Lectures on Symmetric Functions
with a view towards Hessenberg varieties — Draft

Franco Saliola

October 15, 2022

Contents

1 *Introduction*

In 1995, Richard Stanley associated a symmetric function with every finite graph Γ, called the *chromatic symmetric function* of Γ, which is a generating function of the proper colourings of Γ.

In 2011, John Shareshian and Michelle Wachs introduced a quasisymmetric refinement of the chromatic symmetric functions and they identified a significant class of graphs whose chromatic quasisymmetric functions are actually symmetric functions. Moreover, they conjectured a link between these functions and the characters of representations of symmetric groups on the cohomology space of Hessenberg varieties.

The purpose of these lectures is to survey the (quasi)symmetric function background required to understand the constructions and how these ideas tie together. The emphasis is on "how" things work rather than "why"; we will endeavour to include pointers the literature for proofs and further reading. It is hoped that these notes will provide a road map that the interested student or researcher can follow to gain the background required to appreciate some of the beautiful ideas driving the research in this area.

2 What are (chromatic) symmetric functions?

2.1 Symmetric polynomials

Monomials We will denote a family of variables as

$$x = (x_1, x_2, \dots, x_n).$$

For a sequence $\alpha = (a_1, a_2, \dots, a_n)$ of nonnegative integers, we write

$$x^\alpha = x^{(a_1,\dots,a_n)} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Every monomial x^α admits a unique expression of the form

$$x_{i_1}^{a_{i_1}} x_{i_2}^{a_{i_2}} \cdots x_{i_l}^{a_{i_l}} \text{ with } i_1 < \cdots < i_l \text{ and } a_{i_1}, \dots, a_{i_l} \text{ nonzero;}$$

the sequence $(a_{i_1}, \dots, a_{i_l})$ is called its *sequence of (nonzero) exponents*.

Polynomials A polynomial in the variables $(x_1, \dots, x_n)$ is written as

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha = \sum_{(a_1,a_2,\dots,a_n)\in\mathbb{N}^n} \underbrace{\underbrace{c_{(a_1,a_2,\dots,a_n)}}_{\text{coefficient}} \underbrace{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}_{\text{monomial}}}_{\text{term}}$$

where the c_α are coefficients that all belong to the same ring.

Symmetric polynomials Let S_n denote the group of permutations of $[n]$. A polynomial $f(x_1, \dots, x_n)$ is *symmetric* if

$$\underbrace{f(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(n)})}_{\sigma(f)(x_1,\dots,x_n)} = f(x_1, x_2, \dots, x_n) \qquad \text{for all } \sigma \in S_n.$$

Remarks 2.

1. Since S_n is generated by adjacent transpositions $(i, i+1)$, to determine whether a polynomial is symmetric, it suffices to check

$$f(\dots, x_{i+1}, x_i, \dots) = f(\dots, x_i, x_{i+1}, \dots) \quad \text{for all } i \in [n-1].$$

2. Permuting variables has no effect on the *multiset* of exponents of a monomial. It follows that

   if $f(x_1, \dots, x_n)$ is a symmetric polynomial, then the coefficient of x^α and the coefficient of x^β are equal whenever $\{\{\alpha\}\} = \{\{\beta\}\}$.

Algebra of symmetric polynomials The sum and product of two symmetric polynomials in $\mathbb{C}[x_1, \dots, x_n]$ is again a symmetric polynomial in $\mathbb{C}[x_1, \dots, x_n]$; so we have an algebra of symmetric polynomials:

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} = \{f \in \mathbb{C}[x_1, \dots, x_n] : f \text{ is symmetric}\}.$$

Some notational conventions:

- $\mathbb{N} = \{0, 1, 2, \dots\}$
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$
- $x_i^0 = 1$
- $x^{(a_1,\dots,a_n)} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$
- $[n] = \{1, 2, \dots, n\}$
- $\{\{a_1, \dots, a_n\}\}$ denotes a multiset.

Example 1. *Which of the following polynomials are symmetric?*

$$x_1^2 + x_2 x_3$$
$$x_1 x_2 + x_2 x_3 + x_1 x_3$$
$$x_1^2 + x_1 x_2 + x_2^2$$

N.B. *The answer depends on the set of variables!*

2.2 *Monomial symmetric polynomials*

Symmetrizing monomials One way to construct a symmetric polynomial is to start with a monomial and symmetrize:

$$x_1x_2^4 \xmapsto{\text{symmetrize}} x_1x_2^4 + x_2x_1^4 + x_1x_3^4 + x_2x_3^4 + x_3x_1^4 + x_3x_2^4$$
$$x_1x_2^4x_3 \xmapsto{\text{symmetrize}} x_1x_2^4x_3 + x_2x_1^4x_3 + x_1x_3^4x_2$$

These polynomials are determined by the *multiset* of exponents of the starting monomial, which we write as weakly-decreasing sequences:

$$m_{(4,1)}(x_1,x_2,x_3) = x_1x_2^4 + x_1^4x_2 + x_1x_3^4 + x_1^4x_3 + x_2x_3^4 + x_2^4x_3$$
$$m_{(4,1,1)}(x_1,x_2,x_3) = x_1x_2^4x_3 + x_1^4x_2x_3 + x_1x_2x_3^4$$

These are the *monomial symmetric polynomials*:

$$m_{(\lambda_1,\dots,\lambda_l)}(x_1,\dots,x_n) = \sum x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_l}^{\lambda_l},$$

the sum over *all distinct* monomials with exponents $\lambda_1 \geqslant \ldots \geqslant \lambda_l$.

Some other examples:

$$m_3 = x_1^3 + x_2^3 + x_3^3$$
$$m_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2$$
$$m_{111} = x_1x_2x_3$$

Monomial basis It turns out that every symmetric polynomial is a linear combination of the monomial symmetric polynomials.

Theorem 3. *Every symmetric polynomial $f(x_1,\dots,x_n) \in \mathbb{Z}[x_1,\dots,x_n]^{S_n}$ can be written uniquely as*

$$f(x_1,\dots,x_n) = \sum_\lambda c_\lambda m_\lambda(x_1,\dots,x_n) \quad \textit{with} \quad c_\lambda \in \mathbb{Z},$$

where the sum is over all weakly-decreasing sequences $(\lambda_1,\dots,\lambda_l)$ of positive integers of length $l \leqslant n$.

Example 4. *Here is a symmetric polynomial in the variables x_1,x_2,x_3.*

$$f(x_1,x_2,x_3) = 2x_1^2 - 3x_1x_2 + 2x_2^2 - 3x_1x_3 - 3x_2x_3 + 2x_3^2.$$

Pick a monomial and subtract the corresponding symmetric polynomial:

$$f(x_1,x_2,x_3) - (-3)m_{(1,1)}(x_1,x_2,x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2$$

The result is a symmetric polynomial with fewer monomials. Repeat:

$$f(x_1,x_2,x_3) - (-3)m_{(1,1)}(x_1,x_2,x_3) - 2m_{(2)}(x_1,x_2,x_3) = 0,$$

and so

$$f(x_1,x_2,x_3) = 2m_{(2)}(x_1,x_2,x_3) - 3m_{(1,1)}(x_1,x_2,x_3).$$

2.3 *Partitions and Compositions*

A sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

is called

- a *composition* of d if $\lambda_1 + \lambda_2 + \cdots + \lambda_l = d$, in which case we write $\lambda \in \mathrm{Comp}_d$ or $\lambda \models d$;
- a *partition* of d if $\lambda_1 + \lambda_2 + \cdots + \lambda_l = d$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$, in which case we write $\lambda \in \mathrm{Part}_d$ or $\lambda \vdash d$;

We will use the following notation:

- the *size* or *weight* of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_l = d$;
- the integers $\lambda_1, \ldots, \lambda_l$ are called the *parts* of λ;
- the *length* of λ is its number of parts, which we denote by $\ell(\lambda) = l$.
- the *(Young) diagram* associated with λ is a collection of *boxes* or *cells* arranged in left-justified rows, with the first row containing λ_1 cells, the second row containing λ_2 cells, and so on.

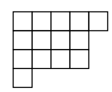


Figure 1: The Young diagram of the partition $\lambda = (5, 4, 4, 1) \vdash 14$.

Corollary 5. *The symmetric polynomials*

$$\{m_\lambda(x_1, \ldots, x_n) : \lambda \in \mathrm{Part}_d \ \& \ \ell(\lambda) \leqslant n\}$$

form a basis of the space of symmetric polynomials in $x_1, \ldots, x_n$ that are homogeneous[1] of degree d.

[1] Recall that a polynomial $f(x_1, \ldots, x_n)$ is *homogeneous of degree d* if the sum of the multiset of exponents of every monomial appearing in f is d. That is, $f = \sum_\alpha c_\alpha x^\alpha$ and $c_\alpha \neq 0$ implies $|\alpha| = d$.

Example 6. *Here is the monomial basis of the space of symmetric polynomials in x_1, x_2, x_3 that are homogeneous of degree 3:*

$$\begin{aligned} m_3 &= x_1^3 + x_2^3 + x_3^3 \\ m_{21} &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2 \\ m_{111} &= x_1 x_2 x_3 \end{aligned}$$

N.B. *This basis is* not multiplicative*:*

$$m_2 m_1 = m_{21} + m_3 \ (\neq m_{21})$$

2.4 *Elementary symmetric polynomials*

The elementary symmetric polynomials appear in the expansion of a product of linear monic polynomials:

$$\begin{aligned}&(t-x_1)(t-x_2)(t-x_3)\\&\quad = t^3 - \underbrace{(x_1+x_2+x_3)}_{e_1(x_1,x_2,x_3)} t^2 + \underbrace{(x_1x_2+x_1x_3+x_2x_3)}_{e_2(x_1,x_2,x_3)} t - \underbrace{x_1x_2x_3}_{e_3(x_1,x_2,x_3)}\end{aligned}$$

Define the *k-th elementary symmetric polynomial* as

$$e_k(x_1,\ldots,x_n) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} x_{i_1}\cdots x_{i_k} = m_{1^k}(x_1,\ldots,x_n).$$

These polynomials are algebraically independently, which implies that

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_l} \qquad (\lambda\vdash n)$$

are linearly independent and give rise to another basis of $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$.

Generating function. After substituting $t\mapsto \frac{1}{t}$ and multiplying by t^3, we get

$$\begin{aligned}&(1+tx_1)(1+tx_2)(1+tx_3)\\&\quad = 1+e_1t^1+e_2t^2+e_3t^3.\end{aligned}$$

In general,

$$\sum_{k\geqslant 0} e_k(x_1,\ldots,x_n)t^k = \prod_{i=1}^n (1+tx_i).$$

Some other examples:

$$\begin{aligned}e_3 &= x_1x_2x_3\\ e_{21} &= (x_1x_2+x_1x_3+x_2x_3)(x_1+x_2+x_3)\\ e_{111} &= (x_1+x_2+x_3)^3\end{aligned}$$

Fundamental theorem of symmetric polynomials

Theorem 7. *Every symmetric polynomial $f(x_1,\ldots,x_n)\in\mathbb{Z}[x_1,\ldots,x_n]^{S_n}$ can be written uniquely as*

$$f(x_1,\ldots,x_n) = \sum_\lambda c_\lambda e_\lambda(x_1,\ldots,x_n) \quad \textit{with} \quad c_\lambda\in\mathbb{Z}.$$

In other words,[2] *every symmetric polynomial is a* polynomial in the elementary symmetric polynomials $e_1, e_2, \ldots, e_n$:

$$\mathbb{Z}[e_1,\ldots,e_n] = \mathbb{Z}[x_1,\ldots,x_n]^{S_n}. \tag{1}$$

Example 8. *We have*[3] *the following expansion of $m_{(2,1)}(x_1,x_2,x_3)$:*

$$m_{(2,1)}(x_1,x_2,x_3) = e_2(x_1,x_2,x_3)e_1(x_1,x_2,x_3) - 3e_3(x_1,x_2,x_3).$$

Thus, $m_{(2,1)} = f(e_1,e_2,e_3)$, where $f(y_1,y_2,y_3) = y_1y_2 - 3y_3$.

Monomial expansion of elementary symmetric polynomials In the expansion of $e_3e_2e_2$, one finds the product

$$\underbrace{(x_1^1x_2^1x_3^0x_4^1)}_{\text{from } e_3}\underbrace{(x_1^0x_2^1x_3^1x_4^0)}_{\text{from } e_2}\underbrace{(x_1^0x_2^1x_3^0x_4^1)}_{\text{from } e_2} \longleftrightarrow \begin{bmatrix}1&1&0&1\\0&1&1&0\\0&1&0&1\end{bmatrix}.$$

These are in bijection with 0/1-matrices whose row sums are $(3,2,2)$.

Proposition 9. *Let $\lambda\in\mathrm{Part}_n$.*

$$e_\lambda = \sum_{\mu\in\mathrm{Part}_n} M_{\lambda,\mu} m_\mu,$$

where $M_{\lambda,\mu}$ is the number of matrices with entries in $\{0,1\}$ whose rows sum to λ and columns sum to μ. Consequently, the transition matrix between the bases $\{m_\lambda : \lambda\vdash n\}$ and $\{e_\lambda : \lambda\vdash n\}$ is symmetric.

[2] There are two ways to interpret the notation on the left of Equation (1), and luckily they agree here.

- $\mathbb{Z}[e_1,\ldots,e_n]$ is the subring of $\mathbb{Z}[x_1,\ldots,x_n]$ generated by the polynomials $e_k(x_1,\ldots x_n)$; the theorem states that it is $\mathbb{Z}[x_1,\ldots,x_n]^{S_n}$.
- $\mathbb{Z}[e_1,\ldots,e_n]$ can also be interpreted as a polynomial ring because

  $$\begin{aligned}\mathbb{Z}[y_1,\ldots,y_n] &\to \mathbb{Z}[e_1,\ldots,e_n]\\ y_k &\mapsto e_k(x_1,\ldots,x_n)\end{aligned}$$

  is an isomorphism of rings.

[3] Here are the details of the expansion:

$$\begin{aligned}&m_{(2,1)}(x_1,x_2,x_3)\\&= x_1^2x_2+x_1x_2^2+x_1^2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2\\&= x_1x_2(x_1+x_2)+x_1x_3(x_1+x_3)+x_2x_3(x_2+x_3)\\&= x_1x_2(x_1+x_2+x_3)+x_1x_3(x_1+x_2+x_3)\\&\qquad +x_2x_3(x_1+x_2+x_3)-3x_1x_2x_3\\&= e_2(x_1,x_2,x_3)e_1(x_1,x_2,x_3)-3e_3(x_1,x_2,x_3)\end{aligned}$$

2.5 *Complete symmetric polynomials*

The *k-th complete homogeneous symmetric polynomial* is the sum of all monomials of degree k:

$$h_k(x_1,\dots,x_n) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_k\leqslant n} x_{i_1}\cdots x_{i_k} = \sum_{\lambda\vdash k} m_\lambda(x_1,\dots,x_n).$$

These polynomials are also algebraically independent, and we define

$$h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}.$$

Theorem 10. *Every symmetric polynomial* $f(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]^{S_n}$ *can be written uniquely as*

$$f(x_1,\dots,x_n) = \sum_\lambda c_\lambda h_\lambda(x_1,\dots,x_n) \quad \textit{with} \quad c_\lambda\in\mathbb{Z}.$$

In other words,

$$\mathbb{Z}[h_1,\dots,h_n] = \mathbb{Z}[x_1,\dots,x_n]^{S_n}.$$

Generating function. Like the e_k, the h_k can also be obtained by expanding a product:

$$\sum_{k\geqslant 0} h_k(x_1,\dots,x_n)t^k = \prod_{i=1}^n \frac{1}{1-tx_i}.$$

Some other examples:

$$\begin{aligned} h_3 &= x_1^3+x_1^2x_2+x_1x_2^2+x_2^3 \\ &\quad x_1^2x_3+x_1x_2x_3+x_2^2x_3+ \\ &\quad x_1x_3^2+x_2x_3^2+x_3^3 \\ h_{21} &= (x_1+x_2+x_3)\times \\ &\quad (x_1^2+x_1x_2+x_2^2+ \\ &\quad x_1x_3+x_2x_3+x_3^2) \\ h_{111} &= (x_1+x_2+x_3)^3 \end{aligned}$$

Monomial expansion of homogeneous symmetric polynomials Let $\lambda\in\mathrm{Part}_n$.

$$h_\lambda = \sum_{\mu\in\mathrm{Part}_n} N_{\lambda,\mu}m_\mu,$$

where $N_{\lambda,\mu}$ is the number of matrices with entries in $\mathbb{N}$ whose rows sum to λ and columns sum to μ. Consequently, the transition matrix between the bases $\{m_\lambda : \lambda\vdash n\}$ and $\{h_\lambda : \lambda\vdash n\}$ is symmetric.

The involution ω Since h_k is a symmetric polynomial in $x_1,\dots,x_n$, it can be expressed as a polynomial in $e_1,\dots,e_n$. It turns out that[4]

$$h_k = f(e_1,\dots,e_n) \quad \text{iff} \quad e_k = f(h_1,\dots,h_n).$$

For example,

$$h_3 = e_{111}-2e_{21}+e_3 \qquad \text{and} \qquad e_3 = h_{111}-2h_{21}+h_3.$$

[4] This can by proved by considering

$$e(t)=\sum_{k\in\mathbb{N}} e_kt^k \text{ and } h(t)=\sum_{k\in\mathbb{N}} h_kt^k;$$

noting that $e(t)h(-t)=1$, which gives

$$\sum_{i+j=k}(-1)^i e_ih_j = 0;$$

and deducing the result by induction.

Proposition 11. *The algebra morphism defined by*

$$\begin{array}{ccc} \mathbb{Z}[e_1,\dots,e_n] & \xrightarrow{\omega} & \mathbb{Z}[h_1,\dots,h_n] \\ e_k & \mapsto & h_k \end{array}$$

is an involution (that is, $\omega^w = \mathrm{Id}$). Consequently,

$$\mathbb{Z}[e_1,\dots,e_n] = \mathbb{Z}[h_1,\dots,h_n] = \mathbb{Z}[x_1,\dots,x_n]^{S_n}.$$

2.6 *Power sum symmetric polynomials*

The *k-th power sum symmetric polynomial* is

$$p_k(x_1,\ldots,x_n) = x_1^k + x_2^k + \cdots + x_n^k = m_k(x_1,\ldots,x_n).$$

These polynomials are also algebraically independent, and we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}.$$

Some other examples:

$$p_3 = x_1^3 + x_2^3 + x_3^3$$
$$p_{21} = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$$
$$p_{111} = (x_1 + x_2 + x_3)^3$$

Theorem 12. *Every symmetric polynomial $f(x_1,\ldots,x_n) \in \mathbb{Q}[x_1,\ldots,x_n]^{S_n}$ can be written uniquely as*

$$f(x_1,\ldots,x_n) = \sum_\lambda c_\lambda p_\lambda(x_1,\ldots,x_n) \quad \text{with} \quad c_\lambda \in \mathbb{Q}.$$

Remark 13. The result does not hold over $\mathbb{Z}$ because $m_{11} = \frac{1}{2}p_{11} - \frac{1}{2}p_2$.

Monomial expansion of power sum symmetric polynomials Let $\lambda \in \text{Part}_n$.

$$p_\lambda = \sum_{\mu \in \text{Part}_n} R_{\lambda,\mu} m_\mu,$$

where $R_{\lambda,\mu}$ is the number of ordered set partitions $(B_1,\ldots,B_k)$ of $[\ell(\lambda)]$ such that $\mu_j = \sum_{i\in B_j} \lambda_i$ for all $j \in [k]$.

Example 14. *For instance, if $\lambda = (2,1)$, then we consider the ordered set partitions of $[2]$:*

$$(\{1,2\}) \qquad (\{1\},\{2\}) \qquad (\{2\},\{1\})$$

which result in the following sequences of partial sums:

$$(\lambda_1+\lambda_2) = (3) \qquad (\lambda_1,\lambda_2) = (2,1) \qquad (\lambda_2,\lambda_1) = (1,2)$$

Thus,

$$p_{21} = m_3 + m_{21}.$$

Other examples:

$$p_3 = m_3$$
$$p_{21} = m_{21} + m_3$$
$$p_{111} = 6m_{111} + 3m_{21} + m_3$$

For $\lambda = (1,1,1)$, there are 13 ordered set partitions of $[3]$:

$$(\{1\},\{2\},\{3\}) \quad (\{1\},\{3\},\{2\})$$
$$(\{2\},\{1\},\{3\}) \quad (\{3\},\{1\},\{2\})$$
$$(\{2\},\{3\},\{1\}) \quad (\{3\},\{2\},\{1\})$$
$$(\{1\},\{2,3\}) \quad (\{2\},\{1,3\})$$
$$(\{3\},\{1,2\}) \quad (\{1,2\},\{3\})$$
$$(\{1,3\},\{2\}) \quad (\{2,3\},\{1\})$$
$$(\{1,2,3\})$$

which result in the following sequences:

$$(1,1,1) \quad (1,1,1) \quad (1,1,1)$$
$$(1,1,1) \quad (1,1,1) \quad (1,1,1)$$
$$(1,2) \quad (1,2) \quad (1,2)$$
$$(2,1) \quad (2,1) \quad (2,1)$$
$$(3)$$

Thus,

$$p_{111} = 6m_{111} + 3m_{21} + m_3.$$

Power sum expansion of elementary and homogeneous symmetric polynomials

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda$$
$$e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{1}{z_\lambda} p_\lambda$$

where $z_\mu = 1^{d_1} d_1! \, 2^{d_2} d_2! \cdots n^{d_1} d_n!$ and $d_i = \text{mult}_i(\mu)$.

Example 15.

$$h_1 = p_1$$
$$h_2 = \tfrac{1}{2}p_{11} + \tfrac{1}{2}p_2$$
$$h_3 = \tfrac{1}{6}p_{111} + \tfrac{1}{2}p_{21} + \tfrac{1}{3}p_3$$

2.7 *Schur polynomials*

Recurring theme: symmetric polynomials from actions of S_n An eloquent way to construct symmetric polynomials is from (certain) group actions of the symmetric group. More precisely, if we have

- a set of combinatorial objects $\mathcal{O}$, a map $\alpha : \mathcal{O} \to \mathbb{N}^n$, and
- an action of S_n on $\mathcal{O}$ satisfying $\alpha_i(\sigma(T)) = \alpha_{\sigma(i)}(T)$,

then the following polynomial is symmetric:

$$f(x_1,\ldots,x_n) = \sum_{T\in\mathcal{O}} x^{\alpha(T)} = \sum_{T\in\mathcal{O}} x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots x_n^{\alpha_n(T)}.$$

We will apply this in a few different settings.

$$\begin{aligned} & f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots x_{\sigma(n)}) \\ &= \sum_{T\in\mathcal{O}} x_{\sigma(1)}^{\alpha_1(T)} x_{\sigma(2)}^{\alpha_2(T)} \cdots x_{\sigma(n)}^{\alpha_n(T)} \\ &= \sum_{T\in\mathcal{O}} x_{\sigma(1)}^{\alpha_1(\sigma T)} x_{\sigma(2)}^{\alpha_2(\sigma T)} \cdots x_{\sigma(n)}^{\alpha_n(\sigma T)} \\ &= \sum_{T\in\mathcal{O}} x_{\sigma(1)}^{\alpha_{\sigma(1)}(T)} x_{\sigma(2)}^{\alpha_{\sigma(2)}(T)} \cdots x_{\sigma(n)}^{\alpha_{\sigma(n)}(T)} \\ &= f(x_1,\ldots,x_n). \end{aligned}$$

Tableaux Let $\lambda \vdash n$.

- A *(Young) tableau* of shape λ is a filling of the cells of the Young diagram of λ by positive integers, called the *entries* of the tableau.
- A tableau is *semistandard* if its entries are *weakly increasing in rows* when read from left-to-right; and *strictly increasing in columns* when read from longest-to-shortest row.
- A tableau of shape λ is *standard* if it is a semistandard whose cells are filled with $1, 2, \ldots, n$ (each occuring exactly once).
- The *weight* or *content* of a tableau T is the vector $(a_1, a_2, \ldots)$ with $a_i = \text{mult}_i(T)$, where $\text{mult}_i(T)$ is the number of times i occurs in T.

$$\begin{array}{|c|c|c|c|}\hline 2&2&4&5\\\hline 4&4&5&7\\\hline 5&7\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

Figure 2: A semistandard tableau of shape $(4,4,2,1)$.

Schur polynomials The *Schur polynomial* indexed by λ is

$$s_\lambda(x_1,\ldots,x_n) = \sum_{T\in\mathsf{SSYT}(\lambda,[n])} x^{\text{weight}(T)},$$

where $\mathsf{SSYT}(\lambda,[n])$ is the set of semistandard tableau of shape λ and with entries in $[n]$.

For example, if $\lambda = (2,1)$ and $n = 2$, then $\mathsf{SSYT}(\lambda,[n])$ consists of

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$$

so that

$$s_{(2,1)}(x_1,x_2) = x_1^2x_2 + x_1x_2^2 = m_{(2,1)}(x_1,x_2);$$

and if $n = 3$, then $\mathsf{SSYT}(\lambda,[n])$ consists of

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

so that

$$\begin{aligned} s_{(2,1)}(x_1,x_2,x_3) &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &= m_{(2,1)}(x_1,x_2,x_3) + 2m_{(1,1,1)}(x_1,x_2,x_3) \end{aligned}$$

Schur polynomials are symmetric First note that it suffices to show that $s_\lambda(x_1, \ldots, x_n)$ is invariant under interchanging x_i and x_{i+1}. Hence, we need a bijection on $\mathsf{SSYT}(\lambda, [n])$ that maps

- a tableau with a occurrences of i and b occurrences of $i+1$ to
- a tableau with b occurrences of i and a occurrences of $i+1$;

and fixes all other elements of the tableau. This is the Bender–Knuth involution:

1. Let $T \in \mathsf{SSYT}(\lambda, [n])$. We say an occurrence of i or $i+1$ in T is *fixed* if it belongs to a column that contains both i and $i+1$.

1	1	1	1	2	2	2	2	2	3
2	2	3	3	3	3				
3									

2 and 3 are fixed
2 and 3 are non-fixed

2. In each row, interchange the number of *non-fixed* is and $(i+1)$s:

1	1	1	1	2	2	2	3	3	3
2	2	2	3	3	3				
3									

row 1 : one 2 and three 3s
$\mapsto$ three 2s and one 3
row 2 : two 2s and one 3s
$\mapsto$ one 2 and two 3s

Monomial expansion of Schur polynomials Let $\lambda \in \mathrm{Part}_n$.

$$s_\lambda = \sum_{\mu \in \mathrm{Part}_n} K_{\lambda,\mu} m_\mu,$$

where $K_{\lambda,\mu}$ is the number of SSYT of shape λ and content μ.

More generally, if λ/ν is a skew partition of size n, then

$$s_{\lambda/\nu} = \sum_{\mu \in \mathrm{Part}_n} K_{\lambda/\nu,\mu} m_\mu,$$

where $K_{\lambda/\nu,\mu}$ is the number of SSYT of shape λ/ν and content μ.

For example, $\mathsf{SSYT}((2,1),[3])$ consists of

1	1	1	1	1	2	1	2	1	3	1	3	2	2	2	3
2		3		2		3		2		3		3		3	

One 3 of these have content that is a partition. Hence,

$$s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}.$$

Note that this formula is independent of the number of variables.

$$s_{(2,1)}(x_1,x_2,x_3) = m_{(2,1)}(x_1,x_2,x_3) + 2m_{(1,1,1)}(x_1,x_2,x_3)$$
$$s_{(2,1)}(x_1,x_2) = m_{(2,1)}(x_1,x_2) + 2\,\underbrace{m_{(1,1,1)}(x_1,x_2,x_3)}_{0}$$

2.8 *Chromatic symmetric polynomials*

- A *graph* $\Gamma = (V, E)$ consists of a finite set of vertices V and a set of edges E, which are unordered pairs of vertices.
- A *colouring* of Γ is a function $\kappa : V \to C$, where C is a set of colours.
- A colouring κ is *proper* if adjacent vertices have different colours:

$$\{i, j\} \in E \implies \kappa(i) \neq \kappa(j).$$

- The *chromatic symmetric function* of Γ is a sum of monomials, one for each proper colouring of Γ:

$$X_\Gamma(x_1, \ldots, x_n) = \sum_{\substack{\text{proper}\\ \text{colourings}\\ \kappa : V \to [n]}} \prod_{v \in V} x_{\kappa(v)}.$$

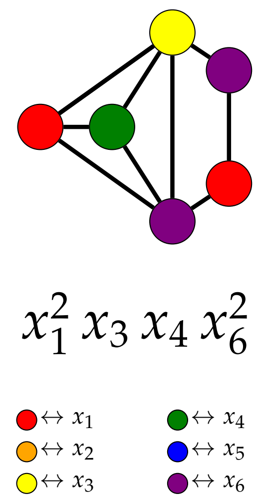
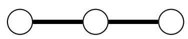


Figure 3: The monomial associated with a proper colouring of a graph

Example 16. *For Γ equal to the path graph*  *there are*

- $3!$ *ways to colour Γ with 3 colours, each giving the monomial $x_1 x_2 x_3$;*
- 2 *ways to colour Γ with 2 colours, giving the monomials $x_i^2 x_j$ and $x_i x_j^2$.*

Thus,

$$\begin{aligned} X_\Gamma(x_1, x_2, x_3) &= 6x_1x_2x_3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &= 6m_{(1,1,1)} + m_{(2,1)} \end{aligned}$$

Proposition 17. *1. $X_\Gamma(x_1, \ldots, x_n)$ is a symmetric polynomial.*

2. $X_\Gamma(\underbrace{1, \ldots, 1}_{n})$ is equal to the number of proper n-colourings of Γ.

Proof.
- The second statement follows from the observation that there is exactly one monomial for each proper n-colouring of Γ.
- The first statement follows from the fact that if $\sigma \in S_n$ and $\kappa : V \to [n]$ is a proper colouring, then $\sigma \circ \kappa$ is again a proper colouring. □

Example 18. *For Γ in Figure 3,*

$$\begin{aligned} X_\Gamma &= 720\, m_{111111} + 144\, m_{21111} + 28\, m_{2211} \\ &= 168\, s_{111111} + 60\, s_{21111} + 28\, s_{2211} \\ &= 28\, e_{42} + 32\, e_{51} + 108\, e_6 \end{aligned}$$

Some numerology:

$$\begin{aligned} 28 + 32 + 108 &= \#\ \textit{acyclic orientations of } \Gamma \\ 28 + 32 &= \#\ \textit{acyclic orientations of } \Gamma \textit{ with } 2 \textit{ sinks} \\ 108 &= \#\ \textit{acyclic orientations of } \Gamma \textit{ with } 1 \textit{ sink} \end{aligned}$$

2.9 *The Stanley–Stembridge conjecture*

e-expansion and acyclic orientations The numerology from the previous example holds in general.

Theorem 19 (Stanley)**.** *Let Γ be a graph and suppose the expansion of X_Γ in the basis of elementary symmetric polynomials is*

$$X_\Gamma = \sum_\lambda c_\lambda e_\lambda.$$

Then

$$\sum_{\substack{\lambda \\ \ell(\lambda)=j}} c_\lambda = \text{number of acyclic orientations of } \Gamma \text{ with exactly } j \text{ sinks}.$$

Example 20. *This holds even if X_Γ is not e-positive. For example, the chromatic symmetric polynomial of the claw graph $K_{1,3}$ is*

$$X_{K_{1,3}} = e_{(2,1,1)} - 2e_{(2,2)} + 5e_{(3,1)} + 4e_{(4)}.$$

And $K_{1,3}$ admits: 1 acyclic orientation with exactly 3 sinks; $3 = 5 - 2$ acyclic orientations with exactly 2 sinks; 6 acyclic orientations with exactly 1 sinks.

Trees and chromatic symmetric polynomials Given that the coefficients of the e-expansion of X_Γ is related to interesting properties of the graph, one can inquire which properties of Γ are encoded by X_Γ. One question posed by Stanley (1995) is whether a tree can be reconstructed from X_Γ. This has been verified for trees up to 29 vertices.

This is not true of every graph, since there are different (non-tree) graphs that share the same chromatic symmetric polynomial.

The Stanley–Stembridge conjecture It is an open problem to characterize graphs Γ for which X_Γ is e-positive. A conjecture of Stanley and Stembridge, which predates the introduction of chromatic symmetric polynomials, posits e-positivity for a certain graphs defined as follows.

The *incomparability graph* of a poset P is the graph on P with edges

$$E(\mathrm{Inc}(P)) = \{\{i,j\} : i \text{ and } j \text{ are incomparable in } P\}.$$

A poset is *$(3+1)$-free* if it does not contain an induced subposet isomorphic to the direct sum of a 3-chain and a 1-chain; see Figure 4.

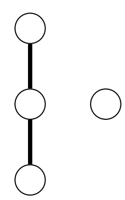


Figure 4: The $(3+1)$-poset is the direct sum of a 3-chain and a 1-chain.

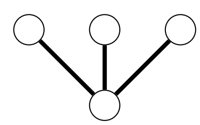


Figure 5: $K_{1,3}$, or the *claw graph*. A graph is *clawfree* if it does not contain an induced subgraph isomorphic to $K_{1,3}$. The incomparability graph of a poset P is clawfree iff P is $(3+1)$-free.

Conjecture 21 (Stanley–Stembridge)**.** *If Γ is the incomparability graph of a $(3+1)$-free poset, then X_Γ is e-positive.*

A significant step towards proving this conjecture was made by Mathieu Guay-Paquet, who proved it is sufficient to prove the conjecture for posets that are both $(3+1)$-free and $(2+2)$-free. This smaller class of posets will help establish the link with Hessenberg varieties.

2.10 *Symmetric polynomials versus symmetric functions*

A word of caution A word of caution is warranted at this point.

- I have deliberately restrained myself from using the terminology "symmetric function". However, many people refer to symmetric polynomials as symmetric functions. (*But they are not the same thing!*)
- One reason for this is that symmetric functions can be viewed as formal power series in countably many variables $\{x_1, x_2, x_3, \ldots\}$ that are invariant under permutations of the variables.
- I find this leads to confusion when one is first learning about about symmetric functions, even more so once one realizes that symmetric functions aren't really functions at all.[5]

[5] At least, not in the obvious sense.

Motivation After working a bit with symmetric polynomials, one starts to recognize that many relations between symmetric polynomials do not depend on the number of variables. For example, we saw that

$$m_{(2,1)}(x_1, x_2, x_3) = e_2(x_1, x_2, x_3)e_1(x_1, x_2, x_3) - 3e_3(x_1, x_2, x_3),$$

which remains valid if we work over any number of variables:

$$m_{(2,1)}(x_1, \ldots, x_n) = e_2(x_1, \ldots, x_n)e_1(x_1, \ldots, x_n) - 3e_3(x_1, \ldots, x_n).$$

We can summarize this observation by dropping any reference to the variables and writing

$$m_{(2,1)} = e_{(2,1)} - 3e_{(3)}.$$

This raises the question of whether there exists a ring that formalizes this idea. We are looking for a ring Sym with the following properties:

1. Sym has various bases $\{m_\lambda\}$, $\{e_\lambda\}$, etc.—all indexed by partitions;
2. symmetric polynomials are "shadows" of elements of Sym:
$$m_\lambda \mapsto m_\lambda(x_1, \ldots, x_n);$$
3. identities in Sym reflect identities among symmetric polynomials that are independent of the number of variables.

The relation between symmetric functions and symmetric polynomials parallels Plato's *Theory of Forms*, which posits that the physical world is only a shadow of the realm of concepts.

Formal power series Let $\mathbb{C}[\![x]\!]$ denote the $\mathbb{C}$-algebra of all formal power series in the variables $x = (x_1, x_2, x_3, \ldots)$. The elements of $\mathbb{C}[\![x]\!]$ are (possibly infinite) linear combinations of the form

$$\sum_{\substack{\alpha=(\alpha_1,\alpha_2,\ldots)\\ \text{finite support}}} c_\alpha \underbrace{x_1^{\alpha_1} x_2^{\alpha_2} \cdots}_{x^\alpha}$$

A power series $\sum_\alpha c_\alpha x^\alpha$ is said to be

1. *homogeneous of degree d* if $c_\alpha \neq 0$ implies $\deg(x^\alpha) = d$ for all α;
2. *of bounded degree* if there is a d such that $c_\alpha \neq 0$ implies $\deg(x^\alpha) \leqslant d$.

Of bounded degree:

$$x_1^2 + x_2^2 + x_3^2 + \cdots$$

Not of bounded degree:

$$1 + x_1 + x_1^2 + x_1^3 + \cdots$$

Each symmetric group S_n acts on $\mathbb{C}[\![x]\!]$ by permuting $\{x_1, \ldots, x_n\}$.

Definition 22. The *algebra of symmetric functions in $x = (x_1, x_2, \ldots)$* is

$$\mathrm{Sym} = \mathrm{Sym}(x) = \left\{ f \in \mathbb{C}[\![X]\!] : \begin{array}{c} f \text{ is of bounded degree,} \\ \text{and for all } n \in \mathbb{N} \text{ we have} \\ \sigma \cdot f = f \text{ for all } \sigma \in S_n \end{array} \right\}.$$

The elements of Sym are called *symmetric functions*. It is a graded ring:

$$\mathrm{Sym} = \bigoplus_{d \in \mathbb{N}} \mathrm{Sym}_d,$$

where

$$\mathrm{Sym}_d(x) = \{ f \in \mathrm{Sym}(x) : f \text{ is homogeneous of degree } d \}.$$

Monomial symmetric functions In analogy with monomial symmetric polynomials, define the *monomial symmetric function m_λ* as the sum over all distinct monomials of exponent λ:

$$\begin{aligned} m_{(3)} &= x_1^3 + x_2^3 + x_3^3 + \cdots + x_{19}^3 + \cdots + x_{23}^3 + \cdots \\ m_{(2,1)} &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + x_{19}^2 x_{23} + x_{19} x_{23}^2 + \cdots \\ m_{(1,1,1)} &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_5 + \cdots + x_{19} x_{23} x_{74} + \cdots \end{aligned}$$

Theorem 23. *Every symmetric function $f \in \mathrm{Sym}$ can be written uniquely as a finite[6] linear combination of the form*

$$f = \sum_{\lambda \in \mathrm{Part}} c_\lambda m_\lambda,$$

where the sum ranges over all partitions λ. If $f \in \mathrm{Sym}_d$, then

$$f = \sum_{\lambda \in \mathrm{Part}_d} c_\lambda m_\lambda.$$

[6] The finiteness follows from the fact that f is of bounded degree.

Other bases of symmetric functions The other bases of Sym are defined in analogy with the other bases of symmetric polynomials.

$$\begin{aligned} e_k &= \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k} &= m_{1^k} \\ h_k &= \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} x_{i_1} \cdots x_{i_k} &= \sum_{\lambda \vdash k} m_\lambda \\ p_k &= \sum_i x_i^k &= m_k \\ s_\lambda &= \sum_{T \in \mathsf{SSYT}(\lambda)} x^{\mathrm{weight}(T)} &= \sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu \end{aligned}$$

Relationship with symmetric polynomials For each $n \in \mathbb{N}$, there is an algebra morphism obtained by setting the variables $x_{n+1}, x_{n+2}, \ldots$ to 0:

$$\mathrm{Sym} \xrightarrow{\mathrm{eval}_n} \mathbb{C}[x_1, \ldots, x_n]^{S_n}.$$

For $d \leqslant n$, this restricts to an isomorphism of vector spaces

$$\mathrm{Sym}_d \xrightarrow{\mathrm{eval}_n} \mathbb{C}[x_1, \ldots, x_n]_d^{S_n}$$

because

- $\{m_\lambda : \lambda \in \mathrm{Part}_d\}$ is a basis of Sym_d, and
- $\{m_\lambda(x_1, \ldots, x_n) : \lambda \in \mathrm{Part}_d\}$ is a basis[7] of $\mathbb{C}[x_1, \ldots, x_n]_d^{S_n}$.

Thus,[8]

identities that are valid in $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ *for all* n *are also valid in* Sym.

[7] Recall from Corollary 5 that the polynomials $m_\lambda(x_1, \ldots, x_n)$ as λ ranges over all partitions of d of length at most n form a basis of $\mathbb{C}[x_1, \ldots, x_n]_d^{S_n}$. Since $d \leqslant n$, we have $\ell(\lambda) \leqslant d \leqslant n$ and so the second condition is superfluous.

[8] In other words, the algebra of symmetric functions is an inverse limit:

$$\mathrm{Sym} = \varprojlim_n \mathbb{C}[x_1, \ldots, x_n]^{S_n}$$

2.11 *Symmetric functions via algebraic independence*

Another way to construct the algebra of symmetric functions is based on the algebraic independence of the elementary symmetric polynomials:

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n].$$

One defines Sym to be a commutative ring generated by countably many indeterminates denoted $e_1, e_2, \ldots$,

$$\mathrm{Sym} = \mathbb{C}[e_1, e_2, e_3, \ldots].$$

For each $n \in \mathbb{N}$, there is an algebra morphism

$$\begin{array}{ccc} \mathrm{Sym} & \xrightarrow{\mathrm{eval}_n} & \mathbb{C}[x_1, \ldots, x_n]^{S_n} \\ e_k & \mapsto & e_k(x_1, \ldots, x_n). \end{array}$$

The other bases of Sym are defined in relation to the e_λ; for example,

$$\begin{aligned} e_k &= m_{1^k} \\ p_k &= m_k \\ h_k &= \sum_{\lambda \vdash k} m_\lambda \\ s_\lambda &= \sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu \end{aligned}$$

Remark 24. This is the approach used in SageMath's implementation of Sym: one defines an algebra with a basis indexed by partitions and then other bases are defined with respect to this basis.

2.12 *Symmetric functions, TL;DR*

There is an abstract algebra of symmetric functions called Sym that admits a basis indexed by partitions,

$$\{e_\lambda : \lambda \in \mathrm{Part}\},$$

and projections onto the algebras of symmetric polynomials

$$\begin{array}{ccc} \mathrm{Sym} & \xrightarrow{\mathrm{eval}_n} & \mathbb{C}[x_1, \ldots, x_n]^{S_n} \\ e_\lambda & \mapsto & e_\lambda(x_1, \ldots, x_n) \end{array}$$

If an identity holds in each of the projections, then it holds in Sym:

$$\text{for } f, g \in \mathrm{Sym},$$
$$f = g \iff \underbrace{f(x_1, \ldots, x_n)}_{\mathrm{eval}_n(f)} = \underbrace{g(x_1, \ldots, x_n)}_{\mathrm{eval}_n(g)} \text{ for all } n \in \mathbb{N}.$$

To verify an identity in Sym, it suffices to verify it in $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ for a sufficiently large n:

$$f = g \iff f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$
$$\text{for some } n \geqslant \max\{\deg(f), \deg(g)\}.$$

2.13 *Some expansions*

Monomial expansion of elementary symmetric functions Let $\lambda \in \text{Part}_n$.

$$e_\lambda = \sum_{\mu \in \text{Part}_n} M_{\lambda,\mu} m_\mu,$$

where $M_{\lambda,\mu}$ is the number of matrices with entries in $\{0,1\}$ whose rows sum to λ and columns sum to μ. Consequently, the transition matrix between the bases $\{m_\lambda : \lambda \vdash n\}$ and $\{e_\lambda : \lambda \vdash n\}$ is symmetric.

Expanding e_λ results in a sum of monomials of the form

$$\underbrace{(x_1^{a_{1,1}} x_2^{a_{1,2}} \ldots x_n^{a_{1,n}})}_{e_{\lambda_1}} \cdots \underbrace{(x_1^{a_{l,1}} x_2^{a_{l,2}} \ldots x_n^{a_{l,n}})}_{e_{\lambda_l}}$$

where $(a_{k,1}, a_{k,2}, \ldots, a_{k,n})$ is a sequence of 0s and 1s that sum to λ_k. Thus, they are in bijection with $0/1$-matrices $[a_{i,j}]_{i,j}$ whose row sums are λ.

Monomial expansion of homogeneous symmetric functions Let $\lambda \in \text{Part}_n$.

$$h_\lambda = \sum_{\mu \in \text{Part}_n} N_{\lambda,\mu} m_\mu,$$

where $N_{\lambda,\mu}$ is the number of matrices with entries in $\mathbb{N}$ whose rows sum to λ and columns sum to μ. Consequently, the transition matrix between the bases $\{m_\lambda : \lambda \vdash n\}$ and $\{h_\lambda : \lambda \vdash n\}$ is symmetric.

Monomial expansion of power sum symmetric functions Let $\lambda \in \text{Part}_n$.

$$p_\lambda = \sum_{\mu \in \text{Part}_n} R_{\lambda,\mu} m_\mu,$$

where $R_{\lambda,\mu}$ is the number of ordered set partitions $(B_1, \ldots, B_k)$ of $[\ell(\lambda)]$ such that $\mu_j = \sum_{i \in B_j} \lambda_i$ for all $j \in [k]$.

Power sum expansion of elementary and homogeneous symmetric functions

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda$$
$$e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{1}{z_\lambda} p_\lambda$$

where

$$z_\mu = 1^{d_1} d_1!\, 2^{d_2} d_2! \cdots n^{d_n} d_n! \quad \text{where} \quad d_i = \text{mult}_i(\mu).$$

Monomial expansion of Schur functions Let $\lambda \in \text{Part}_n$.

$$s_\lambda = \sum_{\mu \in \text{Part}_n} K_{\lambda,\mu} m_\mu,$$

where $K_{\lambda,\mu}$ is the number of SSYT of shape λ and content μ.

More generally, if λ/ν is a skew partition of size n, then

$$s_{\lambda/\nu} = \sum_{\mu \in \text{Part}_n} K_{\lambda/\nu,\mu} m_\mu,$$

where $K_{\lambda/\nu,\mu}$ is the number of SSYT of shape λ/ν and content μ.

2.14 *What to read next*

A very delightful introduction to generating functions that does an excellent job of (visually) presenting the underlying ideas appears in

- **Concrete Mathematics: A Foundation for Computer Science**, by Ronald Graham, Donald Knuth, and Oren Patashnik; see Chapter 7.

Afterwards, I strongly recommend

- **The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions**, by Bruce E.Sagan.
  - Chapter 4, sections 4.1–4.6, on symmetric functions.
  - Chapter 5, section 5, on chromatic symmetric functions.

Other nice introductions to symmetric functions can by found in:

- **Combinatorics: The Art of Counting**, by Bruce E. Sagan.
- **Enumerative Combinatorics, volume 2**, by Richard P. Stanley.

After having read these notes, you should have enough background to start[9] reading Stanley's paper in which X_Γ is introduced and to begin appreciating some of the results. It contains formulas for expansions of the X_Γ in other bases, the connection with acyclic orientiations, and other nice properties.

- **A Symmetric Function Generalization of the Chromatic Polynomial of a Graph**, by Richard P. Stanley (1995).

[9] One does not read a paper in one sitting, but in phases. See

1. *How to Read Mathematics*, by Shai Simonson and Fernando Gouvea
2. *How to Read a Research Paper*, by Matt Baker
3. *How to Read a [Computer Science] Paper*, by S. Keshav

3 Symmetric functions from representations of symmetric groups

3.1 Initiation to representation theory

A *representation (over $\mathbb{C}$)* of a group G is a morphism of groups

$$\rho : G \to \mathrm{GL}(V), \quad \text{where } V \text{ is a } \mathbb{C}\text{-vector space.}$$

By fixing a basis of V, we get a *matrix representation* of G:

$$\rho : G \to \mathrm{GL}_n(\mathbb{C}).$$

G-modules It is important to view ρ as a *linear action* of G on V via

$$g \bullet \vec{v} = \rho(g)(\vec{v}) \qquad (g \in G,\ \vec{v} \in V)$$

With this definition, on has

$$g \bullet (\alpha\vec{v} + \beta\vec{u}) = \alpha(g \bullet \vec{v}) + \beta(g \bullet \vec{u}) \qquad (\vec{v}, \vec{u} \in V,\ g \in G,\ \alpha, \beta \in \mathbb{C}) \tag{2}$$

$$(gh) \bullet \vec{v} = g \bullet (h \bullet \vec{v}) \qquad (\vec{v} \in V,\ g, h \in G) \tag{3}$$

$$e \bullet \vec{v} = \vec{v} \qquad (\vec{v} \in V) \tag{4}$$

A *$\mathbb{C}G$-module*, or more simply a *G-module*, is a $\mathbb{C}$-vector space V equipped with an action of G verifying the conditions (2), (3), (4).

From modules to representations Let V be a G-module with basis $\mathcal{B}$ and action denoted by $g \bullet \vec{v}$. Then the function $\rho : G \to \mathrm{GL}_n(\mathbb{K})$

$$\rho(g) = [g]_{\mathcal{B}},$$

where $[g]_{\mathcal{B}}$ is the matrix of $\vec{v} \mapsto g \bullet \vec{v}$ in $\mathcal{B}$, is a representation of G.

Illustrating example

- Let G be the group of isometries of the triangle in $\mathbb{R}^3$ whose vertices are $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$; *cf.* Figure 6.
- $\mathbb{R}^3$ is a G-module: $g \bullet \vec{v}$ is the image of $\vec{v}$ under the isometry g.
- Each isometry is a linear transformation, so it can be represented by a matrix (with respect to the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$):

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

- Expressing these in the basis $\mathcal{B} = (\vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_3, \vec{e}_1 + \vec{e}_2 + \vec{e}_3)$, we get block diagonal matrices:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}-1&1&0\\-1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&-1&0\\1&-1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\1&-1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}-1&1&0\\0&1&0\\0&0&1\end{bmatrix}$$

- Thus, we have a direct sum decomposition of our vector space

$$\mathbb{R}^3 = \mathrm{vect}\{\vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_3\} \oplus \mathrm{vect}\{\vec{e}_1 + \vec{e}_2 + \vec{e}_3\}$$

into subspaces that are stable[10] for the action of G.

- *General objective:* find the finest such decomposition.

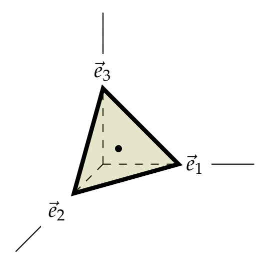


Figure 6: Triangle situated in $\mathbb{R}^3$. There are six isometries: the identity element; counter-clockwise rotations of $2\pi/3$ and $4\pi/3$ degrees about the center of the triangle; and the reflections in the angle bisectors at vertices $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$.

[10] $W \subseteq V$ is *stable* for the action of G if $g \bullet \vec{w} \in W$ for all $g \in G$, $\vec{w} \in W$.

Maschke's Theorem A module/representation V is *irreducible* if

- it is *non-trivial*; and
- it does not admit a *proper subspace* that is stable for the action of G.

Theorem 25. *Let G be a finite group.*
For every complex representation $G \xrightarrow{\rho} \mathrm{GL}(V)$, *there exists*

- *a basis* $\mathcal{B}$ *of V and*
- *irreducible matrix representations* $\rho_1, \ldots, \rho_k$ *of G*

such that, for every $g \in G$,

$$[\rho(g)]_{\mathcal{B}} = \left[\begin{array}{c|c|c|c} \rho_1(g) & 0 & \cdots & 0 \\ \hline 0 & \rho_2(g) & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \rho_k(g) \end{array}\right]. \tag{5}$$

Main problems of representation theory

- Given a group G, determine all its irreducible representations.
- Given a representation, determine its irreducible subrepresentations up to isomorphism (this is its *character*).
- Given a representation V, decompose it into a direct sum of irreducible subrepresentations; in other words, find a basis $\mathcal{B}$ with respect to which the matrices $[\rho(g)]_{\mathcal{B}}$ are block diagonal as in (5).

Characters The *character* of a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is the function $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \mathrm{trace}(\rho(g)) \qquad (g \in G).$$

Here are a few important properties of characters.

1. Representations are isomorphic iff they have the same character.
2. Characters are constant on conjugacy classes:

   if g and h are conjugate in G, then $\chi_\rho(g) = \chi_\rho(h)$.

3. Since conjugacy classes of S_n correspond to partitions of λ of n, it is customary to write[11]

   $\chi(\lambda) = \chi(w)$ *for any permutation w of cycle type* λ.

[11] The *cycle type* of a permutation is the size of the cycles appearing in its (disjoint) cycle decomposition.

$$\begin{aligned} [31524] &= (1\,3\,5\,4\,2) \\ \mathrm{cycletype}([31524]) &= (5) \end{aligned}$$

$$\begin{aligned} [52431] &= (1\,5)\,(2)\,(3\,4) \\ \mathrm{cycletype}([53241]) &= (2,2,1) \end{aligned}$$

3.2 *Frobenius characteristic*

The Frobenius characteristic establishes a correspondence between characeters of symmetric groups and symmetric functions.

Frobenius characteristic Let V be a representation of S_n with character χ_V. The *Frobenius characteristic* of V is

$$\mathrm{Frob}(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma) p_{\mathrm{cycletype}(\sigma)}.$$

By grouping together terms with the same value of cycletype, we get

$$\mathrm{Frob}(V) = \sum_{\mu \vdash n} \chi_V(\mu) \frac{p_\mu}{z_\mu},$$

where[12]

$$z_\mu = 1^{d_1} d_1! \, 2^{d_2} d_2! \cdots n^{d_1} d_n! \quad \text{where} \quad d_i = \mathrm{mult}_i(\mu).$$

[12] $\frac{n!}{z_\mu}$ is the size of the conjugacy class of permutations of cycle type μ; equivalently, z_μ is the size of the centralizer of a permutation of cycle type μ.

Example 26. *Returning to our* illustrating example *of Figure 6, we get*

$$\begin{aligned} &\tfrac{1}{6}\Big(3p_{111} + 0p_3 + 0p_3 + p_{21} + p_{21} + p_{21}\Big) \\ &= \tfrac{1}{2}p_{111} + \tfrac{1}{2}p_{21} \\ &= s_{21} + s_3 \end{aligned}$$

Theorem 27. *Let* $\mathrm{CF}(S_n)$ *be the algebra of characters*[13] *of* S_n*. The Frobenius characteristic* $\mathrm{Frob} : \bigoplus_n \mathrm{CF}(S_n) \to \mathrm{Sym}$ *is an algebra isomorphism.*

1. *The Frobenius characteristic of an irreducible representation of* S_n *is a Schur function* s_λ *with* $\lambda \vdash n$*; and conversely.*
2. *If* χ *and* ψ *are characters of* S_n *and* S_m*, respectively, then*
$$\mathrm{Frob}(\chi)\mathrm{Frob}(\psi) = \mathrm{Frob}\left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(\chi\psi)\right).$$
3. *If* χ *and* ψ *are characters of* S_n*, then*
$$\langle \chi, \psi \rangle_{S_n} = \langle \mathrm{Frob}(\chi), \mathrm{Frob}(\psi) \rangle_{\mathrm{Sym}}.$$

[13] This is usually called the *algebra of class functions*.

$$\langle \chi, \psi \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\overline{\psi(\sigma)}$$

$$\langle s_\lambda, s_\mu \rangle_{\mathrm{Sym}} = \delta_{\lambda,\mu}$$

Remark 28. Notably, the coefficients in the expansion of $\mathrm{Frob}(V)$ in the Schur basis are the multiplicities of the irreducible representations appearing in a direct sum decomposition of V into irreducibles.

Graded Frobenius characteristic We will often work with infinite dimensional representations that break down into finite-dimensional pieces.

A vector space V is *graded* if there are subspaces V_d of V, one for each $d \in \mathbb{N}$, such that

$$V = \bigoplus_{d \in \mathbb{N}} V_d.$$

A *graded representation* is a graded vector space $V = \bigoplus_d V_d$ equipped with an action of the group that maps each component V_d to itself.

In this case, the *graded Frobenius characteristic* of $V = \bigoplus_d V_d$ is

$$\mathrm{Frob}(V)(t) = \sum_d \mathrm{Frob}(V_d) t^d \in \mathrm{Sym}[\![t]\!].$$

3.3 *Example: Coinvariant algebra*

Consider $\mathbb{C}[x_1, x_2, x_3]$ with an action of S_3 given by

$$\sigma f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

The ideal generated by the elementary symmetric polynomials

$$\langle e_1, e_2, e_3 \rangle = \langle x_1 + x_2 + x_3,\ x_1x_2 + x_1x_3 + x_2x_3,\ x_1x_2x_3 \rangle$$

is invariant under the action of S_3 (it consists of symmetric functions). Thus, the quotient space is also a representation of S_3:

$$\mathbb{C}[x_1, x_2, x_3] / \langle e_1, e_2, e_3 \rangle.$$

A basis of this vector space is given by the *Schubert polynomials*.

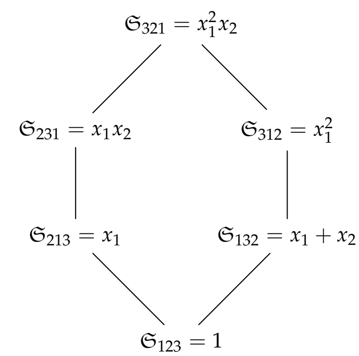


Figure 7: Schubert polynomials for permutations of size $n = 3$.

Schubert polynomials Schubert polynomials are a family of polynomials indexed by permutations. They can be defined recursively as follows.

1. If w_0 is the permutation $[n, n-1, \ldots, 2, 1]$, define

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1.$$

2. If $\mathfrak{S}_w$ is defined and $w(i) > w(i+1)$, define

$$\mathfrak{S}_{ws_i} = \partial_i(\mathfrak{S}_w),$$

where the i-th *divided difference operator* is define as

$$\partial_i(f(x_1, \ldots, x_n)) = \frac{f(x_1, \ldots, x_n) - s_i(f(x_1, \ldots, x_n))}{x_i - x_{i+1}}.$$

Theorem 29. *The Schubert polynomials determine a vector space basis of the quotient space $\mathbb{C}[x_1, \ldots, x_n]/I_n$, where $I_n = \langle e_1, \ldots, e_n \rangle$. More precisely,*

$$\{\mathfrak{S}_w + I_n : w \in S_n\} \text{ is a basis of } \mathbb{C}[x_1, \ldots, x_n]/I_n.$$

Action of S_3 in the Schubert basis It turns out that the subspaces

$$\begin{array}{ll} \mathrm{vect}\{x_1^2x_2 + I_3\} & \mathrm{vect}\{x_1x_2 + I_3, x_1^2 + I_3\} \\ \mathrm{vect}\{1 + I_3\} & \mathrm{vect}\{x_1 + I_3, x_1 + x_2 + I_3\} \end{array}$$

are stable for the action of S_3; *cf.* Figure 8.

For example,

$$s_2(x_1^2x_2) = x_1^2x_3,$$

and since

$$\underbrace{(x_1x_2 + x_1x_3 + x_2x_3)}_{I_3}\, x_1 = x_1^2x_2 + x_1^2x_3 + \underbrace{x_1x_2x_3}_{I_3},$$

we have

$$x_1^2x_3 + I_3 = -x_1^2x_2 + I_3.$$

As another example,

$$s_2(x_1x_2) = x_1x_3,$$

and since

$$\underbrace{(x_1 + x_2 + x_3)}_{I_3}\, x_1 = x_1^2 + x_1x_2 + x_1x_3$$

we have

$$x_1x_3 + I_3 = -(x_1x_2 + I_3) - (x_1^2 + I_3).$$

The following table presents the representation matrices on these subspaces.

S_3	$\mathrm{vect}\{x_1^2x_2\}$	$\mathrm{vect}\{x_1x_2, x_1^2\}$	$\mathrm{vect}\{x_1, x_1+x_2\}$	$\mathrm{vect}\{1\}$
123	$\begin{bmatrix} 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$
132	$\begin{bmatrix} -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$
213	$\begin{bmatrix} -1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$
231	$\begin{bmatrix} 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$
312	$\begin{bmatrix} 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$
321	$\begin{bmatrix} -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$

Graded Frobenius characteristic

$$\mathrm{Frob}\big(\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3\rangle\big) = s_{111}t^3 + s_{21}t^2 + s_{21}t + s_3.$$

	$x_1^2x_2$
123	$x_1^2x_2$
132	$-x_1^2x_2$
213	$-x_1^2x_2$
231	$x_1^2x_2$
312	$x_1^2x_2$
321	$-x_1^2x_2$

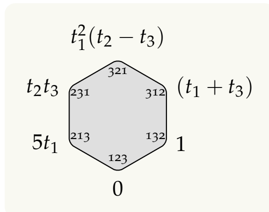
	x_1x_2	x_1^2
123	x_1x_2	x_1^2
132	$-x_1x_2 - x_1^2$	x_1^2
213	x_1x_2	$-x_1x_2 - x_1^2$
231	x_1^2	$-x_1x_2 - x_1^2$
312	$-x_1x_2 - x_1^2$	x_1x_2
321	x_1^2	x_1x_2

	x_1	$x_1 + x_2$
123	x_1	$x_1 + x_2$
132	x_1	$x_1 - (x_1 + x_2)$
213	$-x_1 + (x_1 + x_2)$	$x_1 + x_2$
231	$-x_1 + (x_1 + x_2)$	$-x_1$
312	$-(x_1 + x_2)$	$x_1 - (x_1 + x_2)$
321	$-(x_1 + x_2)$	$-x_1$

Figure 8: The action of S_3 on the Schubert basis of $\mathbb{C}[x_1, x_2, x_3]/I_3$.

3.4 *Example: Hessenberg ring*

(Big) Hessenberg ring Consider the ring whose elements are tuples of polynomials from $\mathbb{C}[t_1, t_2, t_3]$, one for each permutation $w \in S_3$. We visualize elements as



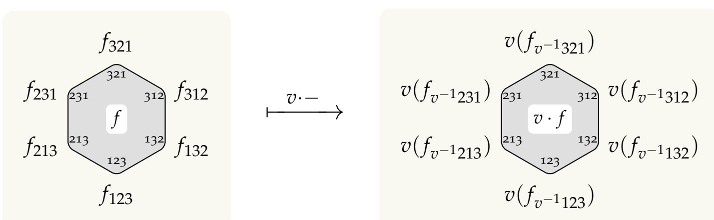
Addition and multiplication are defined component-wise.

Action of S_n If $f(w, t_1, \dots, t_n)$ denotes the polynomial associated with $w \in S_n$, then for all $v \in S_n$, we have

$$(v \cdot f)(w, t_1, \dots, t_n) = f(v^{-1}w, t_{v(1)}, \dots, t_{v(n)}).$$

In other words,

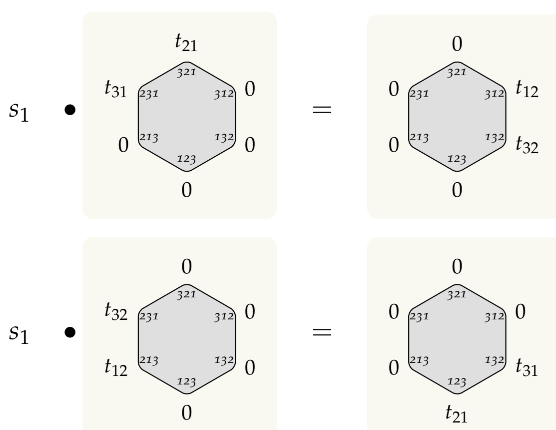
> the polynomial at the vertex w in $v \cdot f$, is obtained from the polynomial at the vertex $v^{-1}w$ by permuting its variables according to v.



Example 30. *To simplify notation, let us write:*

$$t_{ij} = t_i - t_j$$

Then



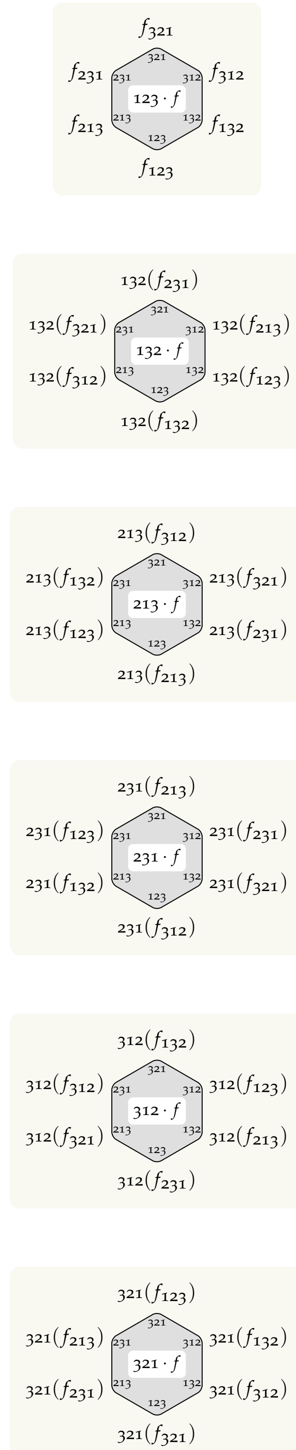
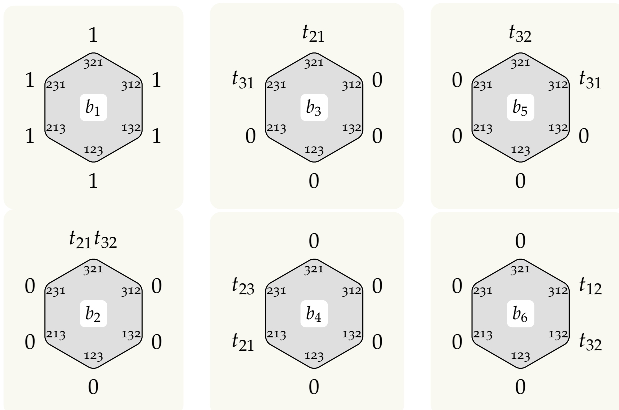
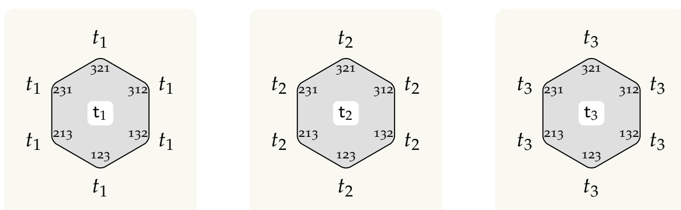


Figure 9: Action of S_3 on elements of the Hessenberg ring.

A subquotient of the Hessenberg ring Consider the quotient ring R/I, where R is the subring generated by



and I is the ideal generated by

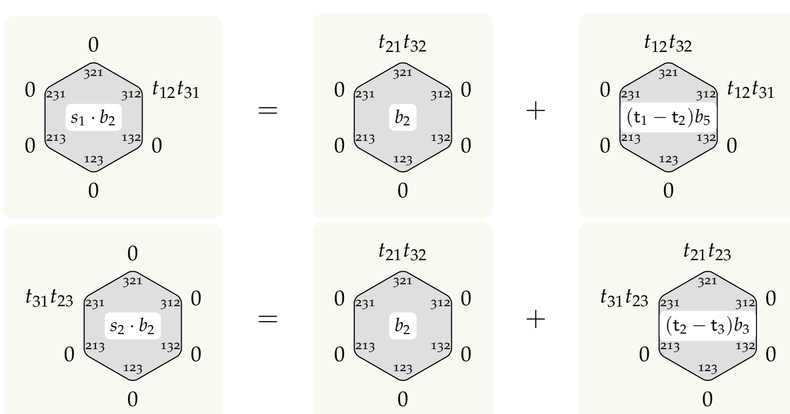


It turns out that $\{b_1 + I, \ldots, b_6 + I\}$ is a vector space basis of R/I.

Action on b_1. We begin by observing $w \cdot b_1 = b_1$ for all $w \in S_3$; thus,

S_3 acts trivially on the vector space $\text{vect}\{b_1 + I\}$.

Action on b_2. We compute



It follows that $w \cdot \big(b_2 + I\big) = b_2 + I$ for all $w \in S_3$, and so

S_3 acts trivially on $\text{vect}\{b_2 + I\}$.

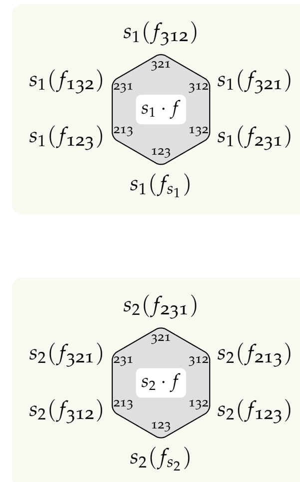


Figure 10: Action of s_1 and s_2 on elements of the Hessenberg ring.

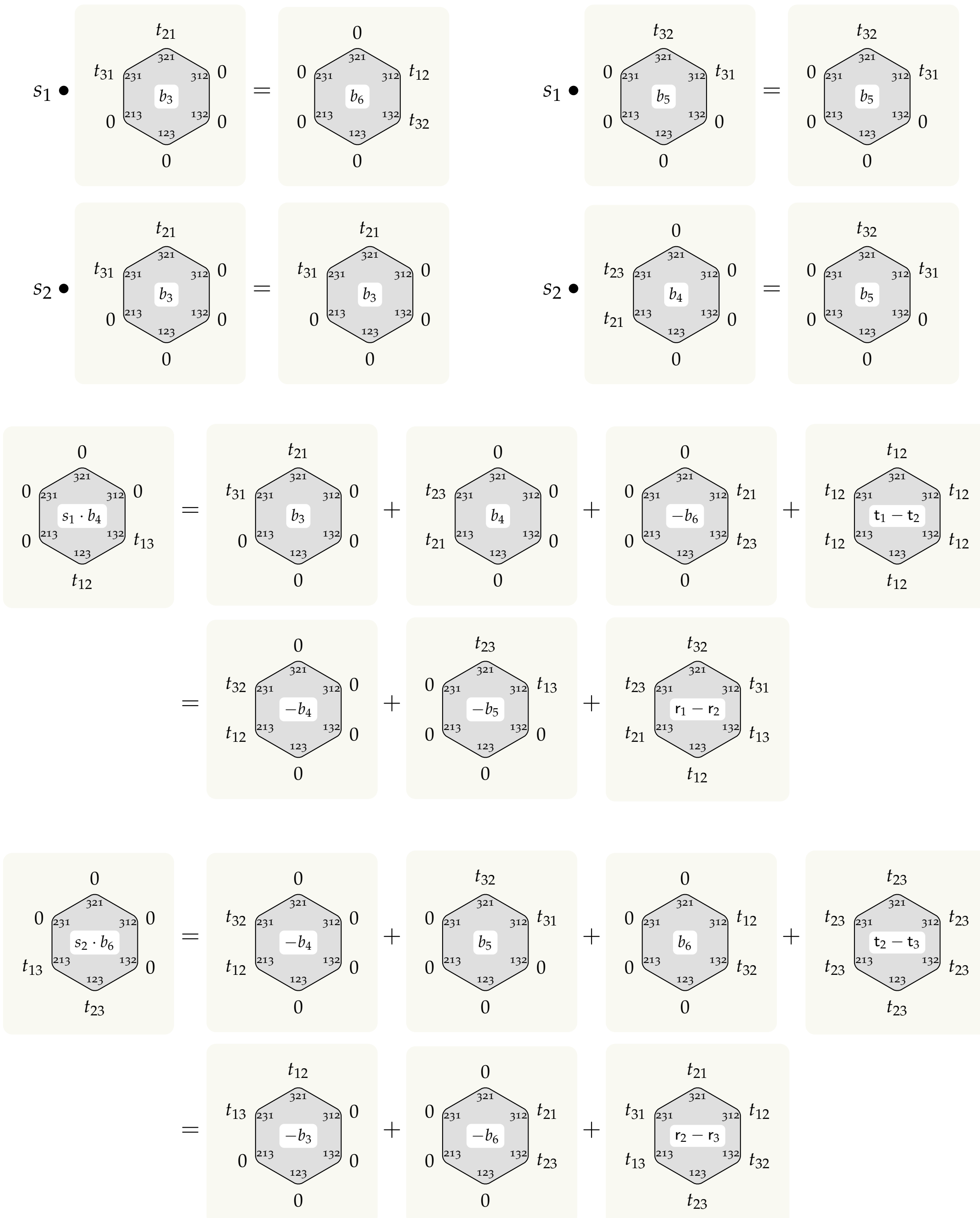


Figure 11: Action of s_i on b_3, b_4, b_5, b_6.

Action on $b_3, \ldots, b_6$. In Figure 11, we have the action of s_1 and s_2 on $b_3, \ldots, b_6$; from which the matrices representing the action of s_1, s_2 and $s_1 s_2$ on $\mathrm{vect}\{b_3 + I, b_4 + I, b_5 + I, b_6 + I\}$ are

$$[s_1] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad [s_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [s_1 s_2] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

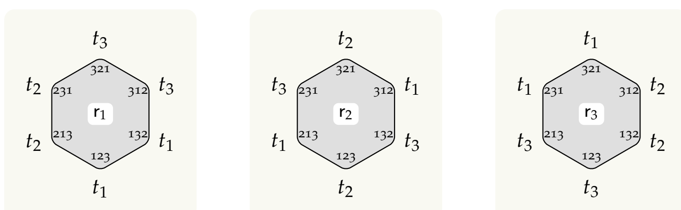
Hence, the Frobenius characteristic of this representation is

$$\begin{aligned} &\frac{1}{6}\left(4p_{111} + (3 \times 2)p_{21} + (2 \times 1)p_3\right) \\ &= \frac{2}{3}p_{111} + p_{21} + \frac{1}{3}p_3 \\ &= 2s_3 + s_{21} \end{aligned}$$

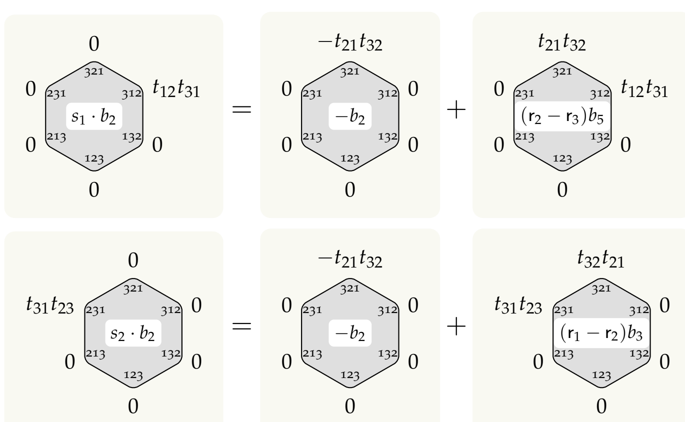
and the graded Frobenius characteristic of R/I is

$$\mathrm{Frob}(R/I) = s_3 t^2 + (2s_3 + s_{21})t + s_3.$$

A second subquotient of the Hessenberg ring Consider instead the quotient ring R/J, with R as above and J the ideal generated by



Combining the identites from Figure 11 with



it follows that the graded Frobenius characteristic of R/J is

$$\mathrm{Frob}(R/J) = s_{111}t^2 + 2s_{21}t + s_3.$$

The matrices representing action of S_3 on $\mathrm{vect}\{b_3 + J, \ldots, b_6 + J\}$ are:

$$[s_1] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[s_2] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$[s_1 s_2] = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

The character χ satisfies

$$\begin{aligned} \chi([1,1,1]) &= 4 \\ \chi([2,1]) &= 0 \\ \chi([3]) &= -2 \end{aligned}$$

whose Frobenius characteristic is $2s_{21}$.

3.5 *What to read next*

For the representation theory of the symmetric group, I recommend

- **The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions**, by Bruce E.Sagan.
  - *Chapter 1, Group Representations*
  - *Chapter 2, Representations of the Symmetric Group*

For a general introduction to representation theory of finite groups, I recommend:

- **Representations and Characters of Groups**, by Gordon James and Martin Liebeck.

There are many other great resources, but these two are excellent starting points into the theory, and include many examples and exercises.

4 What are (chromatic) quasisymmetric functions?

Quasisymmetric functions are refinements of symmetric functions.

4.1 Quasisymmetric functions

Recall that a symmetric function can be viewed as a formal power series of bounded degree such that monomoials with the same *multiset* of nonzero exponents have the same coefficient:

Symmetric functions: *the coefficient of* $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}$ *equals the coefficient of* $x_{j_1}^{\beta_1} x_{j_2}^{\beta_2} \cdots x_{j_l}^{\beta_l}$ *whenever* $\{\{\alpha_1, \ldots, \alpha_l\}\} = \{\{\beta_1, \ldots, \beta_l\}\}$.

A quasisymmetric function requires that monomoials with the same *sequence* of nonzero exponents have the same coefficient:

Quasisymmetric functions: *the coefficient of* $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}$ *equals the coefficient of* $x_{j_1}^{\beta_1} x_{j_2}^{\beta_2} \cdots x_{j_l}^{\beta_l}$ *whenever* $(\alpha_1, \ldots, \alpha_l) = (\beta_1, \ldots, \beta_l)$.

Monomial quasisymmetric functions Let α be a composition. The *monomial quasisymmetric function* M_α is defined as

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_l} \cdots x_{i_l}^{\alpha_l}.$$

For example,

$$\begin{aligned} M_{(2,1)} &= \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots \\ M_{(1,2)} &= \sum_{i<j} x_i x_j^2 = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots \\ m_{(2,1)} &= M_{(2,1)} + M_{(1,2)} \end{aligned}$$

More generally, every m_λ is refined by the M_α:

$$m_\lambda = \sum_{\substack{\text{rearrangements} \\ \alpha \text{ of } \lambda}} M_\alpha.$$

Algebra structure Let QSym denote the set of quasisymmetric functions. It is closed under sums and products, thus it is an algebra.

Proposition 31.

$$M_\alpha M_\beta = \sum_\gamma c_{\alpha,\beta}^\gamma M_\gamma,$$

where $c_{\alpha,\beta}^\gamma$ *is the number of ways of writing* γ *as a shuffle sum of* α *and* β.

Example 32.

$$\begin{aligned} M_{[1,2]} M_{[1]} &= M_{[1,2,1]} + M_{[1,1+2]} + M_{[1,1,2]} + M_{[1+1,2]} + M_{[1,1,2]} \\ &= M_{[1,2,1]} + M_{[1,3]} + M_{[1,1,2]} + M_{[2,2]} + M_{[1,1,2]} \end{aligned}$$

Fundamental quasisymmetric functions The *(Gessel) fundamental quasisymmetric function F_α* indexed by the composition α is

$$F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta.$$

For example,

$$F_{(2,2)} = M_{(2,2)} + M_{(1,1,2)} + M_{(2,1,1)} + M_{(1,1,1,1)}$$
$$F_{(1,3)} = M_{(1,3)} + M_{(1,1,2)} + M_{(1,2,1)} + M_{(1,1,1,1)}$$

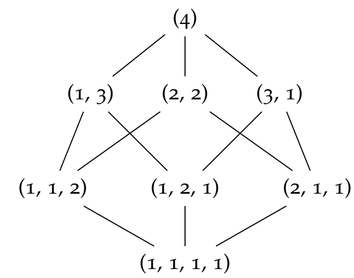


Figure 12: Refinement order on compositions

Some special cases of the fundamental quasisymmetric functions are

$$F_{(1^n)} = M_{(1^n)} = m_{(1^n)} = e_n$$
$$F_{(n)} = \sum_{\alpha \in \mathrm{Comp}_n} M_\alpha = \sum_{\lambda \in \mathrm{Part}_n} = h_n$$

The fundamental quasisymmetric functions form a basis of QSym because

$$M_\alpha = \sum_{\beta \text{ refines } \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} F_\beta.$$

For example,

$$M_{(2,2)} = F_{(2,2)} - F_{(1,1,2)} - F_{(2,1,1)} + F_{(1,1,1,1)}$$
$$M_{(1,3)} = F_{(1,3)} - F_{(1,1,2)} - F_{(1,2,1)} + F_{(1,1,1,1)}$$

Fundamental quasisymmetric expansion of Schur functions There is a bijection between compositions of n and subsets of $[n-1]$:

$$\begin{array}{rcl} \mathrm{Comp}_n & \longleftrightarrow & \mathcal{P}([n-1]) \\ (\alpha_1, \alpha_2, \ldots, \alpha_l) & \xmapsto{\mathrm{Des}} & \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{l-1}\} \\ (i_1, i_2 - i_1, \ldots, n - i_l) & \xleftarrow{\mathrm{comp}_n} & \{i_1 < i_2 < \cdots < i_l\} \end{array}$$

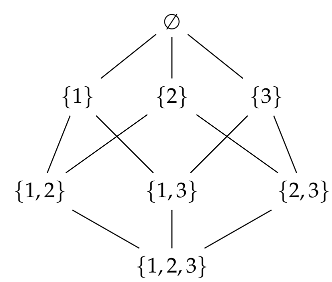


Figure 13: Refinement order on compositions of 4 under the bijection with subsets of [3]

With this bijection, we can express F_α directly as a formal power series:

$$F_\alpha = \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Theorem 33 (Gessel). *For any partition λ,*

$$s_\lambda = \sum_{T \in \mathsf{SYT}(\lambda)} F_{\mathrm{comp}(\mathrm{Des}(T))}$$

where $\mathrm{Des}(T)$ is the set of i in T such that $i+1$ appears below i in T (not necessarily in the same column).

For example, from Figure 14,

$$s_{(3,2)} = F_{(1,2,2)} + F_{(1,3,1)} + F_{(2,2,1)} + F_{(2,3)} + F_{(3,2)}.$$

T	$\mathrm{Des}(T)$	$\mathrm{comp}_5(\mathrm{Des}(T))$
1 3 5 / 2 4	$\{1,3\}$	$(1,2,2)$
1 3 4 / 2 5	$\{1,4\}$	$(1,3,1)$
1 2 4 / 3 5	$\{2,4\}$	$(2,2,1)$
1 2 5 / 3 4	$\{2\}$	$(2,3)$
1 2 3 / 4 5	$\{3\}$	$(3,2)$

Figure 14: Descent sets of standard tableaux of shape $(3,2)$.

In a future version of these notes, I would like to include a section called FUN WITH FUNDAMENTAL QUASISYMMETRIC FUNCTIONS with some complementary results/applications, such as:

- product formula for fundamental quasisymmetric functions
- going from a fundamental expansion to a Schur expansion via the Jacobi-Trudi formula
- the Gessel–Reutenauer symmetric function

4.2 *Chromatic quasisymmetric functions*

Let $\Gamma = (V, E)$ be a finite graph on the vertex set $V = [n]$. We will consider colourings of the vertices of Γ by positive integers.

N.B. *We require here that both the vertex set and the set of colours is ordered.*

Recall that a function $\kappa : [n] \to \mathbb{N}^*$ is a *proper colouring* of Γ if adjacent vertices are assigned different colours:

$$\{i, j\} \in E \implies \kappa(i) \neq \kappa(j).$$

Define the *ascent set* and *ascent statistic* of κ as

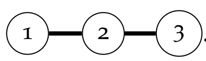
$$\begin{aligned} \mathrm{Asc}_\Gamma(\kappa) &= \{(i,j) \in E : i < j \;\&\; \kappa(i) < \kappa(j)\} \\ \mathrm{asc}_\Gamma(\kappa) &= |\mathrm{Asc}_\Gamma(\kappa)|. \end{aligned}$$

The *chromatic quasisymmetric polynomial*[14] of Γ is

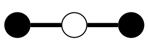
$$X_\Gamma(x;t) = \sum_{\substack{\text{proper} \\ \text{colourings} \\ \kappa:[n]\to\mathbb{N}^*}} t^{\mathrm{asc}_\Gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

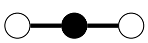
[14] One can think of this as either:

- a quasisymmetric function in $x_1, x_2, \ldots$ with coefficients in $\mathbb{C}[t]$, *i.e.* as an element of $\mathrm{QSym}_{\mathbb{C}[t]}(x_1, x_2, \ldots)$; or
- a polynomial in t with coefficients in QSym, *i.e.* as an element of $\mathrm{QSym}[t]$.

Example Let Γ be the path graph .

- There are two ways to colour Γ with colours {○ < ●}:

 There is one ascent, giving $tM_{(1,2)}$.

 There is one ascent, giving $tM_{(2,1)}$.

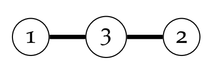
- There are 3! ways to colour Γ with colours $\{1, 2, 3\}$, one for each $w \in S_3$; the number of ascents of the associated colouring is the number of ascents of w. This gives $(t^2 + 4t + 1)M_{(1,1,1)}$.

Thus,

$$\begin{aligned} X_\Gamma(x;t) &= (t^2 + 4t + 1)M_{(1,1,1)} + tM_{(1,2)} + tM_{(2,1)} \\ &= (t^2 + 4t + 1)m_{(1,1,1)} + tm_{(2,1)} \end{aligned}$$

Problem In the above example, $X_\Gamma(x;t)$ turned out to be a *symmetric* function. This is not always that case. [15] An open problem is to characterize the (ordered) graphs Γ for which the quasisymmetric chromatic function $X_\gamma(x;t)$ is symmetric.

[15] *Exercise.* Compute the chromatic quasisymmetric function of the graph



Some other examples

Example 34. *Let* $\Gamma = ([n], \emptyset)$*. Every* $\kappa : [n] \to \mathbb{N}^*$ *is a proper colouring, and* $\mathsf{asc}(\kappa) = 0$ *for all* κ*, so*

$$X_\Gamma(x;t) = \sum_{k_1,k_2,\ldots,k_n} x_{k_1} x_{k_2} \cdots x_{k_n}.$$

Note that the indices $k_1, \ldots, k_n$ *are independent and run through* $\mathbb{N}^*$*; hence*

$$X_\Gamma(x;t) = \Big(\sum_k x_k\Big)^n = e_1^n = e_{(1^n)}.$$

Example 35. *Let* Γ *be the complete graph on n vertices. Every* $\kappa : [n] \to \mathbb{N}^*$ *that is a proper colouring of* Γ *is an injective map, hence*

$$X_\Gamma(x;t) = \sum_{\substack{\text{injective} \\ \kappa:[n]\to\mathbb{N}^*}} t^{\mathsf{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

Since $\mathsf{asc}(\kappa)$ *depends only on the relative order of* $(\kappa(1), \kappa(2), \ldots, \kappa(n))$*, we have in this case that*

$$\mathsf{asc}(\kappa) = \mathsf{asc}(\mathsf{std}(\kappa))$$

where $\mathsf{std}(\kappa)$ *is the unique permutation in* S_n *whose elements appear in the same relative order as* $(\kappa(1), \kappa(2), \ldots, \kappa(n))$*. So we group together the* κ *with the same value for* std*:*

$$X_\Gamma(x;t) = \sum_{\sigma \in S_n} t^{\mathsf{asc}(\sigma)} \sum_{\substack{\text{injective} \\ \kappa:[n]\to\mathbb{N}^* \\ \mathsf{std}(\kappa)=\sigma}} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

Note that the monomials in the inner sum are in bijection with the set of all square-free monomials: if $x_{i_1} x_{i_2} \cdots x_{i_n}$ *is square-free with* $i_1 < \cdots < i_n$*, then* $\kappa(j) = i_{\sigma(j)}$ *is a proper colouring of* Γ *and* $\mathsf{std}(\kappa) = \sigma$*. Thus,*

$$X_\Gamma(x;t) = e_n \sum_{\sigma \in S_n} t^{\mathsf{asc}(\sigma)} = [n]_t!\, e_n,$$

where in the last equality we use a well-known identity.[16]

[16] Notation:

$$[n]_t! = [n]_t [n-1]_t \cdots [1]_t$$
$$[i]_t = 1 + t + t^2 + \cdots + t^{i-1}$$

4.3 *Natural unit interval orders*

Hessenberg vectors $\mathsf{h} = (\mathsf{h}_1, \ldots, \mathsf{h}_n) \in \mathbb{N}^n$ is a *Hessenberg vector* if

$$\mathsf{h}_1 \leqslant \mathsf{h}_2 \leqslant \cdots \leqslant \mathsf{h}_n \qquad \text{and} \qquad i \leqslant \mathsf{h}_i \leqslant n \text{ for all } i \in [n].$$

Given a Hessenberg vector $\mathsf{h} = (\mathsf{h}_1, \ldots, \mathsf{h}_n)$, define a poset P_{h} on $[n]$ by

$$i <_{\mathsf{h}} j \qquad \text{iff} \qquad \mathsf{h}_i < j.$$

and let $\Gamma_{\mathsf{h}} = \mathrm{Inc}(P_{\mathsf{h}})$ be the *incomparability graph* of P_{h}:

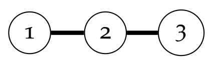
i and j are connected by an edge in Γ_{h} iff
i and j are incomparable in P_{h}.

To simplify notation, for Hessenberg vectors h, we write

$$X_{\mathsf{h}}(x;t) = X_{\Gamma_{\mathsf{h}}}(x;t).$$

Example Let $\mathsf{h} = (2,3,3)$.

- $P_{(2,3,3)}$ is the poset on $\{1,2,3\}$ with exactly one relation $1 <_{\mathsf{h}} 3$.
- Its incomparability graph $\Gamma_{(2,3,3)}$ is



- Thus,
$$X_{(2,3,3)}(x;t) = \left(t^2 + 4t + 1\right)m_{(1,1,1)} + tm_{(2,1)}$$

$X_{\mathsf{h}}(x;t)$ is symmetric

Theorem 36 (Shareshian–Wachs)**.** *If h is a Hessenberg vector, then the chromatic quasisymmetric function $X_{\mathsf{h}}(x;t)$ is a symmetric function.*

The idea of the proof is reminiscent of the Bender–Knuth involution on semistandard tableaux.

1. Note it suffices to prove invariance under exchanging x_c and x_{c+1}.
2. Fix a proper colouring κ of Γ_{h}. It turns out that the (induced) subgraph on the vertices coloured c and $c+1$ is a disjoint union of paths $v_1 - v_2 - \cdots - v_j$ with vertices satisfying $v_1 < v_2 < \cdots < v_j$.
3. For each of these connected components $v_1 - v_2 - \cdots - v_j$:
   - if j is even, then do nothing;
   - if j is odd, then exchange the colours c and $c+1$.
4. The result is a proper colouring with the same ascent statistic.

$$\begin{aligned} X_{(3,3,3)}(x;t) &= (t^3+2t^2+2t+1)m_{(1,1,1)} \\ &= (t^3+2t^2+2t+1)s_{(1,1,1)} \\ &= (t^3+2t^2+2t+1)e_{(3)} \end{aligned}$$

$$\begin{aligned} X_{(2,3,3)}(x;t) &= (t^2+4t+1)m_{(1,1,1)} + tm_{(2,1)} \\ &= (t^2+2t+1)s_{(1,1,1)} + ts_{(2,1)} \\ &= te_{(2,1)} + (t^2+t+1)e_{(3)} \end{aligned}$$

$$\begin{aligned} X_{(2,2,3)}(x;t) &= (3t+3)m_{(1,1,1)} + (t+1)m_{(2,1)} \\ &= (t+1)s_{(1,1,1)} + (t+1)s_{(2,1)} \\ &= (t+1)e_{(2,1)} \end{aligned}$$

$$\begin{aligned} X_{(1,3,3)}(x;t) &= (3t+3)m_{(1,1,1)} + (t+1)m_{(2,1)} \\ &= (t+1)s_{(1,1,1)} + (t+1)s_{(2,1)} \\ &= (t+1)e_{(2,1)} \end{aligned}$$

$$\begin{aligned} X_{(1,2,3)}(x;t) &= 6m_{(1,1,1)} + 3m_{(2,1)} + m_{(3)} \\ &= s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)} \\ &= e_{(1,1,1)} \end{aligned}$$

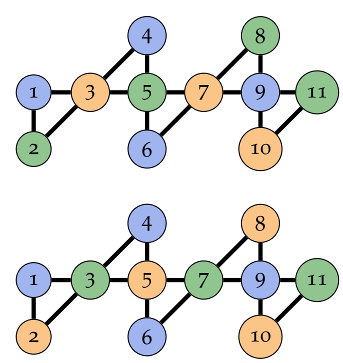
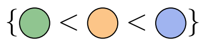
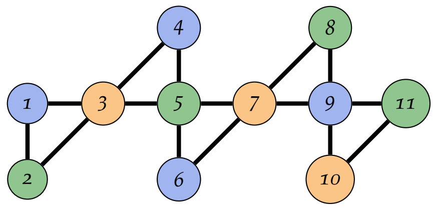


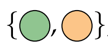
Figure 15: The Shareshian–Wachs involution interchanges these two colourings.

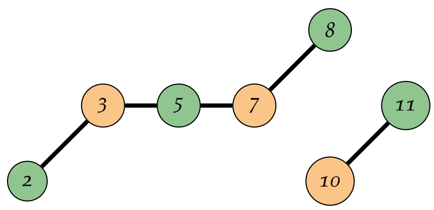
Example 37 (Shareshian–Wachs involution). *Let*

$$\mathsf{h} = (3, 3, 5, 5, 7, 7, 9, 9, 11, 11, 11).$$

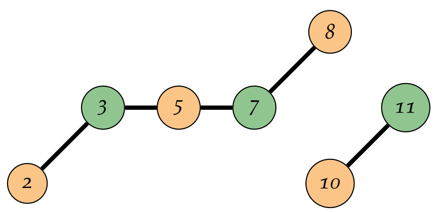
A proper colouring of Γ_{h} *with*  *is*



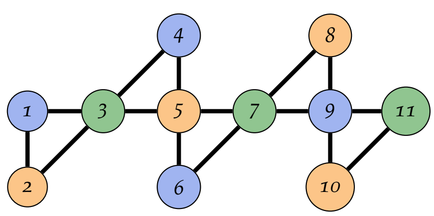
The subgraph induced by the colours *, which we assume adjacent, is*



The Shareshian–Wachs involution maps this subgraph to



and thus the image of Γ_{h} *is*



Note the ascent statistic of Γ_{h} *and that of its image are equal (even though the ascent set changes).*

4.4 *Strengthenings of the Stanley–Stembridge conjecture*

Conjecture 38 (Stanley–Stembridge). *If* Γ *is the incomparability graph of a* $(3+1)$*-free poset, then the chromatic symmetric function* $X_\Gamma(x)$ *is e-positive.*

Theorem 39 (Guay-Paquet). *If* $X_{\mathsf{h}}(x)$ *is e-positive for all Hessenberg vectors* h*, then the Stanley–Stembridge conjecture holds.*

Conjecture 40 (Shareshian–Wachs). *For every Hessenberg vector* h*, the chromatic quasisymmetric function* $X_{\mathsf{h}}(x; t)$ *is e-positive.*

4.5 *Some expansions of chromatic quasisymmetric functions*

Schur-expansion Gasharov proved that the chromatic symmetric function $X_\Gamma(x)$ is Schur-positive whenever Γ is a $(3+1)$-free poset. This result was extended to chromatic quasisymmetric functions of natural unit interval orders by Shareshian–Wachs. [17]

Let P be a poset and λ a partition. A *P-tableau of shape λ* is a filling of the diagram of λ with elements of P such that

1. every element of P appears exactly once;
2. if y appears immediately to the right of x, then $x <_P y$;
3. if y appears immediately below[18] x, then $y \not<_P x$.

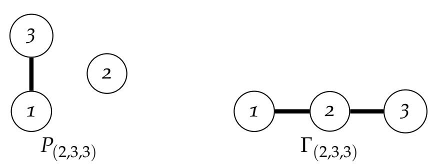
For a P-tableau T and $\Gamma = \mathrm{Inc}(P)$, define

$$\mathrm{inv}_\Gamma(T) = \left|\left\{\{i,j\} \in E(\Gamma) : i < j \text{ and } i \text{ is in a row below } j \text{ in } T\right\}\right|$$

Theorem 41 (Shareshian–Wachs, Theorem 6.3)**.** *Let h be a Hessenberg vector. Recall that Γ_h denotes the incomparability graph of the natural unit interval order P_h. Then*

$$\begin{aligned} X_\mathsf{h}(x;t) &= \sum_{P_\mathsf{h}\text{-tableaux } T} t^{\mathrm{inv}_{\Gamma_\mathsf{h}}(T)} s_{\mathrm{shape}(T)} \\ &= \sum_\lambda \left(\sum_{\substack{P_\mathsf{h}\text{-tableaux } T \\ \mathrm{shape}(T)=\lambda}} t^{\mathrm{inv}_{\Gamma_\mathsf{h}}(T)} \right) s_\lambda. \end{aligned}$$

Example 42. *Let $\mathsf{h} = (2,3,3)$. Then $P_{(2,3,3)}$ and $\Gamma_{(2,3,3)}$ are as follows.*



The P-tableaux, with their inversions highlighted, are

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline \mathbf{2} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \mathbf{2} \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 2 \\ \hline \mathbf{1} \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 3 \\ \hline \mathbf{2} \\ \hline \mathbf{1} \\ \hline \end{array}$$

Thus,

$$X_{(2,3,3)}(x;t) = ts_{(2,1)} + (1+2t+t^2)s_{(1,1,1)}.$$

[17] *N.B.* Incomparability graphs of natural unit interval orders are both $(3+1)$-free and $(2+2)$-free.

$$\begin{array}{|c|c|} \hline x & y \\ \hline \end{array} \implies x <_P y$$

$$\begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array} \implies y \not<_P x$$

Figure 16: Illustration of the conditions defining a P-tableaux

[18] in English notation

Fundamental-expansion Chow described an expansion of $X_\Gamma(x)$ in the F-basis of QSym, which Shareshian and Wachs[19] generalized to an expansion of $X_\Gamma(x;t)$.

[19] Note that the Shareshian–Wachs paper uses a nonstandard definition of F_α; see their footnote 3.

- For a graph Γ with vertex set $[n]$ and a permutation $\sigma \in S_n$, define

$$\mathrm{inv}_\Gamma(\sigma) = \left|\left\{\{u,v\} \in E(\Gamma) : u < v \text{ and } v \text{ appears before } u \text{ in } \sigma\right\}\right|$$

- For a poset P on $[n]$ and a permutation $\sigma \in S_n$, define

$$\mathrm{Asc}_P(\sigma) = \left\{ i \in [n-1] : \begin{array}{c} i \text{ and } i+1 \text{ are incomparable in } P \\ \text{or } \sigma(i) <_P \sigma(i+1) \end{array} \right\}$$

$$\mathrm{asccomp}_P(\sigma) = \mathrm{comp}_n(\mathrm{Asc}_P(\sigma))$$

If we define

$$\mathrm{Des}_P(\sigma) = \left\{ i \in [n-1] : \sigma(i) >_P \sigma(i+1) \right\},$$

then

$$\mathrm{Asc}_P(\sigma) = [n-1] \setminus \mathrm{Des}_P(\sigma).$$

Theorem 43 (Shareshian–Wachs, Theorem 3.1). *Let Γ be the incomparability graph of a poset P on $[n]$.*

$$X_\Gamma(x;t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}_\Gamma(\sigma)} F_{\mathrm{rev}(\mathrm{asccomp}_P(\sigma))}$$

Example 44. *Let P be the poset on $[3]$ whose only relation is $1 <_P 3$. Then $\Gamma = \mathrm{Inc}(P)$ is the graph $1-2-3$ with edges $\{1,2\}$ and $\{2,3\}$.*

	is edge a Γ-inversion?			*is position a P-ascent?*			
S_3	$(1,3)$	$(2,3)$	inv_Γ	1	2	Asc_P	$\mathrm{asccomp}_P$
[123]	*no*	*no*	0	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[132]	*no*	*yes*	1	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[213]	*yes*	*no*	1	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[231]	*yes*	*no*	1	*yes*	*no*	$\{1\}$	$(1,2)$
[312]	*no*	*yes*	1	*no*	*yes*	$\{2\}$	$(2,1)$
[321]	*yes*	*yes*	2	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$

$$\begin{aligned} X_\Gamma(x;t) &= F_{(1,1,1)} + \big(2F_{(1,1,1)} + F_{(2,1)} + F_{(1,2)}\big)t + F_{(1,1,1)}t^2 \\ &= e_{(3)} + \big(e_{(3)} + e_{(2,1)}\big)t + e_{(3)}t^2 \end{aligned}$$

$$\begin{aligned} e_{(3)} &= F_{(1,1,1)} \\ e_{(2,1)} &= F_{(1,1,1)} + F_{(1,2)} + F_{(2,1)} \\ e_{(1,1,1)} &= F_{(1,1,1)} + 2F_{(1,2)} + 2F_{(2,1)} + F_{(3)} \end{aligned}$$

Example 45. *Let P be the poset on $[3]$ whose only relation is $1 <_P 2$. Then $\Gamma = \mathrm{Inc}(P)$ is the graph $1-3-2$ with edges $\{1,3\}$ and $\{2,3\}$.*

	is edge a Γ-inversion?			*is position a P-ascent?*			
S_3	$(1,3)$	$(2,3)$	inv_Γ	1	2	Asc_P	$\mathrm{asccomp}_P$
[123]	*no*	*no*	0	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[132]	*no*	*yes*	1	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[213]	*no*	*no*	0	*no*	*yes*	$\{2\}$	$(2,1)$
[231]	*yes*	*no*	1	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[312]	*yes*	*yes*	2	*yes*	*yes*	$\{1,2\}$	$(1,1,1)$
[321]	*yes*	*yes*	2	*yes*	*no*	$\{1\}$	$(1,2)$

$$\begin{aligned} X_\Gamma(x;t) &= \big(F_{(1,1,1)} + F_{(2,1)}\big)t^0 + 2F_{(1,1,1)}t^1 + \big(F_{(1,1,1)} + F_{(1,2)}\big)t^2 \\ &= (1+2t+t^2)F_{(1,1,1)} + F_{(1,2)} + t^2F_{(2,1)} \end{aligned}$$

Takeway messages:

1. $X_\Gamma(x;t)$ depends on the labelling of the vertices of Γ.
2. If Γ' is a relabelling of Γ, it is possible that $X_\Gamma(x;t)$ is symmetric and $X_{\Gamma'}(x;t)$ is not symmetric.

Add section WHAT TO READ NEXT that
– cites recent trends in quasisymmetric functions https://arxiv.org/pdf/1810.07148.pdf;
– cite recent Symmetric Functions Cataolog;
– cite tutorial on Symmetric Functions in SageMath

5 LLT polynomials / Plethysm

5.1 (Unicellular) LLT polynomials

LLT polynomials are a family of symmetric functions introduced by A Lascoux, B Leclerc, and J-Y Thibon as a means to study *plethysm coefficients*. There are multiple ways to index the LLT polynomials, and we will follow the definition as presented in Haglund–Haiman–Loeher, in which the LLT polynomials are indexed by skew-partitions.

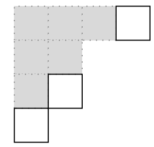


Figure 17: The skew-partition $(4,2,2,1)/(3,2,1)$ consists of the 3 depicted white cells.

Skew-partitions If λ and μ are partitions such that the diagram of λ contains the diagram of μ, then the *skew-partition* λ/μ consists of the cells of λ that do not belong to μ.

If a cell c lies in row i and column j, then *diagonal index*[20] is

$$\mathrm{diag}(c) = j - i.$$

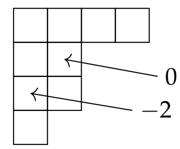


Figure 18: Diagonal index of the cells $(2,2)$ and $(3,1)$.

[20] Sometimes called the "content" of the cell.

Tuple of skew-tableaux A *semistandard skew-tableau of shape λ/μ* is a filling of the cells of the diagram of λ/μ such that the entries are *weakly increasing in rows* and *strictly increasing in columns*.

Given a tuple of semistandard skew-tableaux

$$\underbrace{(T^1,\dots,T^k)}_{\vec{T}} \in \underbrace{\mathsf{SSYT}(\nu^1)\times\mathsf{SSYT}(\nu^2)\times\cdots\times\mathsf{SSYT}(\nu^k)}_{\mathsf{SSYT}(\vec{\nu}),\text{ where } \vec{\nu}=(\nu^1,\nu^2,\dots,\nu^k)}$$

we say that cells $c \in \nu^i$ and $d \in \nu^j$ form an *inversion* of $\vec{T}$ if

- $\vec{T}(c) > \vec{T}(d)$, where $\vec{T}(c)$ denotes the entry in cell c; and either
- $i < j$ and $\mathrm{diag}(c) = \mathrm{diag}(d)$, or $i > j$ and $\mathrm{diag}(c) = \mathrm{diag}(d) + 1$.

Let $\mathrm{Inv}(\vec{T})$ denote the set of inversions of $\vec{T}$, and $\mathrm{inv}(\vec{T}) = |\mathrm{Inv}(\vec{T})|$.

(Unicellular) LLT polynomials The *LLT polynomial* indexed by $\vec{\nu}$ is

$$\mathrm{LLT}_{\vec{\nu}}(x;t) = \sum_{\vec{T}\in\mathsf{SSYT}(\vec{\nu})} t^{\mathrm{inv}(\vec{T})} x^{\vec{T}}.$$

If every ν^i in $\vec{\nu}$ is a single cell, then $\mathrm{LLT}_{\vec{\nu}}(x;t)$ is *unicellular*.

- $\mathrm{LLT}_{\vec{\nu}}(x;t)$ are symmetric in the x variables; see, for example, Theorem 3.3 of HHL2005.
- In HHL2005, this function is denoted $G_{\vec{\nu}}(x;t)$.

Expansion in the fundamental quasisymmetric basis

$$\mathrm{LLT}_{\vec{\nu}}(x;t) = \sum_{\vec{T}\in\mathsf{SYT}(\nu)} t^{\mathrm{inv}(\vec{T})} F_{\mathrm{idescomp}(\vec{T})}(x),$$

where the inverse descent composition is computed from the reading word of $\vec{T}$ (which is obtained by traversing the cells of $\vec{\nu}$ in the opposite order of the one we define below).

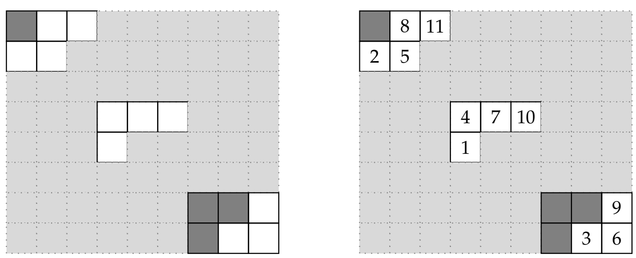
5.2 *Unicellular LLT polynomials and chromatic quasisymmetric functions*

It turns out there is an intriguing relationship between unicellular LLT polynomials and chromatic quasisymmetric functions. To understand this, we begin by reformulating $\mathrm{Inv}(\vec{T})$ in terms of the ascent statistic of a certain graph associated with $\vec{\nu}$.

Order on the cells of $\vec{\nu}$ Given a tuple of skew-partitions

$$\vec{\nu} = (\nu^1, \nu^2, \ldots, \nu^k),$$

align the diagrams on a grid so that cells with the same diagonal index lie on the same diagonal. Order the cells by scanning along diagonals from right-to-left, starting with the bottom-most diagonal:[21]



Graph associated to $\vec{\nu}$ Define a graph $\Gamma_{\vec{\nu}}$ (*cf.* Figure 19 and Figure 20) on the cells of $\vec{\nu}$ with an edge connecting $c \in \nu^i$ and $d \in \nu^j$ whenever

- $i < j$ and $\mathrm{diag}(c) = \mathrm{diag}(d)$; or
- $i > j$ and $\mathrm{diag}(c) = \mathrm{diag}(d) + 1$.

Since these are precisely the conditions from the definition of $\mathrm{Inv}(\vec{T})$,

$$\mathrm{Inv}(\vec{T}) = \{\{u, v\} \in E(\Gamma_{\vec{\nu}}) : u < v \ \&\ \vec{T}(u) < \vec{T}(v)\} = \mathrm{Asc}_{\Gamma_{\vec{\nu}}}(\vec{T}),$$

where we view $\vec{T}$ as a function from the cells of $\vec{\nu}$ into $\mathbb{N}^*$.

Proposition 46. *If $\vec{\nu}$ is unicellular, then*

$$\mathrm{LLT}_{\vec{\nu}}(x;t) = \sum_{\substack{\text{all colourings} \\ \kappa:[k]\to\mathbb{N}^*}} t^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(k)},$$

where $k = \ell(\vec{\nu})$ and $\mathrm{asc}(\kappa) = |\mathrm{Asc}_{\Gamma_{\mathrm{shape}(\vec{\nu})}}(\kappa)|$.

Proof. If $\vec{\nu}$ is unicellular, then $\vec{T} \in \mathsf{SSYT}(\vec{\nu})$ associates a positive integer $\kappa(i)$ to the unique cell in component i of $\vec{\nu}$; in other words, it defines a map $\kappa : [k] \to \mathbb{N}^*$, and conversely. □

There is an operation on the algebra of symmetric functions that has the effect of eliminating the improper colourings in the summation.

cite Haglund–Xin, Novelli–Thibon, Carlsson–Mellit

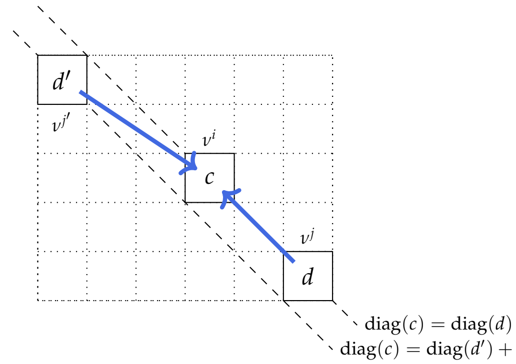


Figure 19: Illustration of the defining conditions of the edges of $\Gamma_{\vec{\nu}}$.

[21] If we write $\xi(c) = (j - i, -a, -i)$ for the cell c in position (i, j) of ν^a, then we are ordering the cells of $\vec{\nu}$ in increasing lexicographic order of $\xi(c)$.

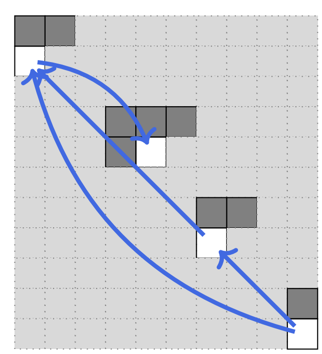


Figure 20: The graph $\Gamma_{\vec{\nu}}$ for $\vec{\nu} = (21/2, 32/31, 21/3, 11/1)$.

Theorem 47. *If $\vec{v}$ is unicellular, then*

$$X_{\Gamma_{\vec{v}}}(x;t) = \frac{1}{(t-1)^{|\ell(\vec{v})|}} \mathrm{LLT}_{\vec{v}}[(t-1)x;t].$$

Example 48. *Consider $\vec{v} = \big((1)/\emptyset, (2)/(1), (3)/(2)\big)$. Then*

$$\Gamma_{\vec{v}} = \text{(1)—(2)—(3)} = \Gamma_{(2,3,3)}.$$

We compute $\Gamma_{\vec{v}}$ explicitly[22] *by working with the variables (x_1, x_2, x_3). There are $3^3 = 27$ possible colourings, which we represent as words.*

κ	$\mathrm{asc}(\kappa)$	κ	$\mathrm{asc}(\kappa)$	κ	$\mathrm{asc}(\kappa)$
111	0	211	0	311	0
112	1	212	1	312	1
113	1	213	1	313	1
121	1	221	0	321	0
122	1	222	0	322	0
123	2	223	1	323	1
131	1	231	1	331	0
132	1	232	1	332	0
133	1	233	1	333	0

Thus,

$$\begin{aligned}
&\mathrm{LLT}_{\vec{v}}(x_1, x_2, x_3; t) \\
&= (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_3^3) \\
&\qquad 2(x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2)t + x_1x_2x_3t^2 \\
&= s_{(3)} + 2s_{(2,1)}t + s_{(1,1,1)}t^2 \\
&= \tfrac{1}{6}(t^2+4t+1)p_{(1,1,1)} - \tfrac{1}{2}(t-1)(t+1)p_{(2,1)} + \tfrac{1}{3}(t-1)^2 p_{(3)}.
\end{aligned}$$

The plethystic substitution *satisfies*

$$p_k[(t-1)x] = (t^k - 1)p_k(x)$$

so that

$$\begin{aligned}
p_{(3)}[(t-1)x] &= (t^3-1)p_{(3)}(x) \\
p_{(2,1)}[(t-1)x] &= (t^2-1)(t-1)p_{(2,1)}(x) \\
p_{(1,1,1)}[(t-1)x] &= (t-1)^3 p_{(1,1,1)}(x)
\end{aligned}$$

and hence

$$\begin{aligned}
&\tfrac{1}{(t-1)^3}\mathrm{LLT}_{\vec{v}}[(t-1)x;t] \\
&\quad = \tfrac{1}{6}(t^2+4t+1)p_{(1,1,1)} - \tfrac{1}{2}(t+1)^2 p_{(2,1)} + \tfrac{1}{3}(t^2+t+1)p_{(3)} \\
&\quad = (t^2+2t+1)s_{(1,1,1)} + ts_{(2,1)} = X_{(2,3,3)}(x;t).
\end{aligned}$$

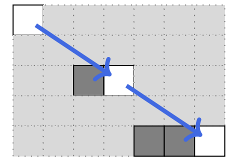


Figure 21: $\Gamma_{(1/\emptyset,2/1,3/1)}$

[22] Or, one can use the formula for the expansion of $\mathrm{LLT}_{\vec{v}}(x;t)$ in the basis of fundamental quasisymmetric functions.

$$\begin{aligned}
&\mathrm{LLT}_{\vec{v}}(x;t) \\
&= \sum_{\vec{T}\in\mathsf{SYT}(\nu)} t^{\mathrm{inv}(\vec{T})} F_{\mathrm{idescomp}(\vec{T})} \\
&= \sum_{w\in S_3} t^{\mathrm{asc}_{\Gamma_{\vec{v}}}(w)} F_{\mathrm{idescomp}(\mathrm{rev}(w))}(x) \\
&= t^2 F_{(1,1,1)} + 2t(F_{(1,2)} + F_{(2,1)}) + F_{(3)} \\
&= t^2 s_{(1,1,1)} + 2ts_{(2,1)} + s_{(3)}
\end{aligned}$$

$\mathrm{rev}(w)$	iDes	comp_3
[321]	$\{1,2\}$	(1, 1, 1)
[231]	$\{1\}$	(1, 2)
[312]	$\{2\}$	(2, 1)
[132]	$\{2\}$	(2, 1)
[213]	$\{1\}$	(1, 2)
[123]	$\{\}$	(3)

w	$\mathrm{asc}_{\Gamma_{\vec{v}}}$
[123]	2
[132]	1
[213]	1
[231]	1
[312]	1
[321]	0

$$\begin{aligned}
\mathrm{LLT}_{(3,3,3)} &= \mathrm{LLT}_{(1/\emptyset,1/\emptyset,1/\emptyset)} \\
&= t^3 s_{(1,1,1)} + (t^2+t)s_{(2,1)} + s_{(3)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{LLT}_{(2,3,3)} &= \mathrm{LLT}_{(1/\emptyset,2/1,3/2)} \\
&= t^2 s_{(1,1,1)} + 2ts_{(2,1)} + s_{(3)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{LLT}_{(2,2,3)} &= \mathrm{LLT}_{(1/\emptyset,1/\emptyset,3/2)} \\
&= ts_{(1,1,1)} + (t+1)s_{(2,1)} + s_{(3)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{LLT}_{(1,3,3)} &= \mathrm{LLT}_{(3/2,1/\emptyset,3/2)} \\
&= ts_{(1,1,1)} + (t+1)s_{(2,1)} + s_{(3)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{LLT}_{(1,2,3)} &= \mathrm{LLT}_{(1/\emptyset,3/2,5/4)} \\
&= s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)}
\end{aligned}$$

5.3 *What is plethysm?*

Characters of $\mathrm{GL}_n(\mathbb{C})$ For each partition λ such that $\ell(\lambda) \leqslant n$, there exists an irreducible $\mathrm{GL}_n(\mathbb{C})$-representation ρ_λ that admits a basis[23]

$$\mathcal{E}_\lambda = \{e_T : T \in \mathsf{SSYT}(\lambda, [n])\}$$

satisfying

$$\mathrm{diag}(x_1, \ldots, x_n) \bullet e_T = x^T e_T.$$

Its character is trace of matrix representing this action:

$$\chi_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathsf{SSYT}(\lambda,[n])} x^T = s_\lambda(x_1, \ldots, x_n).$$

[23] Derutys's construction (1892)

Notation:

$$x_1^2 x_3 \longleftrightarrow \ytableaushort{11,3} \longleftrightarrow \ytableaushort{{x_1}{x_1},{x_3}}$$

Illustrating example

1. For $n = 2$ and $\lambda = (2)$,

$$\rho_{(2)}\left(\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix} = \begin{bmatrix} x^{\ytableaushort{11}} & 0 & 0 \\ 0 & x^{\ytableaushort{12}} & 0 \\ 0 & 0 & x^{\ytableaushort{22}} \end{bmatrix}$$

   whose trace is

$$x^{\ytableaushort{11}} + x^{\ytableaushort{12}} + x^{\ytableaushort{22}} = \sum_{T \in \mathsf{SSYT}((2),[2])} x^T = s_{(2)}(x_1, x_2).$$

2. For $n = 3$ and $\lambda = (2, 1)$,

$$\begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \xmapsto{\rho_\lambda} \begin{bmatrix} y^{\ytableaushort{11,2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & y^{\ytableaushort{11,3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & y^{\ytableaushort{12,2}} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & y^{\ytableaushort{12,3}} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & y^{\ytableaushort{13,2}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & y^{\ytableaushort{13,3}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y^{\ytableaushort{22,3}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y^{\ytableaushort{23,3}} \end{bmatrix}$$

   whose trace is

$$\sum_{T \in \mathsf{SSYT}((2,1),[3])} y^T = s_{(2,1)}(y_1, y_2, y_3).$$

3. Observe that we can compose these representations and that the character of the composition $\rho_{(2,1)} \circ \rho_{(2)}$ is

$$\begin{aligned} s_{(2,1)}\left(x^{\ytableaushort{11}}, x^{\ytableaushort{12}}, x^{\ytableaushort{22}}\right) &= s_{(2,1)}\left(x_1^2, x_1x_2, x_2^2\right) \\ &= \left(s_{(3,2,1)} + s_{(4,2)} + s_{(5,1)}\right)(x_1, x_2). \end{aligned}$$

Composition of GL*-characters* Let

$$\mathrm{GL}_n(\mathbb{K}) \xrightarrow{\rho} \mathrm{GL}_m(\mathbb{K}) \quad \text{and} \quad \mathrm{GL}_m(\mathbb{K}) \xrightarrow{\tau} \mathrm{GL}_\ell(\mathbb{K})$$

be two representations such that the composition $\tau \circ \rho$ is defined.

> *The character* $\chi_\tau[\chi_\rho]$ *of* $\tau \circ \rho : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_\ell(\mathbb{K})$ *is obtained by replacing the i-th variable of* $\chi_\tau(y_1, \ldots, y_m)$ *by the i-th monomial in of* $\chi_\rho(x_1, \ldots, x_n)$.

This operation on polynomials admits an extension to Sym (*i.e.*, so that the result is independent of the number of variables).

Motivation of the axiomatic definition What should $f[g]$ look like?

For all $A \in \mathrm{GL}_n(\mathbb{K})$,

$$(\rho_1 \oplus \rho_2)(\tau(A)) = \rho_1(\tau(A)) \oplus \rho_2(\tau(A))$$
$$(\rho_1 \otimes \rho_2)(\tau(A)) = \rho_1(\tau(A)) \otimes \rho_2(\tau(A))$$

1. If $\rho = \rho_1 \oplus \rho_2$, then $(\rho_1 \oplus \rho_2) \circ \tau = (\rho_1 \circ \tau) \oplus (\rho_2 \circ \tau)$, so that

$$\begin{aligned} \chi_{(\rho_1 \oplus \rho_2)\circ\tau} &= \chi_{\rho_1 \oplus \rho_2}[\chi_\tau] = \left(\chi_{\rho_1} + \chi_{\rho_2}\right)[\chi_\tau] \\ \chi_{(\rho_1 \oplus \rho_2)\circ\tau} &= \chi_{(\rho_1 \circ \tau)\oplus(\rho_2 \circ \tau)} \\ &= \chi_{\rho_1 \circ \tau} + \chi_{\rho_2 \circ \tau} = \chi_{\rho_1}[\chi_\tau] + \chi_{\rho_2}[\chi_\tau]. \end{aligned}$$

   So one would like

$$(f_1 + f_2)[g] = f_1[g] + f_2[g] \qquad (f_1, f_2, g \in \mathrm{Sym})$$

   Similarly, by considering tensor products of representations,[24]

$$(f_1 f_2)[g] = f_1[g] f_2[g] \qquad (f_1, f_2, g \in \mathrm{Sym})$$

[24] If $\rho = \rho_1 \otimes \rho_2$, then

$$(\rho_1 \otimes \rho_2) \circ \tau = (\rho_1 \circ \tau) \otimes (\rho_2 \circ \tau),$$

and so

$$\begin{aligned} &\chi_{(\rho_1 \otimes \rho_2)\circ\tau} \\ &= \chi_{\rho_1 \otimes \rho_2}[\chi_\tau] \\ &= \left(\chi_{\rho_1}\chi_{\rho_2}\right)[\chi_\tau] \end{aligned}$$

and

$$\begin{aligned} &\chi_{(\rho_1 \otimes \rho_2)\circ\tau} \\ &= \chi_{(\rho_1 \circ \tau)\otimes(\rho_2 \circ \tau)} \\ &= \chi_{\rho_1 \circ \tau}\chi_{\rho_2 \circ \tau} = \chi_{\rho_1}[\chi_\tau]\chi_{\rho_2}[\chi_\tau]. \end{aligned}$$

2. Under the projections $\mathrm{Sym} \to \mathbb{C}[x_1, \ldots, x_n]^{S_n}$, we would expect

$$\begin{aligned} p_r[p_s](x_1, \ldots, x_n) &= p_r[x_1^s + \cdots + x_n^s] = x_1^{sr} + \cdots + x_n^{sr} \\ &= p_{rs}(x_1, \ldots, x_n) = p_s[p_r](x_1, \ldots, x_n) \end{aligned}$$

$$\begin{aligned} p_r[p_s + p_t](x_1, \ldots, x_n) &= p_r\left[x_1^s + \cdots + x_n^s + x_1^t + \cdots + x_n^t\right] \\ &= \left(x_1^{sr} + \cdots + x_n^{sr}\right) + \left(x_1^{tr} + \cdots + x_n^{tr}\right) \\ &= \left(p_r[p_s] + p_r[p_t]\right)(x_1, \ldots, x_n) \end{aligned}$$

$$\begin{aligned} p_r[p_s p_t](x_1, \ldots, x_n) &= p_r\left[\left(x_1^s + \cdots + x_n^s\right)\left(x_1^t + \cdots + x_n^t\right)\right] \\ &= \sum_i \sum_j (x_i^s y_j^t)^r = \sum_i x_i^{sr} \sum_j y_i^{tr} \\ &= \left(p_r[p_s] p_r[p_t]\right)(x_1, \ldots, x_n) \end{aligned}$$

These conditions uniquely define an operation on Sym, called *plethysm*.

Plethysm There exists a unique operation on Sym, called *plethysm* and denoted by $f[g]$, satisfying:

(P1) $f \mapsto f[g]$ is an algebra morphism for all $g \in \mathrm{Sym}$.

(P2) $g \mapsto p_r[g]$ is an algebra morphism for all $r \geqslant 1$.

(P3) $p_r[p_s] = p_{rs}$ for all $r, s \geqslant 1$.

Attention: the function $g \mapsto f[g]$ is not in general an algebra map.

Furthermore, this operation satisfies the following properties.

1. $p_r[f](x_1, \ldots, x_n) = f(x_1^r, \ldots, x_n^r)$ for all $f \in \mathrm{Sym}$ and $r \geqslant 1$.
2. $p_r[f] = f[p_r]$ for all $f \in \mathrm{Sym}$ and $r \geqslant 1$.
3. $p_1[f] = f = f[p_1]$ for all $f \in \mathrm{Sym}$.
4. $f[g[h]] = (f[g])[h]$ for all $f, g, h \in \mathrm{Sym}$.

5.4 *Plethystic notation*

Another widespread way to define plethysm makes use of the notion of an "alphabet". The guiding principal here is that:

> *an expression of the form $f[A]$ represents the image of $f \in \mathrm{Sym}$ under a ring morphism $\phi_A : \mathrm{Sym} \to R$ that is determined by an element A of the ring R.*

The rings and ring morphisms here turn out to be quite special.

Lambda rings A *λ-ring* is a commutative ring R equipped with a family of operators $\lambda^k : R \to R$, one for each $k \in \mathbb{N}$, that verify conditions that abstract the operations of direct sum $\oplus$, tensor product $\otimes$, and the exterior product Λ^k on vector spaces. For instance, one has

$$\lambda^k(u+v) = \sum_{i=0}^{n} \lambda^i(u)\lambda^{n-i}(v)$$

because for vector spaces U and V one has

$$\Lambda^k(U \oplus V) = \bigoplus_{i=0}^{n} \Lambda^i(U) \otimes \Lambda^{n-i}(V).$$

One also has

$$\lambda^2(uv) = \lambda^1(u)^2\lambda^2(v) + \lambda^2(u)\lambda^1(v)^2 - 2\lambda^2(u)\lambda^2(v)$$

which parallels the vector space isomorphism

$$[\Lambda^2(U \otimes V)] \oplus 2[\Lambda^2(U) \otimes \Lambda^2(V)] \cong [\Lambda^1(U) \otimes \Lambda^1(U) \otimes \Lambda^2(V)] \oplus [\Lambda^2(U) \otimes \Lambda^1(V) \otimes \Lambda^1(V)]$$

Sym *is a λ-ring* Sym is a λ-ring with operations $\lambda^k(e_1) = e_k$, which parallels the fact that the $\mathrm{GL}_n(\mathbb{C})$-character of $\Lambda^k\mathbb{C}^n$ is e_k. Moreover,

> Sym *is the free λ-ring on one generator; it is generated by e_1.*

Consequently, each element A of a λ-ring R determines a unique morphism of λ-rings[25] $\phi_A : \mathrm{Sym} \to R$ such that

$$\phi_A(e_1) = A \qquad \text{and} \qquad f[g] := \phi_g(f).$$

[25] A morphism of λ-rings is a ring morphism that commutes with the λ^k operations.

$f[g]$ is characterized by $\phi_{f[g]} = \phi_g \circ \phi_f$, which says that plethysm corresponds to the composition of λ-ring endomorphisms of Sym.

Example 49. *Here are some common examples that one finds in the literature.*

- *It is customary to write* $X = x_1 + x_2 + x_3 + \cdots$, *where* $\{x_1, x_2, \ldots\}$ *is the set of variables. In this case,* X *is precisely* e_1 *so that* $\phi = \mathrm{Id}_{\mathrm{Sym}}$ *and*
$$f[X] = f[x_1 + x_2 + x_3 + \cdots] = f[e_1] = f.$$
- *For a finite set of variables, commonly written as* $X_n = x_1 + x_2 + \cdots + x_n$, *one has*
$$f[x_1 + x_2 + \cdots + x_n] = f(x_1, x_2, \ldots, x_n).$$
Here, ϕ *is the canonical projection of* Sym *onto* $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$.
- *The notation* $f[X+Y]$ *corresponds to the* coproduct[26] *of* f, *defined by*
$$f[X+Y] = \Delta(f) = f[p_1 \otimes 1 + 1 \otimes p_1],$$
where $\phi = \Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}$, $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$.
- *The notation* $f[-X]$ *is the image of* f *under the endomorphism of* Sym *defined by* $\phi(p_k) = -p_k$ *(this is the* antipode *of* Sym*).*

[26] Sym is a Hopf algebra, so admits a product, coproduct, antipode, etc.

The $(t-1)$*-transform of* Sym So how does one interpret $f[(t-1)X]$?

- Firstly, $(t-1)X = (t-1)e_1$, so we are working with the λ-ring morphism that maps e_1 to $(t-1)e_1$.
- However, one needs to be careful because the base ring is $\mathbb{C}[t]$, which is a λ-ring via the operations

$$\lambda^k(z) = \binom{z}{k} \quad (z \in \mathbb{C}) \qquad \text{and} \qquad \lambda^k(t) = \begin{cases} 1, & \text{si } k = 0, \\ t, & \text{si } k = 1, \\ 0, & \text{si } k \geqslant 2. \end{cases}$$

With this in mind, we compute $p_2[(t-1)X]$ as follows.

$$\begin{aligned}
&p_2[(t-1)X] \\
&= \phi_{(t-1)e_1}(p_2) \\
&= \phi_{(t-1)e_1}(e_{11}) - 2\phi_{(t-1)e_1}(e_2) \\
&= \phi_{(t-1)e_1}(e_1e_1) - 2\phi_{(t-1)e_1}(\lambda^2(e_1)) \\
&= \phi_{(t-1)e_1}(e_1)^2 - 2\lambda^2(\phi_{(t-1)e_1}(e_1)) \\
&= (t-1)^2e_{11} - 2\lambda^2((t-1)e_1) \\
&= (t-1)^2e_{11} - 2\Big((t-1)^2e_2 + \lambda^2(t-1)e_{11} - 2\lambda^2(t-1)e_2\Big) \\
&= (t-1)^2e_{11} - 2\Big((t-1)^2e_2 - (t-1)e_{11} + 2(t-1)e_2\Big) \\
&= (t-1)^2e_{11} - 2\big((t^2-1)e_2 - (t-1)e_{11}\big) \\
&= (t^2-1)e_{11} - 2(t^2-1)e_2 \\
&= (t^2-1)(e_{11} - 2e_2) \\
&= (t^2-1)p_2.
\end{aligned}$$

$p_2 = e_{11} - 2e_2$

$\lambda^2(uv) = \lambda^1(u)^2\lambda^2(v) + \lambda^2(u)\lambda^1(v)^2 - 2\lambda^2(u)\lambda^2(v)$

$\lambda^2(t-1) = (1)\binom{-1}{2} + (t)(-1) + (0)(1) = 1 - t$

λ-anneaux Voici la définition complète de λ-anneaux.

translate to English...

Definition 50. Un *λ-anneau* est un anneau unitaire commutatif R muni d'une famille d'opérateurs $\lambda^k : R \to R$, un pour chaque $k \in \mathbb{N}$, qui vérifient les axiomes suivants pour tous $x, y \in R$:

1. $\lambda^0(x) = 1$
2. $\lambda^1(x) = x$
3. $\lambda^n(1) = 0$ pour $n \geqslant 2$
4. $\lambda^n(x+y) = \sum_{i=0}^n \lambda^i(x)\lambda^{n-i}(y)$
5. $\lambda^n(xy) = P_n\big(\lambda^1(x), \lambda^2(x), \dots, \lambda^n(x); \lambda^1(y), \lambda^2(y), \dots, \lambda^n(y)\big)$
6. $\lambda^n(\lambda^m(x)) = P_{n,m}\big(\lambda^1(x), \lambda^2(x), \dots, \lambda^{nm}(x)\big)$

où P_n et $P_{n,m}$ sont les polynômes définis par

$$\sum_{n\geqslant 0} P_n\big(e_1(x), \dots, e_n(x); e_1(y), \dots, e_n(y)\big)t^n = \prod_{i\geqslant 1}\prod_{j\geqslant 1}(1 + x_i y_j t)$$

$$\sum_{n\geqslant 0} P_{n,m}\big(e_1(x), e_2(x), \dots, e_{nm}(x)\big)t^n = \prod_{i_1<i_2<\cdots<i_m}(1 + x_{i_1}x_{i_2}\cdots x_{i_m}t)$$

Les axiomes sont des « décatégorifications » de certaines isomorphismes dans la catégorie des espaces vectoriels. Par exemple,

$$\lambda^k(x\oplus y) = \sum_{i=0}^n \lambda^i(x)\lambda^{n-i}(y)$$

est un écho de l'isomorphisme

$$\Lambda^k(X\oplus Y) = \bigoplus_{i=0}^n \Lambda^i(X)\otimes\Lambda^{n-i}(Y);$$

et l'axiome

$$\lambda^2(xy) = \lambda^1(x)^2\lambda^2(y) + \lambda^2(x)\lambda^1(y)^2 - 2\lambda^2(x)\lambda^2(y)$$

est un écho de l'isomorphisme

$$[\Lambda^2(X\otimes Y)] \oplus 2[\Lambda^2(X)\otimes\Lambda^2(Y)] \cong [\Lambda^1(X)\otimes\Lambda^1(X)\otimes\Lambda^2(X)] \oplus [\Lambda^2(Y)\otimes\Lambda^1(Y)\otimes\Lambda^1(Y)].$$

Example 51. *Voici certains des polynômes P_n et $P_{n,m}$:*

$$P_1(x_1; y_1) = x_1y_1$$
$$P_2(x_1, x_2; y_1, y_2) = x_1^2y_2 + x_2y_1^2 - 2x_2y_2$$

Example 52 (λ-anneaux).
- *L'anneau des entiers $\mathbb{Z}$ est λ-anneau, où*

$$\lambda^k(n) = \binom{n}{k}.$$

- *L'anneau de polynômes symétriques à n variables et à coefficients dans $\mathbb{Z}$ est un λ-anneau, où*

$$\lambda^k(f) = e_k[f] \qquad et \qquad \lambda^k(m) = \binom{m}{k}$$

pour tout polynôme symétrique f et tout $m \in \mathbb{Z}$.

- *L'anneau de fonctions symétriques $\Lambda_{\mathbb{Z}}$ est un λ-anneau, où*

$$\lambda^k(f) = e_k[f] \qquad et \qquad \lambda^k(n) = \binom{n}{k}.$$

pour tout $f \in \Lambda_{\mathbb{Z}}$ et tout $n \in \mathbb{Z}$.

Le fait que $\lambda^k(e_1) = e_k$ dans l'anneau de fonctions symétriques est un écho du fait que le $GL_n(\mathbb{C})$-caractère de $\Lambda^k(\mathbb{C}^n)$ est $e_k(x_1, \dots, x_n)$.

Proposition 53. *Λ est engendré comme λ-anneau par e_1; en outre, il est le* λ-anneau libre engendré par un seul générateur. *Ainsi, pour définir un morphisme de λ-anneaux $\phi : \Lambda \to R$, il suffit de préciser l'image de e_1, car*

$$\phi(e_k) = \phi(\lambda^k(e_1)) = \lambda^k(\phi(e_1)).$$

add to:
– use the λ-ring structure to compute images of say e_2 and e_3;
– add information how to do plethysm computations in SageMath – section on "What to read next"