Modern Breakthroughs in Diophantine Problems

Michael Bennett (University of British Columbia)
Nils Bruin (Simon Fraser University)
Samir Siksek (Warwick University)
Bianca Viray (University of Washington)

June 19-24, 2022

1 Overview of the Field and Recent Developments

The subject of Diophantine equations is currently experiencing a rapid succession of breakthroughs. These include:

(i) The work of Rafael von Känel, Benjamin Matschke, Hector Pasten, and others, proving powerful results on classical Diophantine equations by associating solutions to points on modular or Shimura curves.

(ii) Recent successes in making the Chabauty-Kim method effective, explicit and practical, due to Balakrishnan, Dogra, Müller, and others.

(iii) Progress on Manin's conjecture and other quantitative questions by a new generation of analytic number theorists, including Browning, Loughran, Schindler, Tanimoto and many others.

(iv) The introduction of the notion of Campana points which interpolate between rational and integral points, and which give rise to a host of new Diophantine problems.

(iv) Applications of modularity over number fields to the asymptotic Fermat conjecture and other Diophantine problems due to Bennett, Dahmen, Freitas, Kraus, Sengun, Siksek and others.

Whilst these and other successes constitute dramatic progress on problems of tremendous historical importance, there has also been a divergence of methods and approaches, and the subject is undergoing a period of fragmentation. A primary objective of the workshop was to reverse this fragmentation by bringing together researchers belonging to disparate Diophantine traditions, and who would otherwise rarely interact.

2 Presentation Highlights

2.1 Stephanie Chan: Integral points in families of elliptic curves

Given a family of elliptic curves, it is natural to ask how often does they have integral points, and how many integral points there are on average. In this talk Chan gave beautiful answers for two natural families, the congruent number curves, and the cubic twists of a Mordell curve. For example, fix a non-square $k \neq 0$, and consider

$$E_B \;:\; Y^2 \;=\; X^3 + kB^2.$$

The family $\{E_B \;:\; B \in \mathbb{N}\}$ consists of the cubic twists of the Mordell curve $Y^2 = X^3 + k$. Let

$$E_B(\mathbb{Z}) \;=\; \{(X, Y) \in \mathbb{Z}^2 \;:\; Y^2 = X^3 + kB^2\}.$$

Chan sketched proofs of the following results

$$\#\{1 \leq B \leq N \;:\; E_B(\mathbb{Z}) \neq \emptyset\} \;\ll_k\; N \cdot \left(\frac{\log\log N}{\log N}\right)^{1/2}$$

and

$$\sum_{\substack{1 \leq B \leq N \\ B \text{ cubefree}}} \#E_B(\mathbb{Z}) \;\ll_k\; N.$$

For details see [8], [9].

2.2 Levent Alpoge: Integers which are(n't) the sum of two cubes

Thanks to Fermat we have a complete description of which integers are sums of two rational squares. Alpöge sketched the proofs of the following beautiful theorem.

Theorem (Alpöge, Bhargava and Schnidman)**.** *When ordered by their absolute values, a positive proportion of integers are the sum of two rational cubes, and a positive proportion of integers are not.*

The problem of representing an integer n as the sum of two rational cubes is equivalent to deciding if the elliptic curve

$$E_{d,n} \;:\; y^2 = x^3 - dn^2$$

has rational points, for $d = 432$. As torsion is rare in these families, the problem translates into determining for which values of n is the rank of $E_{432,n}$ positive. The main ingredient is the following estimate for the average size of 2-Selmer group of $E_{d,n}$.

Theorem (Alpöge, Bhargava and Schnidman)**.** *Fix $d \neq 0$ and let n range over integers satisfying any finite set (or even "acceptable" infinite sets) of congruence conditions. Then*

$$\mathrm{avg}_n \,\#\,\mathrm{Sel}_2(E_{d,n}) = 3.$$

For details see [2].

2.3 Hector Pasten: On Vojta's conjecture with truncation of rational points

In [36], Vojta proposed a far-reaching generalization of the abc conjecture. Vojta's conjecture is a Diophantine approximation statement in varieties of any dimension and involves truncated counting functions (these are a generalization of the logarithm of the radical of an integer). Pasten sketched the proof of the first unconditional result towards Vojta's conjecture with truncated counting functions in varieties of arbitrary dimension. A striking application is the following corollary, which can be thought of as a subexponential version of the abc conjecture.

Corollary (Pasten)**.** *Let $\epsilon > 0$. There is a number $\kappa_\epsilon > 0$ effectively depending on ϵ such that the following holds: Let a, b, c be coprime positive integers with $a + b = c$. Suppose $a < c^{1-\eta}$ for some $\eta > 0$. Then*

$$c < \exp\left(\eta^{-1} \cdot \kappa_\epsilon \cdot R^{(1+\epsilon)(\log_3^* R)/(\log_2^* R)}\right),$$

where $R = \mathrm{rad}(abc)$.

For details see [29].

2.4 Abbey Bourdon: Sporadic points of odd degree on $X_1(N)$ coming from $\mathbb{Q}$-curves

We say a degree d point x on a curve C is isolated if it does not belong to an infinite family of degree d points parametrized by a geometric object-either $\mathbb{P}^1$ or a positive rank abelian subvariety of the curve's Jacobian. We say x is sporadic if there are only finitely many points on C of degree at most d. Every sporadic point is isolated, but the converse need not hold. It was known from recent work of Bourdon, Ejder, Liu, Odumodu, and Viray [5] that Serre's uniformity conjecture implies that there are only finitely many elliptic curves with j-invariant in $\mathbb{Q}$ which give rise to an isolated point of any degree on $X_1(N)$. On the other hand, by recent work of Bourdon and Najman, an analogous finiteness result on non-CM $\mathbb{Q}$-curves would actually imply Serre's Uniformity Conjecture. The talk highlighted unconditional results in that direction for $\mathbb{Q}$-curves giving rise to sporadic points of odd degree. For details see [7].

2.5 Pip Goodman: Determining cubic and quartic points on modular curves

Let C be a curve over $\mathbb{Q}$, and let $C^{(d)}$ denote the d-th symmetric power. The $\mathbb{Q}$-points of $C^{(d)}$ correspond to degree d rational divisors on C. In particular, if we can determine $C^{(d)}(\mathbb{Q})$ then we know all degree d points on C. Wetherell (unpublished) and Siksek [34] have extended Chabauty's method to determine $C^{(d)}$, under a suitable condition on the rank. One difficulty is that $C^{(d)}(\mathbb{Q})$ might be infinite. For example, if $\rho : C \to D$ has degree d and $D(\mathbb{Q})$ is infinite then $\rho^*(D(\mathbb{Q}))$ is an infinite subset of $C^{(d)}(\mathbb{Q})$. Previous work on symmetric Chabauty focuses on the case where the only infinite source of rational points on $C^{(d)}$ comes from a single degree d map $\rho : C \to D$. In practice, this has turned out to be severely limiting. For example, if C is a modular curve then one often has many maps to modular curves of smaller level, elliptic curves and to quotients by Atkin–Lehner involutions. In this talk Goodman, reporting on joint work with Box and Gajović, consider the most general case where an infinite family within $C^{(d)}$ has the form

$$P + \rho_1^*(C_1^{d_1}(\mathbb{Q})) + \cdots + \rho_r^*(C_r^{d_r}(\mathbb{Q}))$$

where P is a fixed rational divisor of degree d_0, the maps $\rho_i : C \to C_i$ have degrees e_i and

$$d = d_0 + d_1 e_1 + \cdots + d_r e_r.$$

They use their new variant of Chabauty to determine the cubic points on $X_0(N)$ for $N = 53$, 57, 61, 65, 67, 73 and the quadratic points on $X_0(65)$, thereby answering questions posed by Zureick-Brown. For details see [4].

2.6 Adela Gherga: Efficient resolution of Thue–Mahler equations

A Thue–Mahler equation has the form

$$F(X,Y) \;=\; a \cdot p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}, \qquad \gcd(X,Y) = 1, \qquad n_i \geq 0$$

where $F \in \mathbb{Z}[X,Y]$ is an irreducible binary form of degree ≥ 3, a is a non-zero integer, and $p_1, \ldots, p_r$ are distinct primes. The talk highlighted a new algorithm for solving Thue–Mahler equations that makes heavy use of a newly developed "Dirichlet sieve". This allows for the resolution of Thue–Mahler equations of large degree or with a large number of primes. For example, the algorithm determines all solutions to $P(X^4 - 2Y^4) \leq 100$ with $\gcd(X,Y) = 1$, finding that there are precisely 49 solutions. Here $P(m)$ denotes the largest prime divisor of m. This work links with algorithms due to Gherga, Bennett, Rechnitzer, von Känel, Matschke to determine all elliptic curves over $\mathbb{Q}$ with a given set of bad primes, by reducing the problem to the resolution of cubic Thue–Mahler equation. For details see [3], [17].

2.7 Isabel Vogt: Geometry of curves with abundant points

Let X be a curve over a number field k, and let $d \geq 2$. When does X possess infinitely many degree d points? It is known by classification results of Harris–Silverman and Abramovich–Harris that if this happens with $d = 2$, 3, then the curve X admits a non-constant map of degree at most d to either $\mathbb{P}^1$ or an elliptic curve. For $d \geq 4$ the analogous statement is false by work of Debarre and Fahlaoui. Vogt sketched joint work with Kadets that extends the classification of Harris–Silverman and Abramovich–Harris to larger values of d. For a curve X/k we define the arithmetic degree of irrationality $\mathrm{a.irr}_k X$ to be the smallest integer k such that X has infinitely many closed points of degree d. We define the geometric degree of irrationality $\mathrm{a.irr}_{\overline{k}} X$ to be the minimum of the values $\mathrm{a.irr}_L X$ as L ranges over finite extensions of k.

Theorem (Kadets and Vogt)**.** *Suppose X/k is a nice curve. Then the following statements hold:*

1. *If $\mathrm{a.irr}_k X = 2$, then X is a double cover of $\mathbb{P}^1$ or an elliptic curve of positive rank;*
2. *If $\mathrm{a.irr}_k X = 3$, then one of the following three cases holds:*
   (a) *X is a triple cover of $\mathbb{P}^1$ or an elliptic curve of positive rank;*
   (b) *X is a smooth plane quartic with no rational points, positive rank Jacobian, and at least one cubic point;*
   (c) *X is a genus 4 Debarre-Fahlaoui curve;*
3. *If $\mathrm{a.irr}_{\overline{k}} X = d \leqslant 3$, then $X_{\overline{k}}$ is a degree d cover of $\mathbb{P}^1$ or an elliptic curve;*
4. *If $\mathrm{a.irr}_{\overline{k}} X = d = 4, 5$, then either $X_{\overline{k}}$ is a Debarre-Fahlaoui curve, or $X_{\overline{k}}$ is a degree d cover of $\mathbb{P}^1$ or an elliptic curve.*

For details see [22].

2.8 Diana Mocanu: The modular approach to Diophantine equations over totally real fields

Freitas, Kraus and Siksek [16], [15] have related solutions to the Fermat equation $X^p + Y^p + Z^p = 0$ over totally real fields to solutions to a certain S-unit equations using modularity and level lowering. Mocanu extends this to the generalized Fermat equations $X^p + Y^p = Z^2$ and $X^p + Y^p = Z^3$ where the S-unit equations are replaced by equations of the form $\alpha + \beta = \gamma^2$ and $\alpha + \beta = \gamma^3$ where α, β are S-units. Under certain class-field-theoretic assumptions Mocanu can control solutions to these equations. A sample theorem is the following.

Theorem (Mocanu)**.** *Let $d \equiv 5 \pmod 8$ be a rational prime, and write $K = \mathbb{Q}(\sqrt{d})$. There is a constant B_K such that for all primes $p > B_K$, the equation $a^p + b^p = c^2$ has no non-trivial primitive solutions $(a, b, c) \in \mathcal{O}_K^3$ with $2 \mid b$.*

For details see [25].

3 Open Problems

The organizers thank Alex Best for transcribing the open problems.

3.1 Abbey Bourdon: Two Problems on Isolated Points

Let C be a nice curve over a number field k. For the sake of simplicity, assume there exists $P_0 \in C(k)$; for a more general setup, see [5, §4]. We say a closed point $x \in C$ of degree d is **sporadic** if there are only finitely many points of degree at most d. More generally, we say x is **isolated** if it does not belong to an infinite family of degree d points parametrized by $\mathbb{P}^1$ or a positive rank abelian subvariety of the curve's Jacobian. Precisely, to x we can associate the k-rational effective divisor

$$D = P_1 + \cdots + P_d$$

where $P_1, \ldots, P_d$ are the points in the Gal_k-orbit corresponding to x. Thus x gives a k-rational point on the dth symmetric power of C, denoted $C^{(d)}$. With this identification, we can study the image of x under the natural map to the curve's Jacobian

$$\Phi_d : C^{(d)} \to \mathrm{Jac}(C)$$

which sends the effective divisor D of degree d to the class $[D - dP_0]$. We say x is **isolated** if the following conditions are both satisfied:

1. There is no other point $y \in C^{(d)}(k)$ such that $\Phi_d(x) = \Phi_d(y)$.

2. There is no positive rank abelian subvariety $A \subset \mathrm{Jac}(C)$ such that $\Phi_d(x) + A \subset \mathrm{im}(\Phi_d)$.

Any sporadic point is isolated (though the converse need not hold), and any curve has only finitely many isolated points. See [5, Theorem 4.2].

Recent investigations [5, 6, 7, 14, 33] have sought to characterize the elliptic curves producing sporadic and isolated points on $X_1(N)$. We say $j \in X_1(1) \cong \mathbb{P}^1$ is a **sporadic** (resp., **isolated**) j**-invariant** if it is the image of a sporadic (resp., isolated) point on $X_1(N)$ for some positive integer N. If one assumes Serre's Uniformity Conjecture, then there are only finitely many isolated j-invariants in $\mathbb{Q}$ [5, Corollary 1.7], and the subset of non-CM j-invariants in $\mathbb{Q}$ corresponding to isolated points of odd degree has been identified explicitly [6, Theorem 2]. As a first step in the case of even degree, define $J_{\text{isog}}(\mathbb{Q})$ to be the set of j-invariants associated to elliptic curves over $\mathbb{Q}$ with a nontrivial rational cyclic isogeny. This set contains all known examples of isolated j-invariants in $\mathbb{Q}$: those corresponding to CM elliptic curves plus $j = -3^2 \cdot 5^6/2^3$, $3^3 \cdot 13/2^2$, and $-7 \cdot 11^3$. See [5, 6, 27, 21]. By work of Lemos [23], Serre's Uniformity Conjecture holds for all non-CM elliptic curves over $\mathbb{Q}$ possessing a nontrivial cyclic $\mathbb{Q}$-isogeny. Thus, by [5, Corollary 1.7], there are only finitely many isolated j-invariants in $J_{\text{isog}}(\mathbb{Q})$.

Question 1. Can the set of isolated j-invariants in $J_{\text{isog}}(\mathbb{Q})$ be computed explicitly? Are there any isolated j-invariants in $\mathbb{Q}$ which lie outside this set?

We note that there are similarities between this question and the methods used to prove Theorem 2 in [6]. There, an essential observation was that if $x \in X_1(n)$ is a point of odd degree with $j(x) \in \mathbb{Q}$ and $j(x) \neq 3^3 \cdot 5 \cdot 7^5/2^7$, then there exists $y \in X_0(p)(\mathbb{Q})$ with $j(x) = j(y)$ for some odd $p \mid n$; see [6, Thm. 3]. If one follows the approach of [6], it will be necessary to perform a more sophisticated analysis of the possible combinations of simultaneously non-surjective Galois representations associated to elliptic curves $E/\mathbb{Q}$. Partial progress can be made using work of Morrow, Daniels, and González-Jiménez [26], [10] on fiber products of modular curves in combination with results obtained via formal immersions as in work of Darmon and Merel [13, Thm. 8.1] and Lemos [23, Prop. 2.1]. An analysis of certain "entanglement" modular curves was also necessary in [6] and similar computations may be required for Question 1. For more on entanglement modular curves, see [7, 12, 11].

Instead of studying isolated or sporadic points associated to elliptic curves with j-invariant in $\mathbb{Q}$, one could more generally hope to understand isolated points corresponding to $\mathbb{Q}$-curves. Here, by $\mathbb{Q}$-curve, I mean an elliptic curve isogenous (over $\overline{\mathbb{Q}}$) to its Galois conjugates. This class contains all elliptic curves with j-invariant in $\mathbb{Q}$, as well as any curve in the corresponding geometric isogeny class, though there are others not of this form (the so-called "strict" $\mathbb{Q}$-curves). A key motivation for studying sporadic points associated to $\mathbb{Q}$-curves is the following: If all non-CM $\mathbb{Q}$-curves giving rise to a sporadic point on $X_1(N)$ belong to only finitely many geometric isogeny classes—even as we allow N to range over all positive integers—then Serre's Uniformity Conjecture holds. See [7, Theorem 1.3]. We have such a finiteness result for odd degree, where one can show all non-CM $\mathbb{Q}$-curves corresponding to a sporadic point of odd degree on $X_1(N)$ belong to the $\overline{\mathbb{Q}}$-isogeny class of the elliptic curve 162.c3 with j-invariant $-3^2 \cdot 5^6/2^3$ [7, Theorem 1.4]. However, it is unknown whether this isogeny class contains any sporadic j-invariants besides $-3^2 \cdot 5^6/2^3$. This inspires the following question, which also appears as Question 2 in [7]:

Question 2. Does there exist a non-CM $\overline{\mathbb{Q}}$-isogeny class containing infinitely many sporadic j-invariants?

Note that the answer to Question 2 is yes if there exists an elliptic curve producing a sporadic point of sufficiently low degree [7, Proposition 8.1], but the only known examples satisfying this condition are CM elliptic curves.

3.2 Nathan Grieve: Approximation sets for properly intersecting divisors

Consider a polarized projective variety (X, L) defined over a number field $\mathbf{K}$. Let $D_1, \ldots, D_q$ be a collection of nonzero effective and properly intersecting Cartier divisors on X. Fix a finite set of places S, of $\mathbf{K}$, and let $N = q \cdot \#S$.

Expanding on the viewpoint of Schmidt [32], inside of $\mathbb{R}^N$, there is an *approximation set*

$$Approx(X, L; D_1, \ldots, D_q; S) \subseteq \mathbb{R}^N.$$

In defining such approximation sets, a key point is a concept of *density of rational points* with respect to the *subspace topology* on $X(\mathbf{K})$ that is induced by the *linear sections* of the complete linear series $|L|$ and powers thereof (cf. [32, p. 706] and [18, Definition 3.1]).

Arguing as in [32, p. 708], the compactness of such approximation sets follows from the Ru-Vojta Arithmetic General Theorem (see for instance [31, p. 961], [18, Theorem 1.1]). However, it remains an interesting problem to determine defining inequalities of such approximations sets. Such a result would make progress towards a general form of [32, Theorem 2] which, in particular, would treat the case of properly intersecting divisors.

3.3 Hector Pasten: Büchi's problem

Observe: $1, 4, 9, 16, \ldots$ have differences 3, 5, 7, which have differences 2, 2, 2, etc.

You can also take non-consecutive squares such as $0, 49, 100, \ldots$ which have difference 49, 51, and then difference 2. But it seems harder to construct long sequences like this.

The problem is to find how long such a sequence can be, there are infinitely many known examples of length four.

This is known as *Büchi's problem*; show that there exists a uniform M (i.e. constant) such that every sequence of $\geq M$ squares with second differences equal to 2 is trivial (i.e. the squares are consecutive). One expects that $M = 5$ (via heuristic and also from known evidence), but any bound would be interesting.

What is known: A theorem of Vojta shows that the Bombieri–Lang conjecture implies a positive answer to Büchi's problem [35]. Pasten also shows, conditionally on the abc-conjecture, that Büchi's problem admits a positive answer [28]. The challenge is to find something unconditional in this direction.

Nils Bruin remarks that Vojta's approach is via surfaces, showing eventually the surface classifying these is of general type, and so the Bombieri–Lang conjecture is applicable. However, Pasten's approach is less geometric.

3.4 Stanley Xiao: One of the cuboid conjectures

An **perfect Euler brick** is a rectangular prism with side lengths a, b, $c \in \mathbb{N}$ such that all face diagonals (d, e, f) are natural numbers and also the space diagonal g is a natural number. No example of a perfect Euler brick is known, however, it is easy to construct examples of bricks where only the edges and face diagonals are natural numbers (e.g. $(a, b, c, d, e, f) = (44, 117, 240, 125, 255, 267)$.

If we do not insist on the prism being rectangular, and allow non-right angles, examples are also known.

There are conjectures known as the cuboid conjectures, suggested 10 years ago, which appear on the Wikipedia page for the Euler brick. Together the three cuboid conjectures, imply there is no perfect Euler brick. The first conjecture is easy (for experts on invariant theory of quadratic forms), the second seems harder and the third seems to be 99% of the work.

The second cuboid conjecture is as follows:

Conjecture 2. For any two positive coprime integer numbers $p \neq q$ the tenth-degree polynomial

$$\begin{aligned} Q_{pq}(t) = & t^{10} + \left(2q^2 + p^2\right)\left(3q^2 - 2p^2\right)t^8 \\ & + \left(q^8 + 10p^2q^6 + 4p^4q^4 - 14p^6q^2 + p^8\right)t^6 \\ & - p^2q^2\left(q^8 - 14p^2q^6 + 4p^4q^4 + 10p^6q^2 + p^8\right)t^4 \\ & - p^6q^6\left(q^2 + 2p^2\right)\left(-2q^2 + 3p^2\right)t^2 \\ & - q^{10}p^{10} \end{aligned}$$

is irreducible over the ring of integers $\mathbb{Z}$.

Conjecture 2 may be possible using the expertise we have, though it is not completely clear what the motivation for this conjecture is. John Voight suggests looking at Runge's method.

3.5 Drew Sutherland: Modular curves arising in the classification of Galois images

Mazur's vertical uniformity problem asks for the determination of possible ℓ-adic images of Galois representations of elliptic curves $E/\mathbb{Q}$. i.e. given a prime ℓ what are the possibilities for the image of

$$\rho_{E,\ell^\infty} \;:\; G_{\mathbb{Q}} \to \mathrm{GL}(E[\ell^\infty]),$$

as E ranges over all elliptic curves over $\mathbb{Q}$. For $\ell = 2$ the answer is known due to Rouse–Zurieck-Brown. For $\ell = 11, 17$ the answer is known due to Balakrishnan et. al.

Why don't we know more? We need to determine the rational points on certain modular curves such as

$$X_{ns}^+(25), \qquad X_{ns}^+(27), \qquad X_{ns}^+(131), \qquad X_{ns}^+(\ell) \text{ for } \ell \geq 19.$$

There are two other curves that interesting that we would like to know the rational points on (given by their LMFDB labels of the form $L.I.g.n$ where L is the level, I the index, g the genus, and n the curve number). The first is known as $49.147.9.1$, and is a degree 7 cover of $X_{ns}^+(7)$ (which is genus 0), the CM points are above $j = 0$, and the plane model has been computed, of degree 21. The gonality is at least 3, and the Jacobian is geometrically irreducible and of rank 9. The corresponding modular form is
https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/2401.2.a.f/

The second is known as $49.196.9.1$, it is a degree 7 cover of $X_s^+(7)$ the CM points are above $j = 0$, and the plane model has been computed, of degree 14. The gonality is at least 5, and the Jacobian decomposes as the product of a dimension 3 and dimension 6 piece corresponding modular forms are
https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/2401.2.a.b/
and
https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/2401.2.a.c/
The rank of the Jacobian is 9.

Models for these curves can be found at
https://github.com/AndrewVSutherland/ell-adic-galois-images/tree/main/models.

It is conjectured that there are no non-CM non-cuspidal points, the problem is to prove this. Once this is done, the only remaining obstruction to determining the possible ℓ-adic Galois images will the non-split Cartans.

Seeing as there are not many rational points, quadratic Chabauty might be tricky, but still a good option.

3.6 Hector Pasten: Zariski density of rational points on general type surfaces of irregularity 2

Let X be a surface of general type defined over a number field k. Suppose that the irregularity of X is $q = 2$ and that the albanese map is surjective. Prove that $X(k)$ is not Zariski dense in X.

Some context for the problem is given by the following cases:

- If $q > 2$, Faltings's theorem on subvarieties of abelian varieties implies that $X(k)$ is not Zariski dense in X.
- If $q = 2$, but the albanese map is not surjective, then its image is a curve to which one can apply Faltings's theorem. We deduce that $X(k)$ is not Zariski dense.

Thus, it seems that the proposed problem is the next natural case of the Bombieri-Lang conjecture for surfaces.

3.7 Natalia Garcia Fritz: Finding differentials for which a divisor is integral

Given a smooth projective surface $X/\mathbb{C}$, and $D = \sum_{j=1}^{q} D_j$ a reduced divisor on X formed by different irreducible curves D_j, we want to find a non-trivial section $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$ such that every D_j is an ω-integral curve, with $\mathcal{L}$ of low degree on each D_j (or at least degree independent of q). In that way we will be able to prove more instances of Vojta's conjecture or Campana's conjecture for surfaces in the function field case. One can consider Hasse-Schmidt differentials instead of $S^r\Omega^1_{X/\mathbb{C}}$ to cover more cases.

Here are some particular examples:

- In $\mathbb{P}^2$, with D_j in the quadratic family of lines $t^2x+sty+s^2z=0$, one can choose $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(4) \otimes S^2\Omega^1_{\mathbb{P}^2/\mathbb{C}})$ which locally looks like $dxdx - ydxdy + xdydy$.
- In $\mathbb{P}^2$, with D_j lines, we can choose $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(3) \otimes (HS^2_{\mathbb{P}^2/\mathbb{C}})_3)$ which locally looks like $dyd_2x - dxd_2y$.
- In $\mathbb{P}^2$, if we consider D_j in the family $cx^k - (c+1)y^k = c(c+1)z^k$ with $c \in \mathbb{C} \setminus \{-1, 0\}$, $k > 2$ an integer, we can choose $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(k+3) \otimes S^2\Omega^1_{\mathbb{P}^2/\mathbb{C}})$ which locally looks like $x^{k-1}ydxdx + (1 - x^k - y^k)dxdy + xy^{k-1}dydy$.

Problems:

1. Find suitable conditions on D to make this work in more generality
2. Find systematic approach to construct other explicit examples.

4 Outcome of the Meeting

We were fortunate to attract around 38 in person participants for the workshop and another 31 online participants. For many this had been their first face-to-face event in over 2 years. Our foremost priority was to be useful to younger participants, whose careers must have suffered the most during the past two years. We did this in three ways:

- Prioritise talks given by younger participants to allow them to advertise their work.
- With the aim of kick-staring collaborations and new research, we encouraged speakers to suggest open problems during their talks.
- We kept the schedule light (9.00–12.00, 13.30–3.00) to allow ample time for discussions and collaboration.

It is clear from the talks and discussions that the subject remains a very active field. Whilst breakthroughs continue to be made on older problems such as Vojta's conjectures, there are also some newer areas have become prominent in the last few years and provide excellent opportunities for active research and further breakthroughs. These include the following:

- The application of methods from arithmetic statistics to study Diophantine problems in families [2], [8], [9].
- Low degree points on curves, both from the theoretical [22] and from the computational [4], [30] perspectives.
- The recently formulated concepts of sporadic and isolated points [4], [5], [6], [7] on modular curves.
- Quadratic Chabauty used to determine rational points on curves where the Jacobian Mordell–Weil rank equals the genus [1].
- Applications of modularity of elliptic curves over number fields to Fermat-type equations of signatures (p, p, p), $(p, p, 2)$, $(p, p, 3)$ [24], [25].

One of the surprises of the workshop was a talk (based on [20]) by Avinash Kulkarni reporting on how machine learning was successfully used to aid in the computation of the periods of projective hypersurfaces. The use of machine learning in computational arithmetic geometry is certainly an avenue worthy of further exploration and experimentation.

References

[1] N. Adžaga, S. Chidambaram, T. Keller, O. Padurariu, Rational points on hyperelliptic Atkin-Lehner quotients of modular curves and their coverings, *arxiv:arXiv:2203.05541*.

[2] L. Alpöge, M. Bhargava and A. Schnidman, Integers expressible as the sum of two rational cubes, preprint.

[3] M. A. Bennett, A. Gherga and A. Rechnitzer, Computing elliptic curves over $\mathbb{Q}$, *Math. Comp.* **88** (2019), 1341–1390.

[4] J. Box, S. Gajović, P. Goodman, Cubic and Quartic Points on Modular Curves Using Generalised Symmetric Chabauty, *International Mathematics Research Notices* **mab358** (2022).

[5] A. Bourdon, Ö. Ejder, Y. Liu, F. Odumodu, and B. Viray, On the level of modular curves that give rise to isolated j-invariants, *Adv. Math.* **357** (2019), 106824, 1–33.

[6] A. Bourdon, D. Gill, J. Rouse, and L. D. Watson, Odd degree isolated points on $X_1(N)$ with rational j-invariant, *arxiv:2006.14966*.

[7] A. Bourdon and F. Najman, Sporadic points of odd degree on $X_1(N)$ coming from $\mathbb{Q}$-curves, *arxiv:2107.10909*.

[8] S. Chan, Integral points on cubic twists of Mordell curves, *arxiv:2203.11366*.

[9] S. Chan, The average number of integral points on the congruent number curves, *arxiv:2112.01615*.

[10] H. Daniels and E. González-Jimenéz, Serre's constant of elliptic curves over the rationals, to appear in *Exp. Math.*, *arxiv:1812.04133*.

[11] H. Daniels, Á. Lozano-Robledo, and J. S. Morrow, Towards a classification of entanglements of Galois representations attached to elliptic curves, *arxiv:2105.02060*.

[12] H. Daniels and J. S. Morrow, A group theoretic perspective on entanglements of division fields, to appear in *Trans. Amer. Math. Soc. Ser. B.*, *arxiv:22008.09886*.

[13] H. Darmon and L. Merel, Winding quotients and some variants of Fermat's last theorem, *J. Reine Angew. Math.* **490** (1997), 81–100.

[14] Ö. Ejder, Isolated points on $X_1(\ell^n)$ with rational j-invariant, *Res. Number Theory* **8** (2022), no. 1, Paper No. 16, 7.

[15] N. Freitas, A. Kraus and S. Siksek, Local criteria for the unit equation and the asymptotic Fermat's Last Theorem, *Proceedings of the National Academy of Sciences* **118** (2021), No. 12.

[16] N. Freitas and S. Siksek, The Asymptotic Fermat's Last Theorem for Five-Sixths of Real Quadratic Fields, *Compositio Mathematica* **151** (2015), 1395–1415.

[17] A. Gherga and S. Siksek, Efficient resolution of Thue–Mahler equations, *arxiv:2207.14492*.

[18] N. Grieve, On arithmetic inequalities for points of bounded degree, *Res. Number Theory* **7** (2021), no. 1, Paper No. 1, 14.

[19] N. Grieve, Files from BIRS Workshop: 22w5024,
https://www.birs.ca/workshops/2022/22w5024/files/.

[20] K. Heal, A. Kulkarni, and E. C. Sertöz, Deep learning Gauss-Manin connections, *Adv. Appl. Clifford Algebr.* **32(2)**:Paper No. 24, 41, 2022.

[21] M. van Hoeij, Low degree places on the modular curve $X_1(n)$, *arxiv:1202.4355*.

[22] B. Kadets and I. Vogt, Subspace configurations and low degree points on curves, *arXiv:2208.01067*.

[23] P. Lemos, Serre's uniformity conjecture for elliptic curves with rational cyclic isogenies, *Trans. Amer. Math. Soc.* **371** (2019), no. 1, 137–146.

[24] P. Michaud-Jacobs, On some generalized Fermat equations of the form $x^2 + y^{2n} = z^p$, Mathematika **68** (2022), no. 2, 344–361.

[25] D. Mocanu, Asymptotic Fermat for signatures $(p, p, 2)$ and $(p, p, 3)$ over totally real fields, *Mathematika* **68** (2022), Issue 4, 1233–1257

[26] J. S. Morrow, Composite images of Galois for elliptic curves over **Q** and entanglement fields, *Math. Comp.* **88** (2019), no. 319, 2389–2421.

[27] F. Najman, Torsion of rational elliptic curves over cubic fields and sporadic points on $X_1(n)$, *Math. Res. Lett.* **23** (2016), no. 1, 245–272.

[28] H. Pasten, Powerful values of polynomials and a conjecture of Vojta, *J. Number Theory* **133** (2013), no. 9, 2964–2998.

[29] H. Pasten, On the arithmetic case of Vojta's conjecture with truncated counting functions, *arxiv:2205.07841*.

[30] B. Banwait, F. Najman and O. Padurariu, Cyclic Isogenies of Elliptic Curves over a Fixed Quadratic Field, *arxiv:2206.0889*.

[31] M. Ru and P. Vojta, A birational Nevanlinna constant and its consequences, *Amer. J. Math.* **142** (2020), no. 3, 957–991.

[32] W. M. Schmidt, Vojta's refinement of the subspace theorem, *Trans. Amer. Math. Soc.* **340** (1993), no. 2, 705–731.

[33] H. Smith, Ramification in division fields and sporadic points on modular curves, *arxiv:1810.04809*.

[34] S. Siksek, Chabauty for symmetric powers of curves, *Algebra & Number Theory* **3** (2009), No. 2, 209–236.

[35] P. Vojta, Diagonal quadratic forms and Hilbert's tenth problem, Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), *Contemp. Math.*, **270**, Amer. Math. Soc., Providence, RI, 2000, pp. 261–274.

[36] P. Vojta, A more general abc conjecture, *Internat. Math. Res. Notices* (1998) **21**, 1103–1116.