Integrality of unipotent subgroups of Kac-Moody groups

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Arithmetic Aspects of Algebraic Groups

BIRS-Banff (June-2022)

Joint work with Lisa Carbone, Dongwen Liu and Scott S. Murray

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- The second is by generators and relations.

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For infinite dimensional groups, we will use certain representations of the underlying Kac–Moody algebra.

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The Lie algebra $\mathfrak{g}$ can be defined using generators and relations from the data in the generalized Cartan matrix.

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This is constructed from a $\mathbb{Z}$-form of the universal enveloping algebra.

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The *minimal representation-theoretic Kac–Moody group* $G(\mathbb{Q})$ is the group

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We define the *Chevalley subgroup* of $G(\mathbb{Q})$ to be

$$\Gamma(\mathbb{Z}) = \{g \in G(\mathbb{Q}) \mid g(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}}\}.$$

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For Kac–Moody groups, is easy to show that $G(\mathbb{Z}) \subseteq \Gamma(\mathbb{Z})$.

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Here we address the question of integrality of the unipotent subgroup

$$U(\mathbb{Q}) = \langle \exp(t\rho(e_\alpha)) \mid t \in \mathbb{Q},\ \alpha \in \Delta^{\mathrm{re}} \rangle$$

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The question of integrality of $U(\mathbb{Q})$ then becomes

Is $\Gamma(\mathbb{Z}) \cap U(\mathbb{Q}) \subseteq U(\mathbb{Z})$?

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The group $U_{(w)}$ is generated by finitely many real root groups.

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Let $\mathcal{H}$ *be an abelian subgroup of* $U(\mathbb{Q})$ *and* $u \in \mathcal{H}$ *be such that* $uV_{\mathbb{Z}}^{\lambda} \subseteq V_{\mathbb{Z}}^{\lambda}$, *then* $u \in \mathcal{H} \cap U(\mathbb{Z})$.

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When $\mathfrak{g}$ has rank 2 and has a symmetric generalized Cartan matrix, this gives integrality of commutative subgroups U_i of $U(\mathbb{Q})$ for $i = 1, 2$.

Where each U_i is generated by 'half' the positive real roots and $U(\mathbb{Q}) = U_1 * U_2$.

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Thus integrality of $U(\mathbb{Q})$ would be an important step towards proving integrality of $G(\mathbb{Q})$.

Unfortunately our current methods do not extend to a proof of integrality of $U(\mathbb{Q})$, though we conjecture integrality to hold for $U(\mathbb{Q})$, or perhaps a completion of $U(\mathbb{Q})$.

Thank You