Hilbert's 13th Problem for algebraic groups

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Solving polynomials

Classical problem: Solve a polynomial

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of degree n in one variable. Here $a_1, \ldots, a_n$ are elements of some given field K. We fix a base field $k \subset K$. Often $K = k(a_1, \ldots, a_n)$.

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For a general polynomial of degree $n \geqslant 2$, the answer is clearly "no".

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The answer is "yes" is $n \leqslant 4$ and "no" if $n \geqslant 5$ (Ruffini, Abel, Galois).

From polynomials to torsors

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(2) Can every S_n-torsor $\tau\colon\ T \to \mathrm{Spec}(K)$ be split by a solvable field extension L/K? (No, if $n \geqslant 5$.)

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If G is connected, then (2) has a positive answer in many cases but is an open problem in general. For example, for $G = \text{PGL}_n$, the answer is "yes" by the Merkurjev-Suslin Theorem.

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Question 2 (also Tits?) Is it true that every E_8-torsor $T \to \mathrm{Spec}(K)$ is split by a Galois field extension L/K with almost solvable Galois group $\mathrm{Gal}(L/K)$?

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However, for a connected group (and specifically for E_8) this problem turns out to be more accessible.

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In other words, we can obtain every root of $f(x)$ from $a_1, \ldots, a_5$ and elements of the base field k, if we are allowed to apply the four arithmetic operations, extract roots and adjoin roots of polynomials of the form $x^5 + tx + t$.

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Essential dimension of a field extension

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Let K be a field containing a base field k, and L/K be a finite extension. We say that the essential dimension $\mathrm{ed}_k(L/K)$ is $\leqslant d$, if there exists an intermediate field $k \subset K_0 \subset K$ and a field extension L_0/K_0 such that $L = L_0 \otimes_{K_0} K$ and $\mathrm{trdeg}_k(K_0) \leqslant d$.

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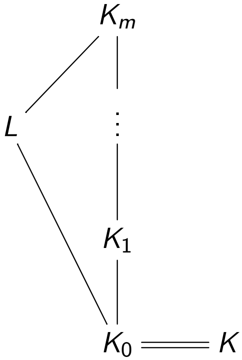
The exact value of $\mathrm{ed}_k(L/K)$ is then the smallest integer d such that $\mathrm{ed}_k(L/K) \leqslant d$.

If L/K is separable, then the inequality $\mathrm{ed}_k(L/K) \leqslant d$ is equivalent to saying that L is generated over K by a single algebraic function in $\leqslant d$ variables.

The level of a finite field extension

We will say that the level $\mathrm{lev}_k(L/K)$ of L/K is $\leqslant d$ if there exists a tower



such that $[K_i : K_{i-1}] < \infty$ and $\mathrm{ed}_k(K_i/K_{i-1}) \leqslant d$ for every $i = 1, \ldots, m$.
The level of L/K is the smallest such d; I will denote it by $\mathrm{lev}_k(L/K)$.

Warning!

It is not known whether or not there exists a finite field extension L/K such that $k \subset K$ and $\text{lev}_k(L/K) > 1$, for any base field k.

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The resolvent degree $\mathrm{rd}_k(G)$ of G is the maximal value of $\mathrm{rd}_k(T)$ as K ranges over field extensions K/k and T ranges over G-torsors $T \to \mathrm{Spec}(K)$.

Hilbert's 13th Problem

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All known upper bounds are of the form $\mathrm{rd}_{\mathbb{C}}(\mathrm{S}_n) \leqslant n - \epsilon(n)$, where $\epsilon(n)$ is an unbounded but very slowly increasing function of n. The latest/strongest are due to Wolfson (2020), Sutherland and Heberle-Sutherland.

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Positive answer to Tits' question $\Longrightarrow$ Conjecture 4 $\Longrightarrow$ Theorem 3.

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Construction of K_2, K_3, K_5 and $K_{\geqslant 7}$

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In characteristic 0 the above argument shows that Serre's conjecture $\Longrightarrow$ positive answer to Tits' Question 1: every E_8-torsor $T \to \mathrm{Spec}(K)$ is split by some solvable extension L/K.