Dynamical Systems and Delays

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Scalar Linear Example

$$\frac{dx}{dt} = \lambda x, \qquad x(0) = x_0 \in \mathbb{R}$$

- This is an *Initial Value Problem*. Initial value is $x_0 \in \mathbb{R}$.
- Solution of IVP is function $x(t)$ that satisfies ODE for $t \geqslant 0$ and initial value.
  Question: How does solution depend on value of x_0?
- $\lambda \in \mathbb{R}$ is a parameter. Does not change in time, but we can consider different values.
  Question: How does behaviour of solution change with λ?

Scalar Linear Example

$$\frac{dx}{dt} = \lambda x, \qquad x(0) = x_0 \in \mathbb{R}$$

Solution

$$x(t) = e^{\lambda t} x_0, \qquad t \geqslant 0 \text{ or } t \in \mathbb{R}$$

This solves IVP, but is not the answer to our questions

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Answers:

- If $\lambda < 0$ then $\lim_{t\to+\infty} x(t) = 0$ and $\lim_{t\to-\infty} |x(t)| = +\infty$
  If $\lambda > 0$ then $\lim_{t\to+\infty} |x(t)| = +\infty$ and $\lim_{t\to-\infty} x(t) = 0$
- $\text{sign}(x(t)) = \text{sign}(x_0)$ for all $t \in \mathbb{R}$. Solutions do not cross $x = 0$
- If $x_0 = 0$ then $x(t) = 0$ for all $t \in \mathbb{R}$ is a solution. Its called a *steady state*.
- Steady state at $x = 0$ is stable if $\lambda < 0$ (other solutions approach it), and unstable if $\lambda > 0$.

Scalar Nonlinear Example: The Logistic Equation

$$\frac{dx}{dt} = f(x, \lambda) = \lambda x(1 - x), \qquad x(0) = x_0 \in \mathbb{R}$$

- There is again an exact formula for solution of IVP. We don't need it.

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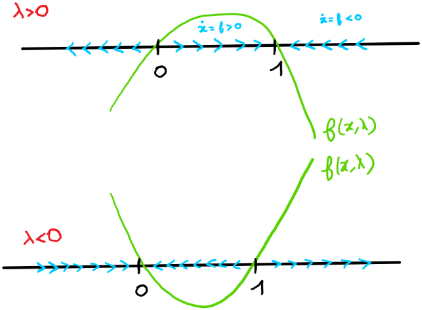
Consider:



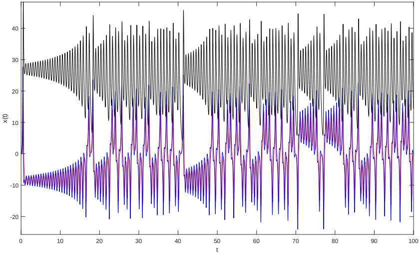
- Plot $f(x, \lambda)$ against x. Then $\text{sign}\left(\frac{dx}{dt}\right) = \text{sign}(f(x, \lambda))$ which allows us to sketch dynamics on $\mathbb{R}$.
- Steady states at $x = 0$ and $x = 1$.
- If $\lambda > 0$ then $x = 0$ is unstable and $x = 1$ is stable with $\lim_{t\to\infty} x(t) = 1$ whenever $x_0 > 0$ and $\lambda > 0$.
- If $\lambda < 0$ then $x = 1$ is unstable and $x = 0$ is stable with $\lim_{t\to\infty} x(t) = 0$ whenever $x_0 < 1$ and $\lambda < 0$.
- Stable steady states are locally but not globally attracting

Lorenz Equations in $\mathbb{R}^3$

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - bz$$

Parameters: $\sigma = 10$, $b = 8/3$, $r = 28$.
Initial condition: $(x_0, y_0, z_0) \in \mathbb{R}^3$
Solution: $\underline{u}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$

Plot solutions components against time t



That's a mess above!

Dynamical Systems in Higher Dimensions

Lorenz Equations in $\mathbb{R}^3$

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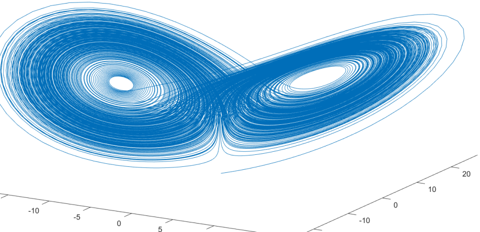

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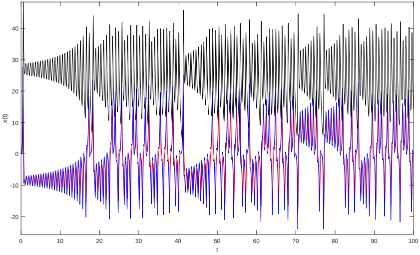
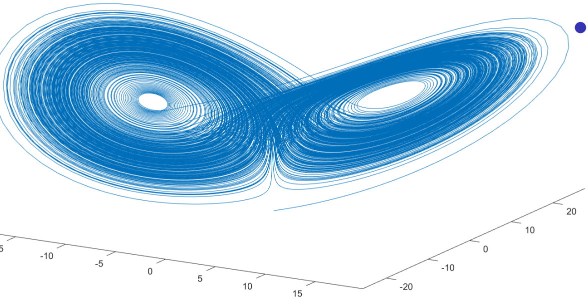


That's a mess above!

Plot solution $(x(t), y(t), z(t))$ as a curve in $\mathbb{R}^3$ parametrised by t.

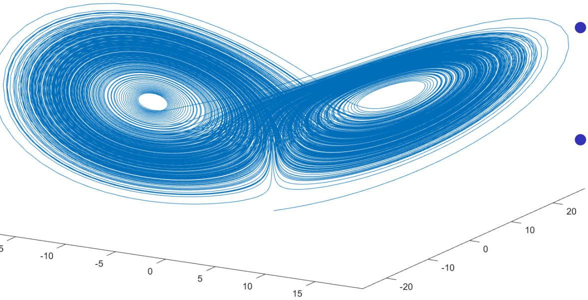
The beautiful Lorenz attractor now appears

Phase Space



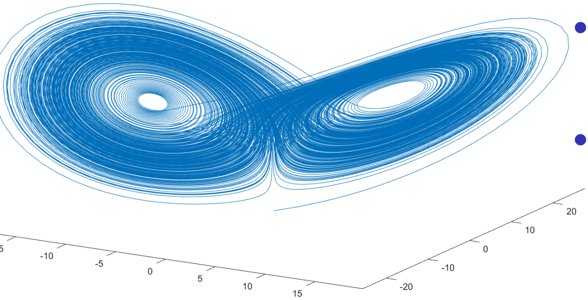
- Why is curve $(x(t), y(t), z(t)) \in \mathbb{R}^3$ so elegant?

Phase Space



- Why is curve $(x(t), y(t), z(t)) \in \mathbb{R}^3$ so elegant?
- Because
  - $(x_0, y_0, z_0) \in \mathbb{R}^3$ also
  - Initial condition specifies a unique solution of ODE.
  - Uniqueness ensures that solutions do not cross.

Phase Space



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Phase Space

Phase space is the space that the initial conditions belong to.

- Set up needs to ensure that solution of IVP for any $(x_0, y_0, z_0) \in \mathbb{R}^3$ is unique
- Crucial feature: dynamics depends only on position, not on time.

Systems with delay, noise, forcing are excluded (for now).

Evolution Operator

Let U be phase space ($\mathbb{R}^n$ for now).
Evolution operator $S(t)$ maps initial condition $u_0 \in \mathbb{R}^n$ to solutions t time units later,

Commutative Semigroup Property

1. $S(t_1)S(t_2) = S(t_2)S(t_1) = S(t_1 + t_2)$ for all $t_1, t_2 \geqslant 0$ (associative and commutative)
2. $S(0) = I$ (identity operator; so a commutative monoid)

Evolution operator allows us to define invariant sets $A \subset U$.

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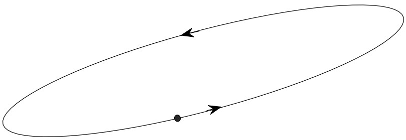
Invariant Sets Under Dynamics

A is *forward invariant* if $S(t)u \in A$ for all $u \in A$ and all $t \geqslant 0$.
A is *backward invariant* if $S(-t)u \in A$ for all $u \in A$ and all $t \leqslant 0$.
A is *invariant* if it is both forward and backward invariant.

Invariant Sets and Stability



Invariant Sets include

- Steady states
- Periodic Orbits
- More exotic things, including invariant tori and strange attractors (inc. Lorenz attractor).

Stability of Steady States

For a steady state $u^* \in \mathbb{R}^n$ let $v(t) = u(t) - u^*$ and linearize to obtain

$$\frac{dv}{dt} = Av,$$

where $A \in \mathbb{R}^{n \times n}$ is the $n \times n$ Jacobian matrix of f evaluated at u^*.

- Steady-state is stable if all eigenvalues λ have negative real parts.
- Floquet theory generalises technique to periodic orbits.

Parameter Continuation and Bifurcations

Recall $\frac{du}{dt} = f(u, \mu)$ has parameter(s).

Implicit Function Theorem

If all eigenvalues of Jacobian matrix at steady-state u^* have $Re(\lambda) \neq 0$ then as parameter μ is varied

- u^* varies continuously in phase space
- Number of eigenvalues with positive and negative real parts is constant, so no change in stability.

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Bifurcation is qualitative change in dynamics as parameter μ is varied.

Bifurcations

Occur when

- Steady-state bifurcation: Real eigenvalue crosses 0. Number and stability of steady states close to u^* changes
- Hopf bifurcation: Complex conjugate pair of eigenvalues cross the imaginary axis. A Periodic orbit is born from the steady state.

There are plenty of more complicated bifurcations

Delays arise in Physics/Engineering

due to

- Transport
- Communication
- Processing Time

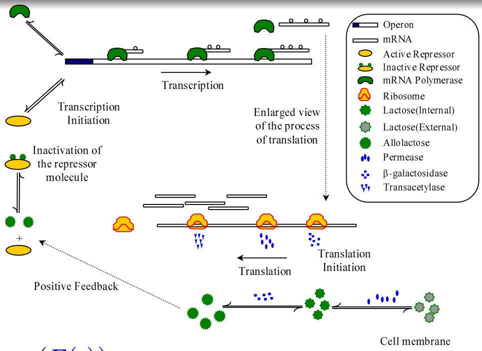
Delays in Physiology

Often blend all three

- Hormone/Antigen must be produced and transported to receptor before signal received
- Maturation/incubation delays often significant
- Its a modelling choice to incorporate a delay, rather than model the entire process leading to that delay.

Goodwin Operon Model

- Protein Production
- mRNA Transcription & Translation
- [GOODWIN 1963,1965] without delay
- τ constant: 1970s, 1980s
- [GEDEON,ARH ET AL, JMB 2022]:



MRNA: $$\frac{dM}{dt}(t) = \beta_M e^{-\mu\tau_M(t)} \frac{v_M(E(t))}{v_M(E(t-\tau_M(t)))} f(E(t-\tau_M(t))) - \bar{\gamma}_M M(t),$$

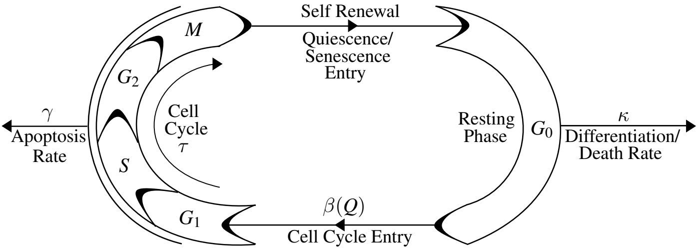
Intermediate: $$\frac{dI}{dt}(t) = \beta_I e^{-\mu\tau_I(t)} \frac{v_I(M(t))}{v_I(M(t-\tau_I(t)))} M(t-\tau_I(t)) - \bar{\gamma}_I I(t),$$

Effector: $$\frac{dE}{dt}(t) = \beta_E I(t) - \bar{\gamma}_E E(t).$$

Threshold delays : $$a_j = \int_{t-\tau_j(t)}^{t} v_j(E(s))ds, \qquad j = M, I.$$

Burns and Tannock G_0 Cell Cycle Model

Cell cycle model: [Burns & Tannock 1970]:



Stem Cell DDE: [Mackey Blood 1978]

$$Q'(t) = -(\kappa + \beta(Q(t)))Q(t) + A\beta(Q(t-\tau))Q(t-\tau),$$

$$\beta(Q) = f\frac{\theta^s}{\theta^s + Q^s}, \qquad A = 2e^{-\gamma\tau}$$

- Describes cell division
- Non-monotone delayed feedback

Hematopoiesis

Body produces more than 10^{11} blood cells per day

- Thats 10^{11} Burns-Tannock cell cycles per day
- Numerous proteins needed for each cell cycle (Goodwin Model)
- A macro-model is needed that simplifies these processes

Granulopoiesis Model [Craig, ARH, Mackey BMB 16]:

$$\text{Stem Cells}: \frac{dQ}{dt} = -\left(\kappa_N(G(t)) + \kappa_\delta + \beta(Q(t))\right) + A_Q(t)\beta\left(Q(t-\tau_Q)\right)Q(t-\tau_Q)$$

$$\text{Reservoir}: \frac{dN_R}{dt} = A_N(t)\kappa_N(G(t-\tau_N))Q(t-\tau_N)\frac{V_{N_M}(G(t))}{V_{N_M}(G(t-\tau_{N_M}(t)))} - (\gamma_{N_R} + \varphi_{N_R}(G(t)))N_R(t)$$

$$\text{Circulating}: \frac{dN}{dt} = \varphi_{N_R}(G(t))N_R(t) - \gamma_N N(t)$$

Maturation Threshold Condition and Velocity Ratio

Constant V:

$$\frac{dN_R}{dt} = K_N(G(t-\tau_N \quad))Q(t-\tau_N \quad)A_N(t) \\ -(\gamma_{N_R} + \varphi_{N_R}(G(t)))N_R$$

Maturation Threshold Condition and Velocity Ratio

Variable V. Tempting to write

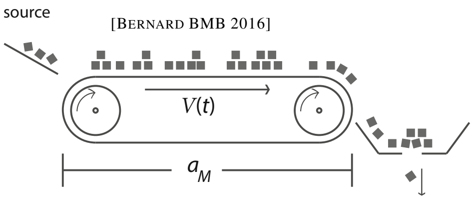
$$\frac{dN_R}{dt} = K_N(G(t-\tau_N(t)))Q(t-\tau_N(t))A_N(t) - (\gamma_{N_R} + \varphi_{N_R}(G(t)))N_R$$

But wrong

Maturation Threshold Condition and Velocity Ratio

Variable V. With velocity correction:

$$\frac{dN_R}{dt} = K_N(G(t-\tau_N(t)))Q(t-\tau_N(t))A_N(t)\frac{V_{N_M}(G(t))}{V_{N_M}(G(t-\tau_{N_M}(t)))}$$
$$-(\gamma_{N_R}+\varphi_{N_R}(G(t)))N_R$$



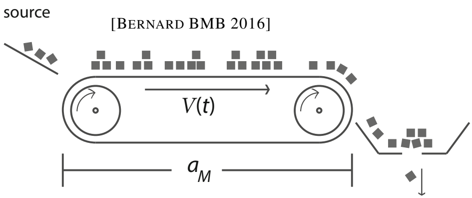
- add bags to conveyor belt at constant rate
- For any constant belt speed they exit at same rate
- Not true if belt speed varies

Maturation Threshold Condition and Velocity Ratio

Variable V. With velocity correction:

$$\frac{dN_R}{dt} = K_N(G(t-\tau_N(t)))Q(t-\tau_N(t))A_N(t)\frac{V_{N_M}(G(t))}{V_{N_M}(G(t-\tau_{N_M}(t)))} - (\gamma_{N_R} + \varphi_{N_R}(G(t)))N_R$$

[BERNARD BMB 2016]



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- Differentiate Threshold condition: $\int_{t-\tau_{N_M}(t)}^{t} V_{N_M}(G(s))ds = a_{N_M}$,

  $\frac{d}{dt}\tau_{N_M}(t) = 1 - \frac{V_{N_M}(G(t))}{V_{N_M}(G(t-\tau_{N_M}(t)))}$ and $\frac{d}{dt}(t-\tau_{N_M}(t)) > 0$
- Same correction term derived in [CRAIG,ARH,MACKEY BMB 2016] from age structured PDE, also as far back as [SMITH MATH BIOSCI '93]. Generalized to random maturation age in [CASSIDY,CRAIG,ARH MATH BIOSCIENG '19]

Phase Space With Delays

Constant Delay DDE IVP

$$\dot{u}(t) = f(t, u(t), u(t-\tau)), \qquad u(0) = u_0 \in \mathbb{R}^d, \quad u(t) \in \mathbb{R}^d, t \geqslant t_0$$

For unique IVP solution for $t \geqslant t_0$

- it is *not* sufficient to specify $u(t_0)$
- To evaluate RHS at t_0 require $u(t_0 - \tau)$
- $\forall s \in [t_0 - \tau, t_0]$ require a value of $u(s)$ to evaluate RHS of DDE at $t = s + \tau \in [t_0, t_0 + \tau]$.

For uniqueness of IVP solution need an initial function

$$u(t) = \varphi(t), \quad \forall t \in [t_0 - \tau, t_0]$$

Provided φ is Lipschitz and $f = f(t, u, v)$ is Lipschitz in its arguments this is sufficient for local existence and uniqueness.

- Recall that phase space is space of initial functions

Breaking Points and Smoothing

$$\dot{u}(t) = f(t, u(t), u(t-\tau)), \quad t \geqslant t_0$$
$$u(t) = \varphi(t), \qquad t \in [t_0 - \tau, t_0]$$

Breaking Point at t_0

Usually $\quad \dot{\varphi}(t_0) \neq f(t_0, \varphi(t_0), \varphi(t_0 - \tau))$
so $\dot{u}(t_0^-) \neq \dot{u}(t_0^+)$. This is a *breaking point*.

Breaking Points at $t_0 + k\tau$

$$\ddot{u}(t) = f_t(t, u(t), u(t-\tau)) + \dot{u}(t) f_u(t, u(t), u(t-\tau)) + \dot{u}(t-\tau) f_v(t, u(t), u(t-\tau)).$$

So $\ddot{u}$ generically discontinuous at $t_0 + \tau$ and similarly,
$u^{(k+1)}(t)$ discontinuous at $t = t_0 + k\tau$ for $k \geqslant 0$.

- Smoothing: $u(t) \in C^{k+1}$ for $t \geqslant t_0 + k\tau$
- *No* such smoothing for neutral problems

DDEs as Dynamical Systems

Phase space of DS is set of (initial) states of system:

$$\{u_t : u_t(\theta) = u(t+\theta),\ \theta \in [-\tau, 0]\}$$

But for $t \in (t_0, t_0+\tau)$ $\exists\, \theta \in (-\tau, 0)$ s.t. $t+\theta = t_0$.
$u_t(\theta)$ is not differentiable at this θ.

Phase Space of continuous functions

$$\{\varphi : \varphi \in C([-\tau, 0], \mathbb{R}^d)\}$$

Includes all polynomials, so phase space is infinite dimensional even for scalar $d = 1$ problems

Retarded Functional Differential Equations

$$\dot{u}(t) = F(t, u_t), \qquad F : \mathbb{R} \times C \to \mathbb{R}^d$$

- Lack of differentiability is a serious hindrance to theory

Linearization for Autonomous Constant Delay DDEs

Scalar Example

Suppose $f(u,v)$ satisfies $f(0,0)=0$ so $u=0$ is a steady state then

$$\dot{u}(t)=f(u(t),u(t-\tau))=f_u(0,0)u(t)+f_v(0,0)u(t-\tau)+h.o.t$$

and linearization is

$$\dot{u}(t)=f_u(0,0)u(t)+f_v(0,0)u(t-\tau)=\mu u(t)+\sigma u(t-\tau)$$

Positing $u(t)=e^{\lambda t}$ gives transcendental *characteristic equation*

$$\lambda-\mu-\sigma e^{-\tau\lambda}=0.$$

Let $\lambda=x+iy$ and take real and imaginary parts:

$$x-\mu-\sigma e^{-\tau x}\cos(y\tau)=y+\sigma e^{-\tau x}\sin(y\tau)=0$$

Infinitely many roots, all lie on curve $y=\pm\sqrt{\sigma^2e^{-2\tau x}-(x-\mu)^2}$

- Laplace transforms show all solutions are exponentials
- Finitely many roots to right of any vertical line in $\mathbb{C}$;
- All characteristic roots satisfy $x<|\mu|+|\sigma|$
- Stable manifolds is infinite dimensional

Linerization for DDEs in $\mathbb{R}^d$

$$\dot{u}(t) = f(u(t), u(t-\tau_1), \ldots, u(t-\tau_m))$$

Let $f(u, v_1, \ldots, v_m) : \mathbb{R}^d \times \mathbb{R}^{md} \to \mathbb{R}^d$ satisfy $f(0, 0, \ldots, 0) = 0$, so $u = 0$ is a steady state.
Linearization is variational equation

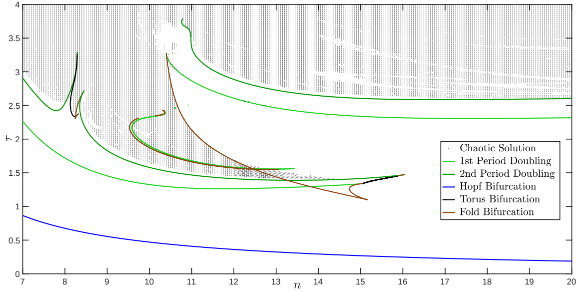
$$\dot{u}(t) = A_0 u(t) + \sum_{j=1}^m A_j u(t-\tau_j),$$

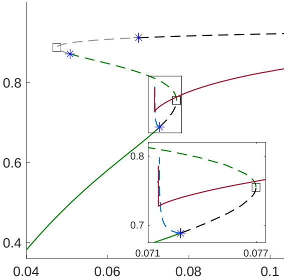
where $A_0 = f_u$ and $A_j = f_{v_j}$ are $d \times d$ matrices evaluated at the steady-state (essentially a Jacobian matrix for each 'delay').
There is nontrivial solution $u(t) = e^{\lambda t}\underline{v} \in \mathbb{R}^d$ with $\Delta(\lambda)\underline{v} = 0$ if

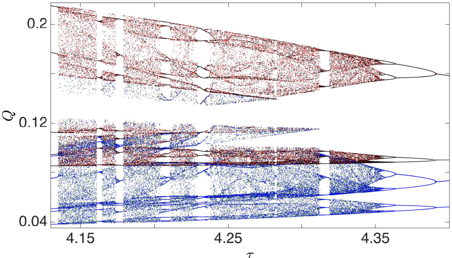
$$0 = \det(\Delta(\lambda)), \quad \Delta(\lambda) = \lambda I_d - A_0 - \sum_{j=1}^m A_j e^{-\lambda\tau_j}.$$

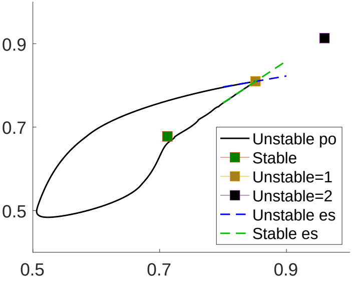
- Characteristic equation has infinitely many roots
- Variational equation soln: $u(t) = \sum_i \alpha_i e^{\lambda_i t}\underline{v}_i$
- Finitely many λ_i with $Re(\lambda_i) > \beta$ for any $\beta \in \mathbb{R}$.
- State-dependent DDEs are linearized by freezing the delays

Numerical tools: DDE-Biftool and DDE23 in Matlab for solution and bifurcation computation









Distributed Delays

Threshold delays are example of distributed delays. Such delays hidden in many models this week. Lets consider infinite delay:

Model Distributed Delay DE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f\left(t, u(t), \int_{-\infty}^{t} u(s)g(t-s)ds\right) = f\left(t, u(t), \int_{0}^{\infty} u(t-\sigma)g(\sigma)d\sigma\right).$$

- PDF:

$$g(t) \geqslant 0, \qquad \int_0^\infty g(t)dt = 1, \qquad \int_0^\infty tg(t)dt = \tau.$$

- Dynamics of $u(t)$ determined by a distribution of previous values.

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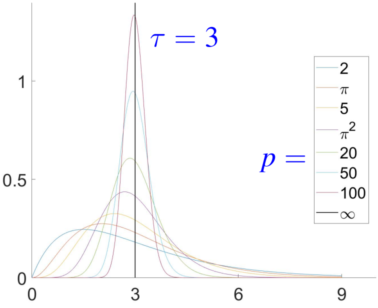
- Dynamics of $u(t)$ determined by a distribution of previous values.
- Problem: Such problems not covered by off the shelf numerical packages for simulation or bifurcation detection
- Should specify a particular PDF.



PDF: $g_a^p(t) = \frac{a^p}{\Gamma(p)} t^{p-1} e^{-at}$,

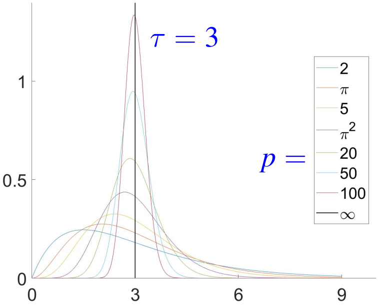
Mean delay: $\tau = p/a$.

Standard deviation: $\sigma^2 = p/a^2$.

$\Gamma(n) = (n-1)! \quad n \in \mathbb{N}$.

$\Gamma(p) = (p-1)\Gamma(p-1), \quad p \in \mathbb{R}/\mathbb{Z}_-$.

Erlang distribution is special case of Gamma distribution with $p \in \mathbb{N}$



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$\Gamma(n) = (n-1)! \quad n \in \mathbb{N}$.

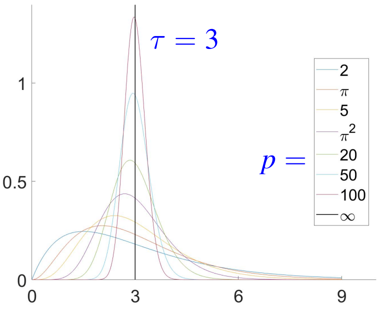
$\Gamma(p) = (p-1)\Gamma(p-1), \quad p \in \mathbb{R}/\mathbb{Z}_-$.

Erlang distribution is special case of Gamma distribution with $p \in \mathbb{N}$

In limit $p \to \infty$ with $\tau = p/a$ constant, $\sigma^2 \to 0$ so $g_a^p(t) \to \delta(t-\tau)$.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f\left(t, u(t), \int_{-\infty}^{t} u(s)g(t-s)ds\right) \to \frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u(t), u(t-\tau))$$

The Gamma Distribution



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Differentiation Property

$$\frac{\mathrm{d}}{\mathrm{d}t} g_a^p(t) = \begin{cases} a(g_a^{p-1}(t) - g_a^p(t)), & p \neq 1 \\ -a g_a^1(t), & p = 1. \end{cases}$$

Gives closed system if $p \in \mathbb{Z}_+$.

Linear Chain Trick

Distributed Delay DE

$$\dot{u}(t) = f\left(t, u(t), \int_0^\infty u(t-\sigma) g_a^n(\sigma) d\sigma\right)$$

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Where we let

$$T_j(t) = \int_0^\infty u(t-s) g_a^j(s)\, ds, \qquad j = 1, \dots, n.$$

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Equivalent ODE System

$$\dot{u}(t) = f(t, u(t), T_n(t))$$

$$\frac{\mathrm{d}T_j}{\mathrm{d}t} = \begin{cases} a(T_{j-1}(t) - T_j(t)), & j = \{2, 3, \ldots, n\}, \\ a(u(t) - T_1(t)), & j = 1. \end{cases}$$

$\tau = n/a$ and $\sigma^2 = n/a^2$.

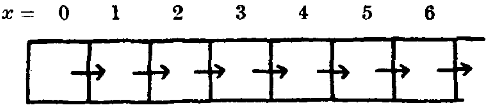
- This is linear chain technique

[Vogel Proc. Int. Symp. Nonlinear Vib. '61],

[MacDonald Time Lags in Biological Models '78],...

Linear Chain Trick

- Equivalent ODE is a **transit compartment model**. They have a long history:



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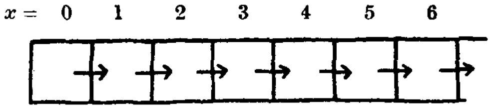

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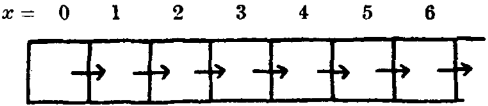

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- These models are finite dimensional, but become discrete delays in limit of infinitely many compartments
- Compartment model requires n integer : distributed DDE does not.
- Given estimates of $\tau \in \mathbb{R}_+$ and $\sigma^2 \in \mathbb{R}_+$,
  No reason to suppose $n = \tau^2/\sigma^2 \in \mathbb{Z}_+$.

Will's model (he says not)

Ultradian Model

$$\frac{\mathrm{d}G}{\mathrm{d}t} = f_4(h_n) + I_G(t) - f_2(G) - f_3(I_i)G$$

$$\frac{\mathrm{d}h_j}{\mathrm{d}t} = \begin{cases} a(h_{j-1}(t) - h_j(t)), & j = \{2, 3, \ldots, n\}, \\ a(I_p(t) - h_1(t)), & j = 1. \end{cases}$$

with $n = 3$ and $a = 1/t_d$

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Linear Chain Trick Equivalence

- Delay $\tau = n/a = 3t_d$
- Standard deviation: $\sigma^2 = n/a^2 = 3t_d^2$
- These are there in the model in whichever formulation, just obscured in ODE formulation.
- Q?: Why $n = 3$ Will?

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Linear Chain Trick Equivalence

- Equivalent Distributed DDE:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = f_4(h_n) + I_G(t) - f_2(G) - f_3(I_i)G$$

$$h_n(t) = \int_0^\infty I_p(t-s) g_a^n(s)\, ds,$$

- In this direction its equivalent to solving the linear ODEs

Transit Compartment Models: Hidden Delays

An ODE Neutrophil Model generalised from [QUARTINO ET AL, PHARM RES 2014] (who had $a = k_{tr}$)

$$\begin{aligned}
\dot{P} &= P(k_P(1 - E_{Drug})(G/G_0)^{\gamma} - k_{tr}(G/G_0)^{\beta}) \\
\dot{T}_1 &= a(G/G_0)^{\beta}(k_{tr}P - aT_1) \\
\dot{T}_j &= a(G/G_0)^{\beta}(T_{j-1} - T_j), \qquad j = 2, \ldots, n \\
\dot{N} &= a(G/G_0)^{\beta}T_n - k_{circ}N \\
\dot{G} &= k_{in} - (k_e + k_{ANC}N)G,
\end{aligned}$$

- If $G = G_0$, constant, rewrite model as a distributed DDE using linear chain technique, with mean **delay** $\tau = n/a$.
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- If $G = G_0$, constant, rewrite model as a distributed DDE using linear chain technique, with mean **delay** $\tau = n/a$.
- Original paper has wrong delay and wrong production rate.
- For general $G(t)$, compartment transit rate is $a(G(t)/G_0)^{\beta}$, state-dependent and linear chain trick does not apply.
- [CAMARA,...,ARH, JPKPD 2018] rescale time and apply linear chain trick to get distributed DDE even for state-dependent delay [CASSIDY,CRAIG,ARH, MATH BIOSCI & ENG 2019] apply generalised linear chain technique to avoid inelegant time rescaling.

Summary

References

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[Craig,ARH,Mackey Bull Math Biol 2016],
[Craig,ARH,Mackey Blood 2016], [Camara,Craig,...,ARH JPKPD 2018],
[Camara,ARH SIADS 2019], [Cassidy,Craig,ARH Math Biosci & Eng 2019],
[Gedeon,ARH et al JMB 2022], [Duruisseaux,ARH JCD 2022]

Conclusions

- Delays allow to simplify physiological modelling
- Delay Differential Equations Define Infinite Dimensional Dynamical Systems. These are tractable numerically and theoretically
- Even scalar DDEs can display very interesting dynamics
- Equations which depend on a distribution of past state values, or where the value of the delay is discrete but depends on state of the system are interesting and tractable.

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