An A_∞ category from instantons

BIRS Workshop: Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4

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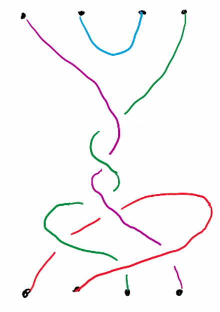
The goal

- Given a disk with n points, we build an A_∞ category.
- We show that there is a finite set of objects such that all objects in the category can be generated with exact triangles.

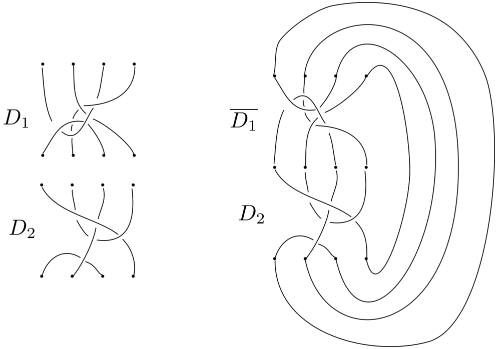
Objects of the category

Tangles in $D^2 \times [0, 1]$, with n incoming strands coming in through $D_2 \times 0$ and n outgoing strands going out of $D_2 \times 1$.

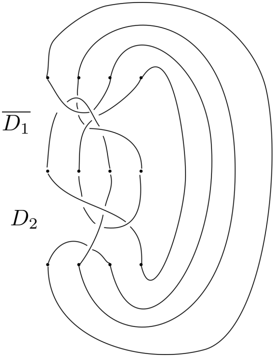


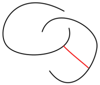
The morphisms

To define $\mathrm{Hom}(D_1, D_2)$ consider



The morphisms pt 2





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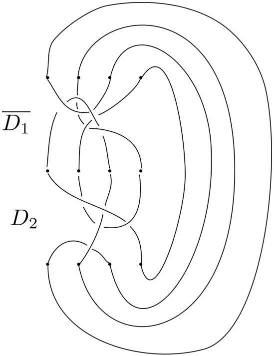

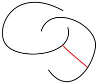

$$\mathrm{Hom}(D_1, D_2) = IC^\sharp(L_{\overline{D_1}D_2}, \mathcal{P}_{\overline{D_1}D_2}),$$

the instanton complex of $(S^3, L_{\overline{D_1}D_2} \amalg H)$ with metric and perturbation data given by $\mathcal{P}_{\overline{D_1}D_2}$.

Instanton Floer homology

- $\mathbb{Z}/4$ graded chain complex $IC^\sharp$ generated by flat connections with a singularity at the link (and the added Hopf link)
- d map generated by ASD connections.
- $IC^\sharp(U) = \mathbb{F}u_+ \oplus \mathbb{F}u_-$
- $m(u_+ \otimes x) = x$ for $m : IC^\sharp(U_2) \to IC^\sharp(U)$.

Composition (μ_2) step 1: excision

$$IC^\sharp(L_{\overline{D_2}D_3}, \mathcal{P}_{\overline{D_2}D_3}) \otimes IC^\sharp(L_{\overline{D_1}D_2}, \mathcal{P}_{\overline{D_1}D_2}) \to IC^\sharp(L_{\overline{D_1}D_3}, \mathcal{P}_{\overline{D_1}D_3}),$$

will be induced by a composition of maps: excision and then some merging maps.

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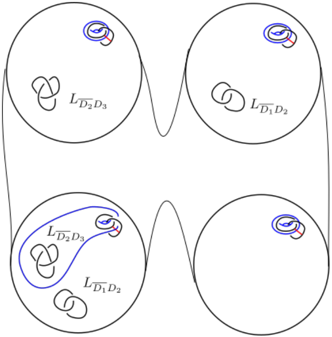
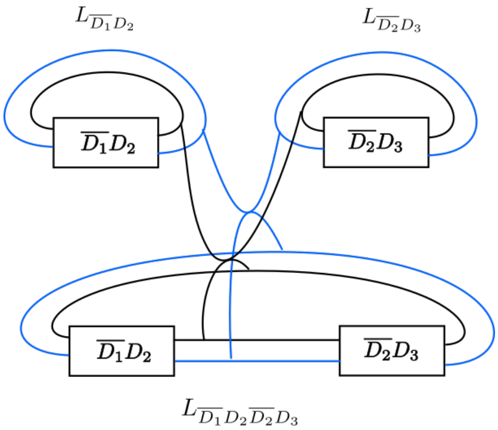
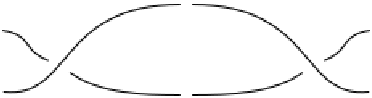


Composition (μ_2) step 2: joining the links



Composition (μ_2) step 3: cancelling D_2

For a braid, crossings in D_2 cancel with corresponding ones in $\overline{D_2}$, by R2 moves.



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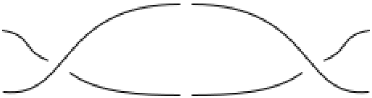
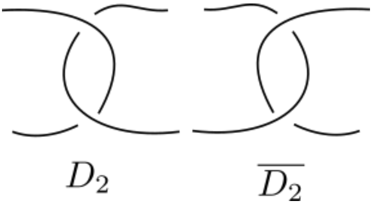
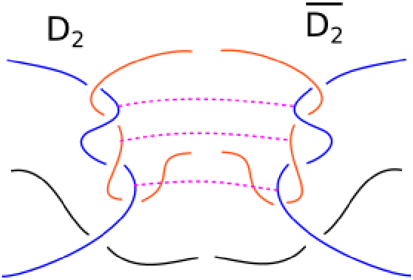


In general, can't.



Composition (μ_2) step 3: cancelling D_2 - contd

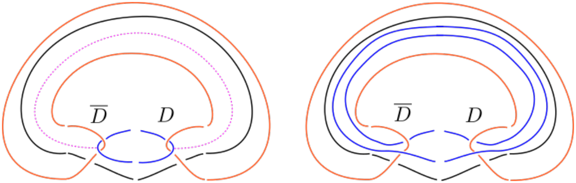
When this happens, add bands



and cap off resulting unlinked unknotted components.
(Then do $R2$ moves.)

identity construction

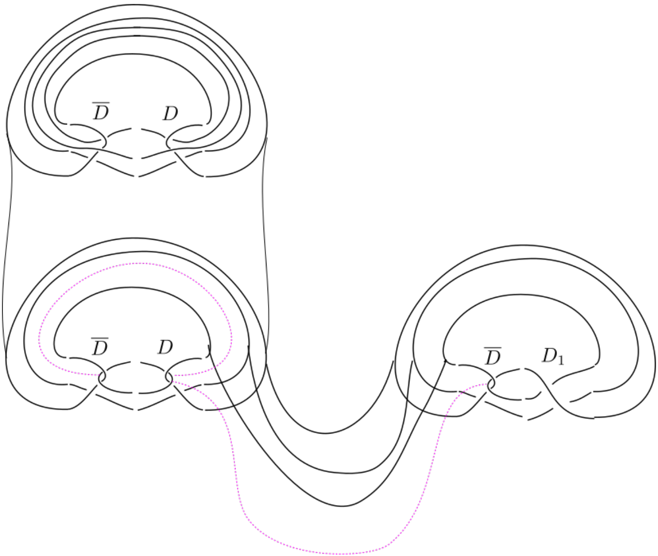
We construct a homotopy identity.
First we add a band for each maximum.



Let Σ_D be the corresponding map from the picture on the right to the one on the left.

$$\mathrm{Id}_D = \Sigma_D(u_+ \otimes \cdots \otimes u_+) \in IC^\sharp(L_{\overline{D}D}).$$

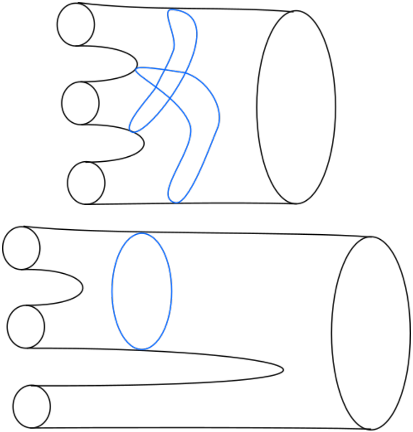
Composing with homotopy identity



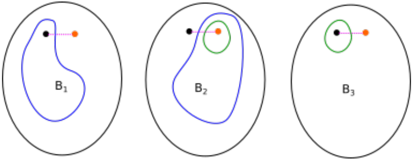
μ_3 and higher maps - basics

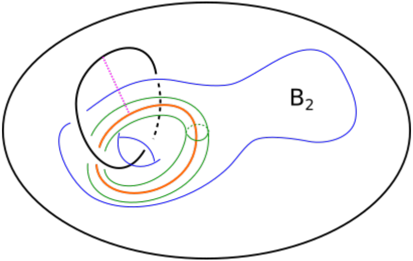
$$\mu_1(\mu_3(x_1, x_2, x_3)) + \mu_3(\mu_1(x_1), x_2, x_3) + \mu_3(x_1, \mu_1(x_2), x_3) + \mu_3(x_1, x_2, \mu_1(x_3)$$

$$= \mu_2(x_1, \mu_2(x_2, x_3)) + \mu_2(\mu_2(x_1, x_2), x_3)$$



μ_3 and higher maps - excision tori





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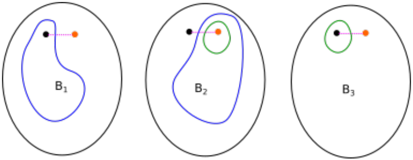


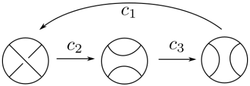
Then $\mu_2(\mu_2(B_1, B_2), B_3)$ is induced by the cobordism

- $(1,2)e \amalg Cyl(B_3)$
- $(1,2)m \amalg Cyl(B_3)$
- $(12,3)e$
- $(12,3)m$.

and $\mu_2(B_1, \mu_2(B_2, B_3))$ is induced by the cobordism

- $Cyl(B_1) \amalg (2,3)e$
- $Cyl(B_1) \amalg (2,3)m$
- $(1,23)e$
- $(1,23)m$.

Finite generation - basics



Exact if and only if there are $h_1 \in \mathrm{Hom}(D_1, D_0)$, $h_2 \in \mathrm{Hom}(D_2, D_1)$ and $k \in \mathrm{Hom}(D_1, D_1)$ satisfying

- $\mu_1(h_1) = \mu_2(c_3, c_2)$
- $\mu_1(h_2) = \mu_2(c_1, c_3)$
- $\mu_1(k) = -\mu_2(c_1, h_1) + \mu_2(h_2, c_2) + \mu_3(c_1, c_3, c_2) - e_Y$,

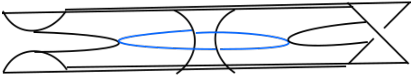
where e_Y is a chain representative for the identity and for each object D, the following chain complex is acyclic:

$$\mathrm{Hom}(D, D_2)[1] \oplus \mathrm{Hom}(D, D_0)[1] \oplus \mathrm{Hom}(D, D_1),$$

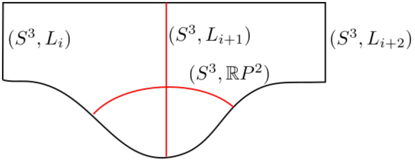
$$\partial = \begin{bmatrix} \mu_1 & 0 & 0 \\ \mu_2(c_3, -) & \mu_1 & 0 \\ \mu_2(h_2, -) + \mu_3(c_1, c_3, -) & \mu_2(c_1, -) & \mu_1 \end{bmatrix}.$$

Construction of h_i

This is a construction Kronheimer and Mrowka used to establish the spectral sequence from Khovanov homology.

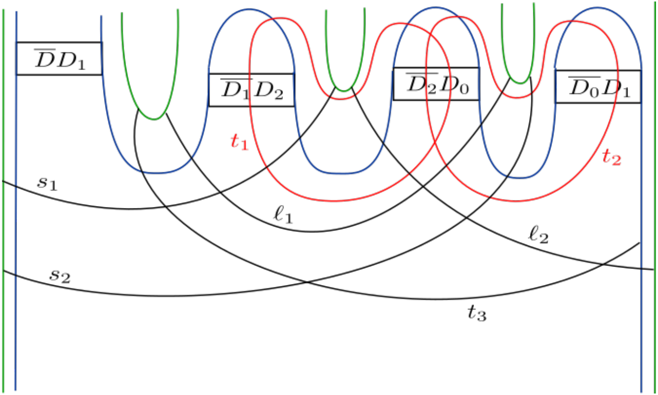


Consider a path of metrics with fully stretching out the $(S^3, \mathbb{R}P^2)$ on one end, and stretching out along the middle (S^3, L) on the other.

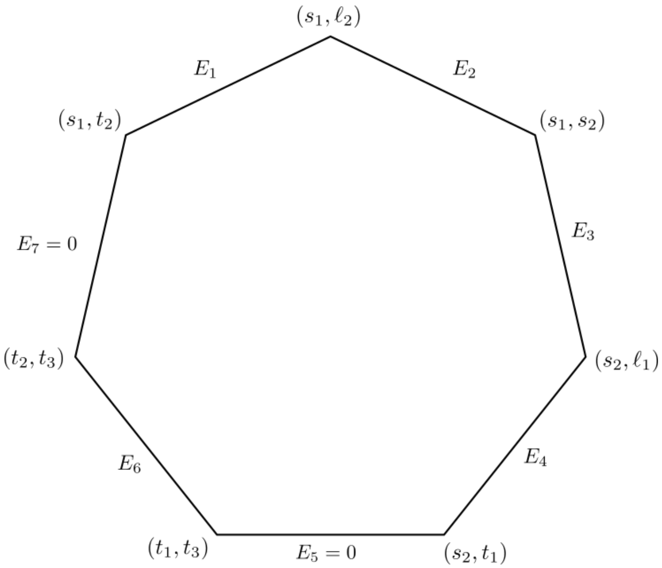


Construction of k_i: stretching curves

This is similar (but not the same) as a construction Kronheimer and Mrowka used to establish the spectral sequence from Khovanov homology.



Construction of k_i: heptagon of metrics



Thank you!

Thank you for the invitation and thank you for listening!