Chromatic Symmetric Functions and Sign-Reversing Involutions

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joint work with Zachary Hamaker and Vincent Vatter

BIRS Workshop on Interactions between Hessenberg Varieties, Chromatic Functions, and LLT Polynomials

Sign-reversing involutions

The $(3 + 1)$-free Conjecture

The coefficient of e_n

Other results and future work

Outline

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$$\sum_{s \in S} (\mathrm{sgn}\, s)(\mathrm{wt}\, s) = \sum_{s \in S^{\iota}} \mathrm{wt}\, s.$$

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Stanley's *chromatic symmetric function* is

$$X(G) = X(G; \mathbf{x}) = \sum_\kappa \mathbf{x}^\kappa$$

where the sum is over all proper $\kappa : V \to \mathbb{P}$.

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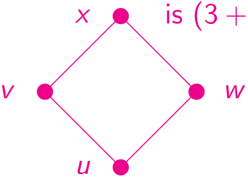

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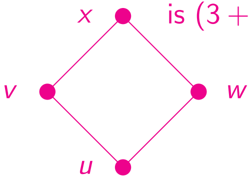
Ex. $P =$ 

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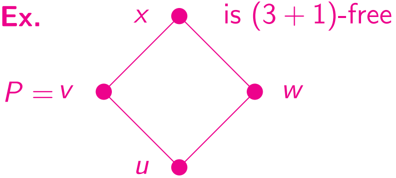

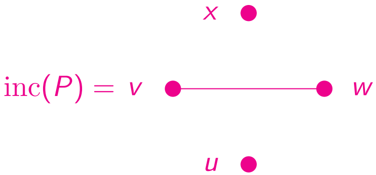

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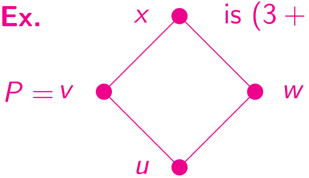

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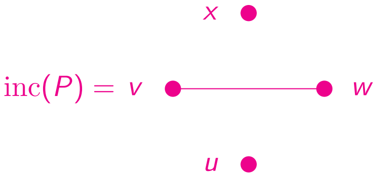

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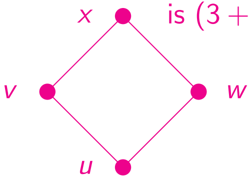
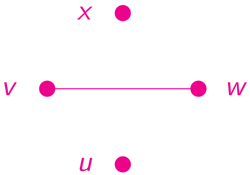
Ex. $P =$  is $(3+1)$-free

$\mathrm{inc}(P) =$ 

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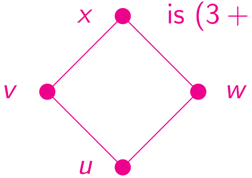

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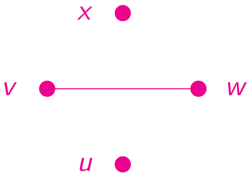

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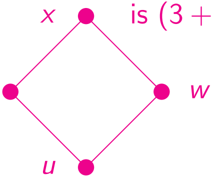

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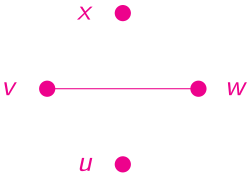

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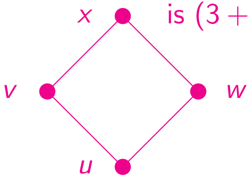

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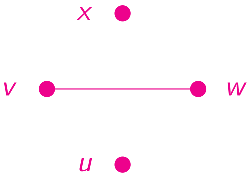

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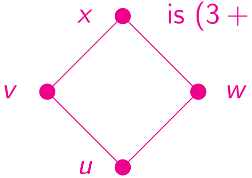
Ex. $P =$ 

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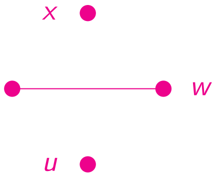
$\text{inc}(P) =$ 

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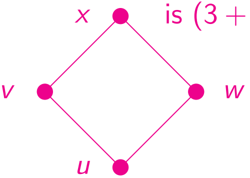
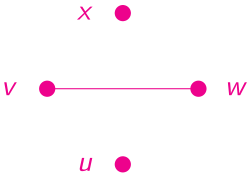
Ex. $P =$  is $(3+1)$-free

$\mathrm{inc}(P) =$ 

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3. Use a sign-reversing involution to cancel the negative terms.

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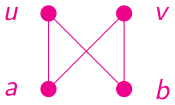
$P = P_{2,2} =$ 

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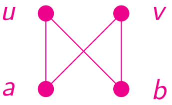

Some P-tableaux:

a	u
b	v

b	v
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u	

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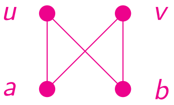

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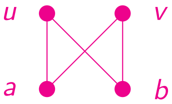

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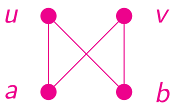
$P = P_{2,2} =$ 

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If P is $(3+1)$-free and $X(\mathrm{inc}(P)) = \sum_\lambda c_\lambda s_\lambda$ then

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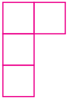
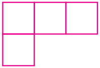
Ex. If $\lambda =$  then $\lambda^t =$ .

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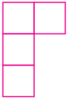
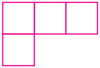
Ex. If $\lambda =$  then $\lambda^t =$ .

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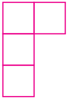
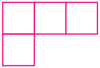
Ex. If $\lambda =$  then $\lambda^t =$ .

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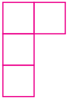
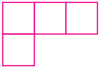
Ex. If $\lambda =$  then $\lambda^t =$ .

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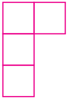
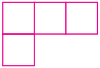
Ex. If $\lambda =$  then $\lambda^t =$ .

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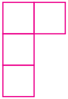
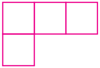
Ex. If $\lambda =$  then $\lambda^t =$ .

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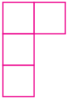
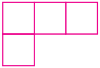
Ex. If $\lambda =$  then $\lambda^t =$ .

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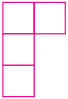
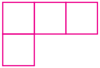
Ex. If $\lambda =$  then $\lambda^t =$ .

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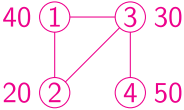

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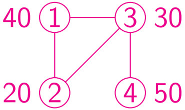

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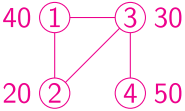

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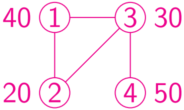

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Conjecture (Shareshian-Wachs)

If P is a NUIO then $X(\mathrm{inc}(P); \mathbf{x}, t)$ is e-positive.

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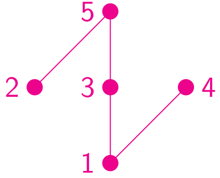

$T =$

1	3	5
2		
4		

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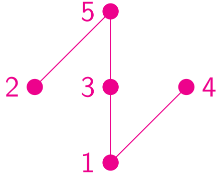
$P =$ 

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Outline

Sign-reversing involutions

The $(3+1)$-free Conjecture

The coefficient of e_n

Other results and future work

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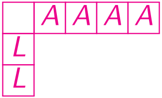
Ex. $\lambda =$ 

$$s_\lambda = \begin{vmatrix} e_3 & e_4 & e_5 & e_6 & e_7 \\ e_0 & e_1 & e_2 & e_3 & e_4 \\ 0 & e_0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_0 & e_1 & e_2 \\ 0 & 0 & 0 & e_0 & e_1 \end{vmatrix}$$

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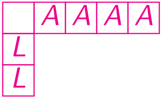
Ex. $\lambda =$ 

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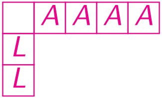
Ex. $\lambda =$ 

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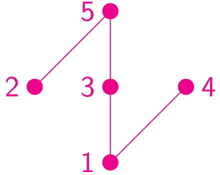
$P =$ 

$T =$

1	3	5
2		
4		

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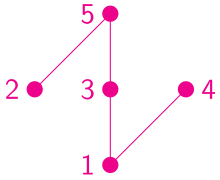
$P =$ 

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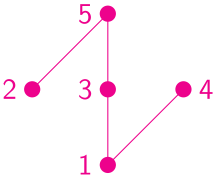
$P =$ 

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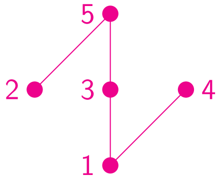
$P =$ 

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Outline

Sign-reversing involutions

The $(3+1)$-free Conjecture

The coefficient of e_n

Other results and future work

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