phi(0, X) = \int_0^t \mathcal{D}_t\phi\, dt + \nabla_x\phi \cdot dX_t + \frac{1}{2}\nabla_x^2\phi : d\langle X\rangle_t, \quad \mathbb{P}\text{-a.s.}$$

The functions $\mathcal{D}_t\phi, \nabla_x\phi, \nabla_x^2\phi$ are known as the time derivative, first order space derivative and second order space derivative of ϕ, respectively.

Note that $A : B = \mathrm{tr}(A^T B)$ for matrices A and B.

Langrange multiplier for "semimartingale measures"

Lemma

Suppose that $\mu \in \mathcal{M}_+(\Omega)$ *and* $\nu \in \mathcal{M}_+(\Lambda)$*. Then we have the equality*

$$\int_\Omega \phi(1,\cdot)\, d\mu - \int_{\Omega_0} \phi(0,\cdot)\, d\rho_0 = \int_\Lambda \mathcal{D}_t\phi + \alpha \cdot \nabla_x \phi + \frac{1}{2}\beta : \nabla_x^2 \phi\, d\nu \tag{1}$$

holds for all $\phi \in C^{1,2}(\Lambda)$ *if and only if all of the following hold:*
(a) $d\mu \times dt = d\nu$,
(b) $\mu \in \mathcal{P}(\rho_0)$ *and*
(c) X *is a* μ*-semimartingale with characteristics* (α, β).

Let us rewrite (1) using the shorthand $\mathcal{L}(\phi, \mu, \nu, \alpha, \beta) = 0$

Path-dependent optimal transport

Our problem is

$$
\begin{aligned}
V &= \inf_{\mathbb{P}\in\mathcal{P}(\rho_0)} \int_\Lambda H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt d\mathbb{P} \quad \text{s.t. } \mathbb{P}\in\mathcal{N},\ X \text{ is a } \mathbb{P}\text{-semimartingale} \\
&= \inf_{\mathbb{P}\in\mathcal{P}(\rho_0)} F^*(\mathbb{P}) + \int_\Lambda H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt d\mathbb{P} \quad \text{s.t. } X \text{ is a } \mathbb{P}\text{-semimartingale} \\
&= \inf_{\substack{\mu\in\mathcal{M}_+(\Omega),\\ \nu\in\mathcal{M}_+(\Lambda),\\ (\alpha,\beta)\in L^1(\Lambda,\nu)}} \sup_{\substack{\phi\in\bar{C}_0^{1,2}(\Lambda),\\ \psi\in C_b(\Omega)}} \int_\Omega \psi\, d\mu - F(\psi) + \int_\Lambda H(\alpha,\beta)\, d\nu - \mathcal{L}(\phi,\mu,\nu,\alpha,\beta)
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We want to swap the inf with the sup.

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We want to swap the inf with the sup.

Use the Fenchel-Rockafellar duality theorem, operating on dual pairings of the form

$$(\mu,\nu,\bar{\nu},\tilde{\nu}) \quad \text{and} \quad (\phi_1+\psi,\mathcal{D}_t\phi,\nabla_x\phi,\nabla_x^2\phi),$$

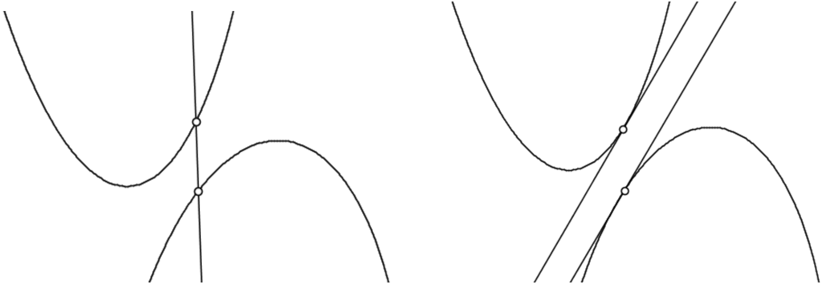
where $d\bar{\nu}=\alpha d\nu$ and $d\tilde{\nu}=\beta d\nu$.

Fenchel-Rockafellar duality theorem

Let f be convex and g be concave. Let f^* and g_* be the respective convex and concave conjugates. Under some conditions,

$$\inf_{x \in \mathcal{X}} f(x) - g(x) = \sup_{x^* \in \mathcal{X}^*} g_*(x^*) - f^*(x^*),$$

$$\inf_{x \in \mathcal{X}} \sup_{x^* \in \mathcal{X}^*} f(x) + g_*(x^*) - \langle x, x^* \rangle = \sup_{x^* \in \mathcal{X}^*} \inf_{x \in \mathcal{X}} g_*(x^*) + f(x) - \langle x, x^* \rangle.$$



Main duality result

Theorem

$$V = \inf_{\mathbb{P}\in\mathcal{P}(\rho_0)} \int_\Lambda H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dtd\mathbb{P} \quad \textit{s.t. } \mathbb{P}\in\mathcal{N}, \ X \textit{ is a } \mathbb{P}\textit{-semimartingale}$$

$$= \inf_{\mu,\nu,\alpha,\beta} \sup_{\phi,\psi} \int_\Omega \psi\, d\mu - F(\psi) + \int_\Lambda H(\alpha,\beta)\, d\nu - \mathcal{L}(\phi,\mu,\nu,\alpha,\beta)$$

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$$= \sup_{\psi\in C_b(\Omega),\phi\in C^{1,2}(\Lambda)} -F(\psi) - \int_{\Omega_0} \phi(0,\cdot)\, d\rho_0,$$

$$\textit{s.t.} \quad \phi(1,\cdot) \geq -\psi \quad \textit{and} \quad \mathcal{D}_t\phi + H^*\left(\nabla_x\phi, \frac{1}{2}\nabla_x^2\phi\right) \leq 0.$$

Properties of optimiser

$$\inf_{\mathbb{P}\in\mathcal{P}(\rho_0)\cap\mathcal{N}} \int_\Lambda H(\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})\, dt d\mathbb{P} = \sup_{\psi\in C_b(\Omega), \phi\in C^{1,2}(\Lambda)} -F(\psi) - \int_{\Omega_0} \phi(0,\cdot)\, d\rho_0,$$

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- If (ψ^n, ϕ^n) is a maximising sequence of the dual problem, then

$$\phi^n + \psi^n \overset{d\tilde{\mathbb{P}}}{\to} 0, \quad \mathcal{D}_t\phi^n + H^*\left(\nabla_x\phi^n, \frac{1}{2}\nabla_x^2\phi^n\right) \overset{d\tilde{\mathbb{P}}\times dt}{\to} 0,$$

$$\nabla H^*\left(\nabla_x\phi^n, \frac{1}{2}\nabla_x^2\phi^n\right) \overset{d\tilde{\mathbb{P}}\times dt}{\to} (\tilde{\alpha}, \tilde{\beta}).$$

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- Under some conditions, via partial comparison principles for PPDEs, we recover the HJB equation without using the dynamic programming principle.

Optimal transport for volatility calibration

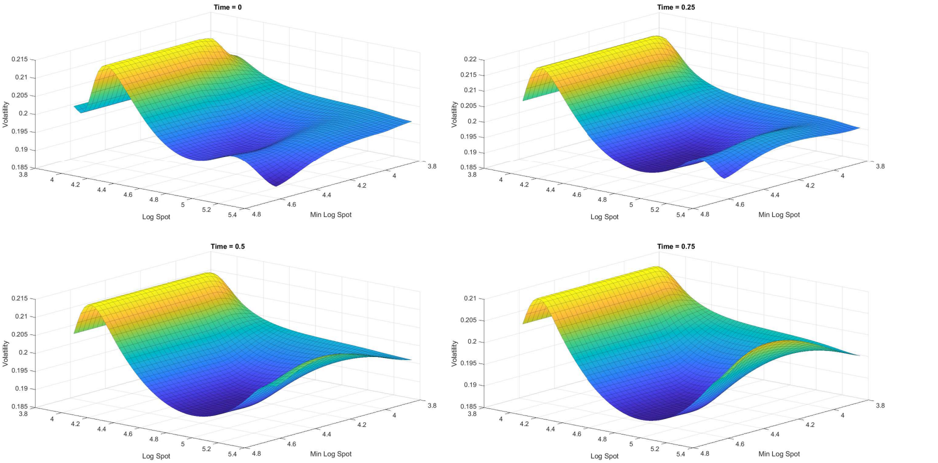

Figure: Volatility $\sigma(t, x, y)$ (y is the running minimum) calibrated to European puts, down-and-out puts (all possible barriers) and lookback puts, at all strikes and four different maturities. The figure shows the t cross sections.

Section 2

Robust hedging in continuous time

Robust hedging: model uncertainty

Consider a market with stocks X and some European claims g which WLOG have initial prices of 0. We are allowed to trade X dynamically and g statically.

Let $\mathcal{Q} \subset \mathcal{P}$ be the set of possible "models", i.e., X is martingale, g has zero expectation, etc.

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Consider a European claim Z. Worst case model price:

$$\sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Super-hedging price:

$$\pi(Z) := \inf\{x : \exists (q, h), \text{ s.t. } x + \int_0^1 q \cdot dX_t + h \cdot g \geq Z, \ \mathcal{Q}\text{-q.s.}\}.$$

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It is easy to check that

$$\pi(Z) \geq \sup_{\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z.$$

Duality (equality) results are obtained in various settings by Denis & Martini (2006); Soner, Touzi & Zhang (2013); Neufeld & Nutz (2013); and Possamaï, Royer & Touzi (2013); Hou & Obłój (2018) and many more.

Hedging duality via Path-dependent OT

Theorem

Let $H : \Lambda \times \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\}$ *satisfy some assumptions and* $H^*(t, \omega, \cdot)$ *be the convex conjugate of* $H(t, \omega, \cdot)$*. Define*

$$V := \sup_{\mathbb{P}} \inf_{h \in \mathbb{R}^m} \mathbb{E}^{\mathbb{P}}(-h \cdot g + Z) - \mathbb{E}^{\mathbb{P}} \int_0^1 H(\beta_t^{\mathbb{P}})\, dt,$$

$$\mathcal{V} := \inf_{h \in \mathbb{R}^m, \phi \in C^{1,2}(\Lambda)} \phi(0, X_0),$$

subject to $\phi(1, \cdot) \geq Z - h \cdot g$ *and* $\mathcal{D}_t \phi + H^* \left(\frac{1}{2} \nabla_x^2 \phi \right) \leq 0.$

Then $V = \mathcal{V}$*. Moreover, if* V *is finite, then the supremum is attained.*

Hedging duality via Path-dependent OT

Let $H(\beta)$ be 0 if $\beta \in D$ (volatility constraint), or ∞ otherwise. Then the dual is

$$\mathcal{V} = \inf_{h \in \mathbb{R}^m, \phi \in C^{1,2}(\Lambda)} \phi(0, X_0),$$

$$\text{subject to} \quad \phi(1, \cdot) \geq Z - h \cdot g \quad \text{and} \quad \mathcal{D}_t \phi + \sup_{\beta \in D} \frac{1}{2} \nabla_x^2 \phi : \beta \leq 0. \qquad (2)$$

Each ϕ is actually a super-hedge. For every $\mathbb{P} \in \mathcal{Q}$

$$\begin{aligned} Z - h \cdot g - \phi(0, X_0) &\leq \phi(1, X) - \phi(0, X_0) \\ &= \int_0^1 (\mathcal{D}_t \phi + \frac{1}{2} \beta^{\mathbb{P}} : \nabla_x^2 \phi) dt + \nabla_x \phi \cdot dX_t, \quad \mathbb{P}\text{-a.s.} \\ &\leq \int_0^1 \nabla_x \phi \cdot dX_t. \end{aligned}$$

Hence $\phi(0, X_0) \geq \pi(Z)$. Since this works for all ϕ satisfying (2), it implies

$$\mathcal{V} = \inf_{\phi \in C_0^{1,2}(\Lambda), (2)} \phi(0, X_0) \geq \pi(Z) \geq \sup_{\mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z = V = \mathcal{V},$$

Robust hedging American options

Let Z be an American-style claim. Worst case model price:

$$\sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$$

Super-hedging price:

$$\pi^A(Z) := \inf\{x : \exists (p, q, h) \text{ s.t.}$$
$$x + \int_0^\tau p \cdot dX_t + \int_\tau^1 q^\tau \cdot dX_t + hg \geq Z_\tau, \mathcal{Q}^D\text{-q.s.}, \forall \tau \in \mathcal{T}\}.$$

Again, it is easy to check $\pi^A(Z) \geq \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot)$.

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When the set of statically traded European options is non-empty, there may be a duality gap, which can be eliminated by enlarging the probability space.

In discrete time, various duality results for American options are obtained by Dolinsky (2014); Hobson & Neuberger (2017); Bayraktar & Zhou (2017); Aksamit, Deng, Obłój & Tan (2019); and more. Some relevant works in continuous time include Herrmann & Stebegg (2017); Tiplea (2019); Grigorova, Quenez & Sulem (2021) etc.

Overview

The main idea of Aksamit et al. (2019) is to enlarge the space Ω with the stopping decisions to obtain $\bar{\Omega}$. Then the American option can be seen as a European option under the enlarged space.

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In the case where there is no statically traded European options g.

$$\bar{\pi}(Z) = \pi^A(Z) \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}(Z),$$

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When g does exist, then we have to introduce a second enlarged space $\hat{\Omega}$ which includes the price process of g as another martingale.

$$\bar{\pi}_g(Z) = \pi_g^A(Z) \geq \widehat{\pi}^A(Z) = \bar{\widehat{\pi}}(Z) \geq \sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} Z \geq \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}_g} \mathbb{E}^{\bar{\mathbb{P}}} Z = \bar{\pi}_g(Z),$$

$$\sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} Z = \sup_{\widehat{\tau} \in \widehat{\mathcal{T}}, \widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}} \mathbb{E}^{\widehat{\mathbb{P}}} Z(\widehat{\tau}, \cdot).$$

Continuous time issues

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The equality $\bar{\pi}(Z) = \pi^A(Z)$ can also be argued in mostly the same way.

However, the equality

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot),$$

creates difficulties in continuous time. Possible approaches include approximating with discrete time, Doob-Meyer type decomposition of non-linear Snell envelopes, reflected 2BSDEs, etc.

Section 3

Duality for American options: Convexifying stopping times and martingale measures

Enlarged space

The original space for our model is $\Omega := C([0,1];\mathbb{R}^d)$ with canonical process X. We enlarge it to $\bar{\Omega} := \Theta \times \Omega$ where

$$\Theta := \{\vartheta \in C([0,1],\mathbb{R}) : \vartheta_t = \theta \wedge t, \text{ for some } \theta \in [0,1]\}.$$

Θ is isometric to $[0,1]$.

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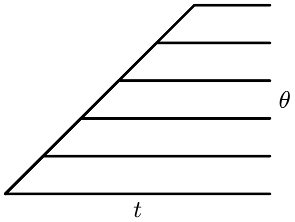

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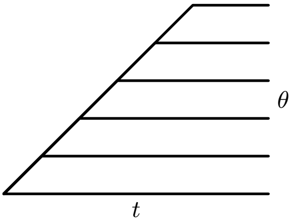

Most aspects of Ω can be naturally extended to $\bar{\Omega}$, include semimartingale measures (since ϑ semimartingale with characteristics $(\mathbb{1}(t \leq \theta), 0)$). E.g., we define $\bar{\mathcal{Q}}$ to be the set of measures under which X is a martingale.

Path-dependent optimal transport on $\bar{\Omega}$

Also define the "stopped paths" of $\bar{\Omega}$, by

$$\bar{\Lambda} := \{(t, \bar{\omega}_{\cdot\wedge t}) : t \in [0,1], \bar{\omega} \in \bar{\Omega}\}.$$

So elements of $\bar{\Lambda}$ are $(t, \bar{\omega}_{\cdot\wedge t}) = (t, \vartheta_{\cdot\wedge t}, \omega_{\cdot\wedge t}) = (t, \theta \wedge t, \omega_{\cdot\wedge t})$.
Functional Itô calculus and PPDEs can be extended in the same way.

The path-dependent optimal transport duality results (Guo and Loeper (2021)) can be applied here.

Theorem

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f = \inf_{\phi \in C_0^{1,1,2}(\bar{\Lambda})} \phi(0,0,X_0),$$

subject to $\phi(1,\cdot,\cdot) \geq f$ *and* $\mathcal{D}_t \phi + \mathbb{1}(t \leq \theta) \nabla_\theta \phi + \sup_{\beta \in D} \frac{1}{2} \beta : \nabla_x^2 \phi \leq 0.$

European duality on the enlarged space

Theorem

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By the functional Itô formula, for each ϕ and $\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D$, the following holds $\bar{\mathbb{P}}$-a.s.

$$\begin{aligned} f - \phi(0, 0, X_0) &\leq \phi(1, \cdot, \cdot) - \phi(0, 0, X_0) \\ &= \int_0^1 (\mathcal{D}_t \phi + \mathbb{1}(t \leq \theta) \nabla_\theta \phi + \frac{1}{2} \beta^{\mathbb{P}} : \nabla_x^2 \phi) dt + \nabla_x \phi \cdot dX_t \\ &\leq \int_0^1 \nabla_x \phi \cdot dX_t. \end{aligned}$$

Hence $\phi(0, 0, X_0) \geq \bar{\pi}(f)$. Since this holds for all ϕ, it implies

$$\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}^D} \mathbb{E}^{\bar{\mathbb{P}}} f \geq \bar{\pi}(f).$$

Convexifying measures and stopping times

We still need to address $\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau\in\mathcal{T},\mathbb{P}\in\mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau,\cdot)$.

Given a pair $(\tau,\mathbb{P}) \in \mathcal{T} \times \mathcal{P}(\Omega)$, we can associate it with $\bar{\mathbb{P}} \in \mathcal{P}(\bar{\Omega})$ so that $\mathbb{E}^{\mathbb{P}} Z(\tau,\cdot) = \mathbb{E}^{\bar{\mathbb{P}}} Z$. So $\mathcal{P}(\bar{\Omega})$ contains the "convex hull" of $\mathcal{T} \times \mathcal{P}(\Omega)$.

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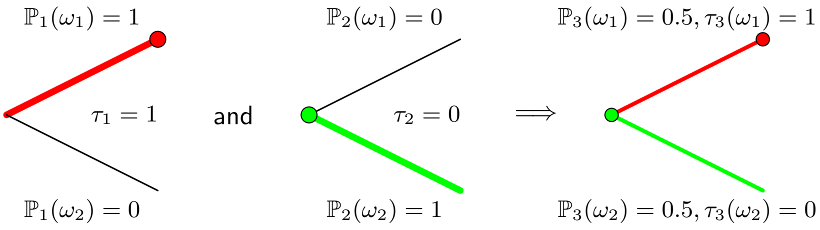
But $\bar{\mathbb{P}} \in \mathcal{P}(\bar{\Omega})$ in general may correspond to "proper" random times, not even randomised stopping times.



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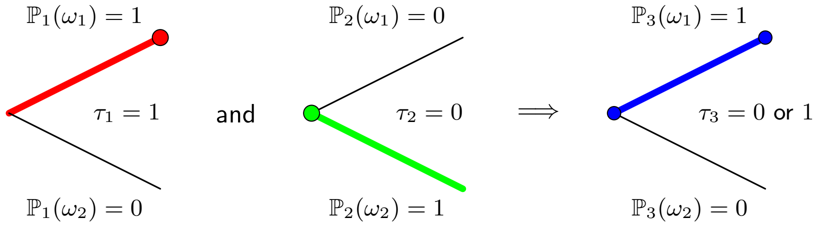

If we choose another measure carefully and *only test against non-anticipative functions*, we can recover randomised stopping times. We also want to preserve martingale properties which is tricky.

A few concepts

Definition

Given a random time ρ, let $R := \mathbb{1}(\rho \le t)$. Then the *Azéma supermartingale* is defined as

$$Y = 1 - {}^{o}R = \mathbb{P}(\rho > t \,|\, \mathcal{F}_t).$$

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Theorem (Itô-Watanabe)

Let Y be a non-negative càdlàg supermartingale with $Y_0 > 0$. Then we have the decomposition

$$Y = M(1 - A)$$

where M is a positive local martingale, A is a right-continuous, increasing process with $A_0 = 0$. The decomposition is unique up to $\tau = \inf\{t : Y_t = 0\}$.

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Definition (Shmaya & Solan)

A *randomised stopping time* corresponds to an adapted, right-continuous, increasing process A with $A_0 = 0$ and $A_1 = 1$.

From random times to stopping times

Lemma

(a) For every $\mu \in \mathcal{P}(\bar{\Omega})$, there exists an increasing and adapted A with $A_0 = 0$ and $A_1 = 1$, and $\mathbb{P} \in \mathcal{P}(\Omega)$ with $\mathbb{P} \ll \mu_\Omega$, such that for every (non-anticipative) $\psi \in L^\infty(\Lambda)$,

$$\mu(\psi(\theta, \omega_{\cdot \wedge \theta})) = \mathbb{E}^{\mathbb{P}} \int_0^1 \psi(t, \omega_{\cdot \wedge t}) \, dA_t.$$

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Lemma

(a) For every $\mu \in \mathcal{P}(\bar{\Omega})$*, there exists an increasing and adapted* A *with* $A_0 = 0$ *and* $A_1 = 1$*, and* $\mathbb{P} \in \mathcal{P}(\Omega)$ *with* $\mathbb{P} \ll \mu_\Omega$*, such that for every (non-anticipative)* $\psi \in L^\infty(\Lambda)$*,*

$$\mu(\psi(\theta, \omega_{\cdot\wedge\theta})) = \mathbb{E}^{\mathbb{P}} \int_0^1 \psi(t, \omega_{\cdot\wedge t})\, dA_t.$$

Proof outline: Define the raw IV process R from μ, for any $E \in \mathcal{F}_s$,

$$\int_{\bar{\Omega}} \mathbb{1}(\theta > s, \omega \in E)\, d\mu = \mathbb{E}^{\mu_\Omega}\big((1 - R_s)\mathbb{1}(\omega \in E)\big).$$

Then $1 - {}^oR$ is a non-negative μ_Ω-supermartingale (the Azéma supermartingale of the random time associated with μ). It has the (Itô-Watanabe) multiplicative decomposition $1 - {}^oR = M(1 - A)$, so

$$\mathbb{E}^{\mu_\Omega}\big((1 - R_s)\mathbb{1}(\omega \in E)\big) = \mathbb{E}^{\mu_\Omega}\big((1 - {}^oR_s)\mathbb{1}(\omega \in E)\big)$$
$$= \mathbb{E}^{\mu_\Omega}\big(M_s(1 - A_s)\mathbb{1}(\omega \in E)\big) = \mathbb{E}^{\mathbb{P}}\big((1 - A_s)\mathbb{1}(\omega \in E)\big),$$

where $d\mathbb{P}/d\mu_\Omega = M_1$.

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(b) For every $\mu \in \mathcal{P}(\bar{\Omega})$*, there exists a family of true stopping times* τ_r *and probability measures* $\mathbb{P}^r$*, indexed by* $r \in [0, 1]$*, such that for every* $\eta \in L^\infty(\bar{\Omega})$*,*

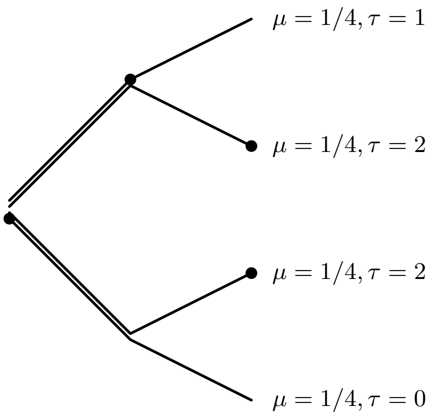
$$\mu(\eta(\theta, \omega)) = \int_0^1 \mathbb{E}^{\mathbb{P}^r} \eta(\theta = \tau_r, \omega)\, dr.$$

(c) For any $a \in [0, 1]$ *and any bounded and* $\mathcal{F}_{\tau_a}$*-measurable function* γ*,*

$$\int_a^1 \mathbb{E}^{\mathbb{P}^r} \gamma\, dr = (1 - a)\mathbb{E}^{\mathbb{P}} \gamma.$$

Roughly speaking, we obtain r by disintegrating μ according to the value of A.

Why so complicated?



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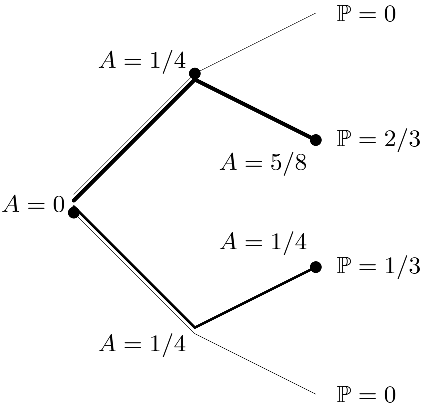

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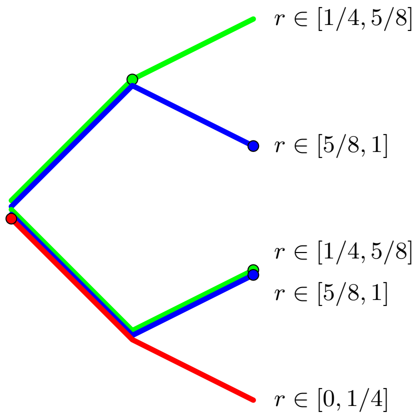

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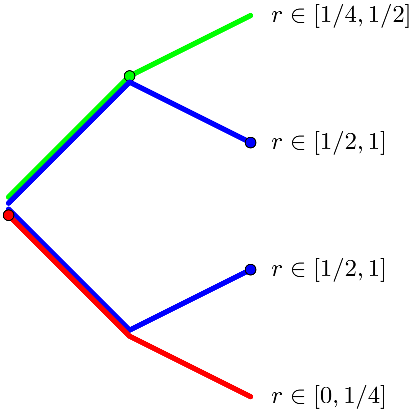
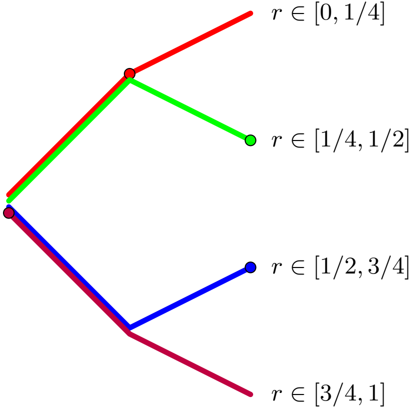
Why not disintegrate via t?

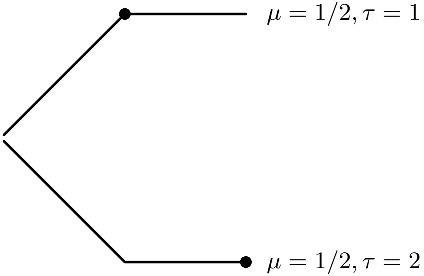


Why so complicated?

Why not disintegrate via t? Or even just ω?



...because we want to preserve martingale measures!



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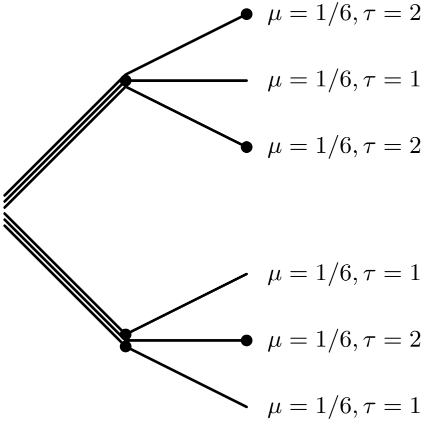

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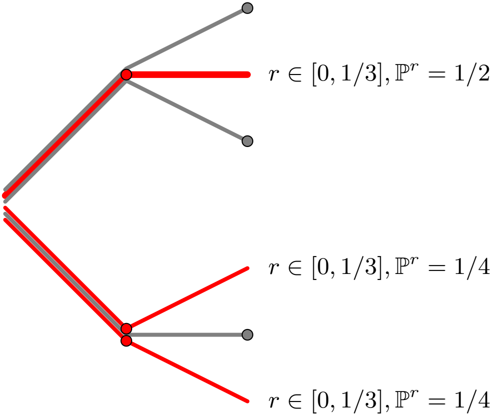

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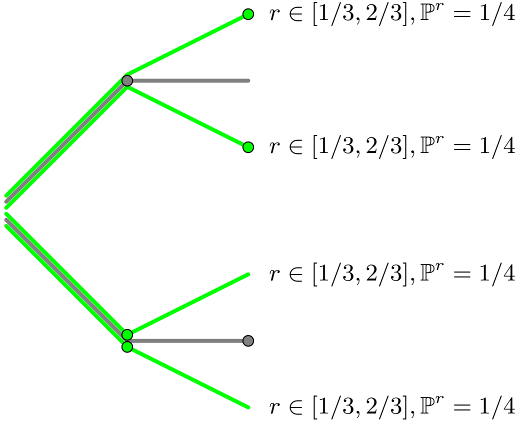

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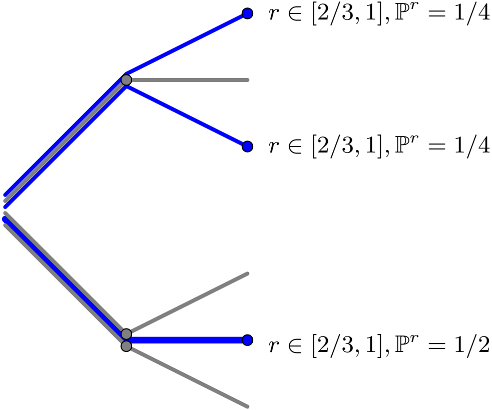

Disintegrating martingale measures

Lemma

Suppose that X is a martingale under $\mu \in \mathcal{P}(\bar{\Omega})$ with characteristic $(0, \beta)$. Then there exists a family of true stopping times τ_r and probability measures $\mathbb{P}^r \in \mathcal{P}(\Omega)$, indexed by $r \in [0, 1]$, such that for every $\eta \in L^\infty(\bar{\Omega})$,

$$\mu(\eta(\theta, \omega)) = \int_0^1 \mathbb{E}^{\mathbb{P}^r} \eta(\theta = \tau_r, \cdot)\, dr.$$

Moreover, each $\mathbb{P}^r \in \mathcal{P}(\Omega)$ is a martingale measure with characteristic $(0, \beta(t, t \wedge \tau_r(\omega), \omega_{\cdot \wedge t}))$.

Corollary

For any $Z \in L^\infty(\bar{\Omega})$, and any $E \subseteq \Omega$,

$$\sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}, \mathbb{P}(E)=1} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}, \bar{\mathbb{P}}(\Theta \times E)=1} \mathbb{E}^{\bar{\mathbb{P}}} Z.$$

Proof outline: For every $s, a \in [0,1]$ and $E \in \mathcal{F}_s$, we show that

$$\int_a^1 \mathbb{E}^{\mathbb{P}^r}(\mathbb{1}(E)(X_1 - X_s))\, dr = 0.$$

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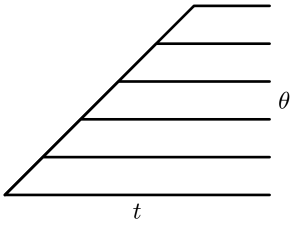

This is done by breaking into two terms: before τ_r:

$$\mathbb{E}^{\mathbb{P}^r}(\mathbb{1}(E)\mathbb{1}(\tau_r > s)(X_{\tau_r} - X_s))$$

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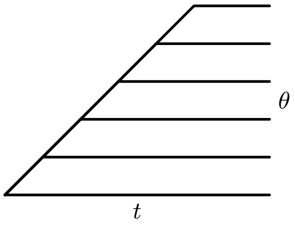

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Then $\mathbb{E}^{\mathbb{P}^r}(X_1 \,|\, \mathcal{F}_s) = X_s$ holds for a countable dense set $s \in [0,1]$, dr-a.e.. The continuity of X implies that it's a martingale.

The diffusion coefficient can be proven in the same way, replacing X by $X^2 - \int_0^{\cdot} \beta(t, t \wedge \tau_r(\omega), \omega_{\cdot \wedge t})\, dt$.

Pricing hedging duality for American options

Combining all previous elements together, we have the main result.

Theorem

Suppose $Z \in C_b(\bar{\Omega})$*, then*

$$\pi^A(Z) = \bar{\pi}(Z) = \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{Q}}} \mathbb{E}^{\bar{\mathbb{P}}} Z = \sup_{\tau \in \mathcal{T}, \mathbb{P} \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} Z(\tau, \cdot).$$

Similarly, in the case where the statically traded European options g *are present,*

$$\pi_g^A(Z) = \widehat{\pi}^A(Z) = \bar{\widehat{\pi}}(Z) = \sup_{\bar{\widehat{\mathbb{P}}} \in \bar{\widehat{\mathcal{Q}}}} \mathbb{E}^{\bar{\widehat{\mathbb{P}}}} Z = \sup_{\widehat{\tau} \in \widehat{\mathcal{T}}, \widehat{\mathbb{P}} \in \widehat{\mathcal{Q}}} \mathbb{E}^{\widehat{\mathbb{P}}} Z(\widehat{\tau}, \cdot).$$

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Further ongoing works include complexity reduction on the measurability of the robust price and optimal super-hedging strategies, hedging multiple American options and numerical implementations.

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Thank you for listening!