Equivalence relations arising from general Polish group actions

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BIRS Workshop: Interactions between Descriptive Set Theory and Smooth Dynamics

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In a way, Polish groups can be treated a far-reaching generalization of the Lie groups

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Every Polish group admits a (bounded) left-invariant metric.

Examples

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- the group Iso(X) of all **isometries** of a Polish metric space X
- the group LIso(X) of all **linear isometries** of a separable Banach space X
- the group Aut(μ) of all **measure-preserving automorphisms** of a standard Borel probability space.

Fact

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Proof

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Proof

Recall that if X and Y are Polish spaces and $X \subseteq Y$, then X is G_δ (i.e. $\mathbf{\Pi}^0_1$) in X.

The latter can be deduced from a theorem (Kuratowski) about extensions of partial continuous functions on complete metric spaces to G_δ sets

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This implies that $G = H$, by the **Baire category theorem**.

Definition

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Theorem (Uspenskij)

The group $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ is a universal Polish group.

Proof sketch

Consider the **Banach space** $C_b(G)$ of bounded real-valued functions on G, equipped with the sup norm:

$$\|f\|_\infty = \sup\{f(g) : g \in G\}$$

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While $C_b(G)$ is not a separable Banach space, for each $g \in G$ we have the function

$$f_g(h) = d(g, h)$$

and $f_g \in C_b(G)$. Take

$$X = \mathrm{cl}\{f_g : g \in G\}$$

and note that X **is a separable Banach space**.

The group G **acts on of** X by linear isometries via

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This way, we have **embedded** G **into LIso**(X) for some separable Banach space.

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Theorem (Keller)

The unit ball of the dual space of an infinite-dimensional separable Banach space is homeomorphic to the **Hilbert cube**.

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Theorem (Keller)

The unit ball of the dual space of an infinite-dimensional separable Banach space is homeomorphic to the **Hilbert cube**.

This gives the desired embedding of G into Homeo$([0,1]^{\mathbb{N}})$.

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The group $\mathrm{Aut}(\mu)$ is **not** a universal Polish group.

Definition

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The group $\mathrm{Aut}(\mu)$ embeds into $U(\ell_2)$ via the Koopman representation, and so its **subgroups cannot be not exotic**.

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Theorem (Becker–Kechris)

For every Polish group G there exists a universal Borel G-space.

Notation

We denote by

$$F(G) = \{C \subseteq G : C \text{ is closed}\}$$

This space carries a standard Borel structure called the **Effros Borel space**.

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This space carries a standard Borel structure called the **Effros Borel space**.

If G is compact, then $F(G)$ is the space of compact subsets of G with the **Hausdorff distance**.

There is a **natural action** $G \curvearrowright F(G)$ by left multiplication.

Consider the countable power $F(G)^{\mathbb{N}}$ as a Borel G-space, with the **coordinate-wise action**

$$g \cdot (F_0, F_1, \ldots) = (gF_0, gF_1, \ldots)$$

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Take any Borel G-space $G \curvearrowright X$. For simplicity assume that G **acts by homeomorphisms and X is a zero-dimensional Polish space**.

Fix a countable **basis of topology** $\{U_n : n \in \mathbb{N}\}$ on X, consisting of clopen sets.

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For each $x \in X$ and $n \in \mathbb{N}$ let

$$f_n(x) = \{h \in G : hx \in U_n\}^{-1}$$

and note that $f_n(x)$ is a **closed set**.

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Define $f : X \to F(G)^{\mathbb{N}}$ as

$$f(x) = (f_0(x), f_1(x), \ldots)$$

The function f is an **injection and is G-invariant**.

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To see the latter, it is enough to note that

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which rewrites to

$$\{h \in G : hx \in U_n\}g^{-1} = \{h \in G : hgx \in U_n\}$$

and thus is satisfied.

Extensions of actions

Suppose G is a closed subgroup of a Polish group H and $G \curvearrowright X$ is a Borel G-space. Can we **extend the action of G to an action of H**?

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Theorem (Mackey–Hjorth)

Let H be a Polish group and G a closed subgroup of H. Let $G \curvearrowright X$ be a Borel G-space. Then there is a Borel H-space Y such that

- X is a Borel subset of Y
- the actions $G \curvearrowright X$ and $G \curvearrowright Y$ agree,
- every H-orbit in Y contains exactly one G-orbit in X.

Proof sketch

For simplity assume X **is Polish** and the action $G \curvearrowright X$ is **continuous by homeomorphisms**.

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Consider **the set** $X \times H$ and the action $G \curvearrowright (X \times H)$ defined as

$$g \cdot (x, h) = (gx, gh)$$

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Consider **the set** $X \times H$ and the action $G \curvearrowright (X \times H)$ defined as

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Write Y **for the quotient** $(X \times H)/G$. It turns out (Hjorth) that Y **is a Polish space**.

For $(x, h) \in X \times H$ write $[x, h]$ for its image in the quotient Y.

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Define the **action** $H \curvearrowright Y$ as follows

$$k \cdot [x, h] = [x, hk^{-1}]$$

It is easy to check that with this action we have (i) and (ii) satisfied

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To see (iii), note that **the orbit of** $[x, h]$ **contains the unique** G**-orbit of** $[x, e]$.

Definition

By an **orbit equivalence relation** we will mean an equivalence relation induced by orbits of an action of a Polish group on a standard Borel space.

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Definition

An orbit equivalence relation E is **complete** (or maximal) if every orbit equivalence relation can be Borel-reduced to E.

Fact

Complete orbit equivalence relations **exist**.

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Proof

This follows from the existence of a **universal Polish group** G and a **universal** G**-space**

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This follows from the existence of a **universal Polish group** G and a **universal** G**-space**

as well as the **Mackey–Hjorth extension theorem**, which implies that an orbit equivalence relation induced by a group G can be reduced to an orbit equivalence relation induced by a universal Polish group.

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Consider the relation of **homeomorphism on** $K([0,1]^{\mathbb{N}})$.

Recall that if K_1, K_2 are so-called **Z-sets** in the Hilbert cube, then every homeomorphism between K_1 and K_2 extends to a homeomorphism of the Hilbert cube.

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If we write

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then any compact **subset of the first coordinate is a Z-set**.

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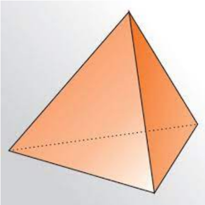
Thus, identifying $K(0,1^{\mathbb{N}})$ with subsets of the first cooridinate of $[0,1]^{\mathbb{N}} = [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$, we see that the relation of **homeomophism on** $K(0,1^{\mathbb{N}})$ **is induced by the action of Homeo**$(0,1^{\mathbb{N}})$, and thus is an orbit equivalence relation.

Definition

A convex compact set C (of a locally convex space) is a **Choquet simplex** if every point of C is the barycenter of a unique probabilty measure concentrated on the extreme points of C.



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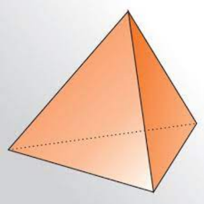

Any Choquet simplex is an inverse limit of a sequence of finite-dimensional simplices (we only talk about metrizable simplices)

Poulsen simplex

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The Poulsen simplex can be realized as the **simplex of invariant measures** on the (full) shift $\{0,1\}^{\mathbb{Z}}$.

Every (separable) Choquet simplex can be affinely embedded as **a face** into **the Poulsen simplex**.

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Thus, we can take the space of **all closed proper faces of the Poulsen simplex** as the space of all Choquet simplices.

Consider the relation of **affine homeomorphism** on the space of Choquet simplices.

Write

$$\mathrm{Aff}(P) = \{f : P \to P : f \text{ is an affine homeomorphism}\}$$

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Thus, the affine homeomorphism of Choquet simplices is an **orbit equivalence relation** induced by the action of Aff(P) on the space of its closed proper faces.

Theorem (S.)

Affine homeomorphism of Choquet simplices is a complete orbit equivalence relation.

Theorem (Zielinski)

Homeomorphism of compact metric spaces is a complete orbit equivalence relation

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Proof

For each Choquet simplex C write

$$\zeta(C) = (C, \{(x,y,z) \in C^3 : z = \frac{1}{2}x + \frac{1}{2}y\}$$

Note that $\zeta(C)$ is **of the form** (K, R) where K is a compact space and $R \subseteq K^3$ is a closed set.

Consider the relation $\simeq_3$ on pairs (K_1, R_1) and (K_2, R_2) as above defined by $(K_1, R_1) \simeq_3 (K_2, R_2)$ if there **exists a homeomorphism** $f : K_1 \to K_2$ **such that** $f^3(R_1) = R_2$.

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Two Choquet simplices C_1 and C_2 are **affinely homeomorphic if and only if** $\zeta(C_1) \simeq_3 \zeta(C_2)$

There exists a Borel reduction of $\simeq_3$ to the relation of **homeomorphism of compact metric spaces**

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Precomposing such reduction with ζ gives a **reduction** of affine homeomorphism of Choquet simplices to the homeomorphism of compact metric spaces

Thus, homeomorphism of compact metric spaces is also complete.

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Let us say a **topological dynamical system** is a compact metric space X with an action of $\mathbb{Z}$ on it via a homeomorphism $T : X \to X$.

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Let us say a **topological dynamical system** is a compact metric space X with an action of $\mathbb{Z}$ on it via a homeomorphism $T : X \to X$.

Two such systems (X_1, T_1) and (X_2, T_2) are **topologically conjugate** if there is a homeomorphism of X_1 and X_2 which conjugates T_1 to T_2.

One can consider the space of all (metric) topological systems by embedding them into a universal system and the **relation of topological conjugacy** on them is an analytic equivalence relation

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There is an easy reduction of the homeomorphism of compact metric spaces to the conjugacy of topological systems

For a compact space X consider the **shift** $X^{\mathbb{Z}}$

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If $X^{\mathbb{Z}}$ is conjugate to $Y^{\mathbb{Z}}$, then X and Y are homeomorphic as X is identified with the set of fixed points

$$\{(\ldots, x, x, x, \ldots) : x \in X\}$$

of $X^{\mathbb{Z}}$.

Thus

$$X \mapsto X^{\mathbb{Z}}$$

is a **reduction** from homeomorphism of compact spaces to topological conjugacy of topological systems.

Let us end with a question.

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Consider dynamical systems of the form (X, mu, T) where X is a **compact metric spac**e, $T : X \to X$ is a homeomorphism and μ is an **atomless probability measure**.

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Consider dynamical systems of the form (X, mu, T) where X is a **compact metric spac**e, $T : X \to X$ is a homeomorphism and μ is an **atomless probability measure**.

Say that (X, μ, T) is **conjugate** to (Y, ν, S) if there is a homeomorphism from X to Y which maps μ to ν and conjugates T to S.

Question

Suppose (X, μ) and (Y, ν) are two compact spaces with atomless probability measures. Are the following equivalent?

- there exists a **homeomorphism from X to Y which maps μ to ν**
- the shifts $(X^{\mathbb{Z}}, \mu^{\mathbb{Z}}, S)$ and $(Y^{\mathbb{Z}}, \nu^{\mathbb{Z}}, S)$ are **conjugate**