Dynamical Systems and Countable Structures

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Banff Workshop on Interactions between
Descriptive Set Theory and Smooth Dynamics
March 28–31,2022

Plan of the Talk(s)

- $=, E_0, E_\infty, E_0^\omega, =^+$
- S_∞-actions
- Borel complete classes of countable structures
- Conjugacy of $\mathrm{Hom}^+[0,1]$
- Cantor systems
- Pointed Cantor minimal systems
- Borel S_∞-orbit equivalence relations
- Subshifts and subflows
- Turbulence and anti-idealisticity

Benchmarks

- $=$ on $2^{\mathbb{N}}$
- E_0 on $2^{\mathbb{N}}$:

$$xE_0y \iff \exists N\ \forall n \geq N\ x(n) = y(n)$$

- E_∞ on $2^{\mathbb{F}_2}$:

$$\begin{aligned} xE_\infty y &\iff \exists\gamma \in \mathbb{F}_2\ (\gamma \cdot x = y) \\ &\iff \exists\gamma \in \mathbb{F}_2\ \forall g \in \mathbb{F}_2\ x(\gamma^{-1}g) = y(g) \end{aligned}$$

- E_0^ω on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$(x_n)E_0^\omega(y_n) \iff \forall n\ (x_nE_0y_n)$$

- $=^+$ on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$(x_n)=^+(y_n) \iff \{x_n \mid n \in \mathbb{N}\} = \{y_n \mid n \in \mathbb{N}\}$$

By early work of

- Silver
- Harrington–Kechris–Louveau
- Dougherty–Jackson–Kechris
- Jackson–Kechris–Louveau

We now understand their relative complexity

- $= \leq_B E_0$:
  Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by
  $$\theta(x)(\langle m, n \rangle) = x(n)$$
  Then
  $$x = y \iff \theta(x) E_0 \theta(y)$$
- $E_0 \leq_B =^+$:
  Fix a bijection $s : \mathbb{N} \to 2^{<\mathbb{N}}$. Define $\theta : 2^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by
  $$\theta(x)_n = x \text{ with the first } |s(n)| \text{ many digits replaced by } s(n)$$
  i.e. $\theta(x)$ enumerates all elements of $[x]_{E_0}$.

- $E_0 \not\leq_B =$:
  Assume there is a Borel reduction $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from E_0 to $=$. For each $n \in \mathbb{N}$, let

$$S_n = \{x \in 2^{\mathbb{N}} \mid \theta(x)(n) = 0\}$$

  Then S_n is an E_0-invariant Borel set. Thus S_n is either meager or comeager. Let $A_n = S_n$ or $2^{\mathbb{N}} - S_n$ so that A_n is comeager. Then

$$\bigcap_n A_n$$

  is comeager. But $\bigcap_n A_n$ is a single E_0-class, hence countable and meager. Contradiction.

S_∞-actions

The infinite permutation group

$$S_\infty = \mathrm{Sym}(\mathbb{N}) = \{g \in \mathbb{N}^{\mathbb{N}} \mid g \text{ is a bijection}\}$$

S_∞ is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$, hence a Polish space.

The group operations on S_∞ are continuous, thus S_∞ is a Polish group.

If S_∞ acts on a Polish space X continuously, for each $x \in X$, the orbit of x is

$$[x] = [x]_{S_\infty} = S_\infty \cdot x = \{g \cdot x \mid g \in S_\infty\}$$

The orbit equivalence relation

$$x E^X_{S_\infty} y \iff [x] = [y] \iff \exists g \in S_\infty \, (g \cdot x = y)$$

These can be generalized

Polish space	$\longrightarrow$	standard Borel space
continuous action	$\longrightarrow$	Borel action

Example The isomorphism relation of all countably infinite digraphs

Consider all countably infinite digraphs $G = (V(G), E(G))$ where $V(G) = \mathbb{N}$. Then $E(G) \subseteq \mathbb{N} \times \mathbb{N}$.

Let

$$\mathcal{DG} = 2^{\mathbb{N}\times\mathbb{N}}$$

We view $\mathcal{DG}$ as the space of all (codes of) countably infinite digraphs. This is a compact metrizable space, hence a Polish space.

Consider the action of S_∞ on $\mathcal{DG}$ by

$$g \cdot E(G) = \{(g \cdot x, g \cdot y) \mid (x, y) \in E(G)\}$$

Then the orbit equivalence relation $E^X_{S_\infty}$ is exactly the isomorphism relation:

$$G \cong G' \iff \exists g \in S_\infty\ (g \cdot E(G) = E(G'))$$

The logic action

A signature L is a collection of

- relation symbols $R, R', \cdots$
- function symbols $F, F', \cdots$
- constant symbols $c, c', \cdots$

each relation symbol or function symbol is associated with a positive integer (arity), denoted $a(R)$ or $a(F)$, etc.

If L is a signature as above, an L-structure M is a tuple

$$M = (X^M, R^M, R'^M, \cdots, F^M, F'^M, \cdots, c^M, c'^M, \cdots)$$

where X^M is a set, and

- if $R \in L$, then $R^M \subseteq (X^M)^{a(R)}$ is a relation;
- if $F \in L$, then $F^M : (X^M)^{a(F)} \to X^M$ is a function;
- if $c \in L$, then $c^M \in X^M$ is a constant.

Let L be a countable signature. The collection of all countably infinite L-structures can be thought of as the set of all L-structures M where $X^M = \mathbb{N}$.

Let

$$\mathrm{Mod}_L = \prod_{R\in L} 2^{\mathbb{N}^{a(R)}} \times \prod_{F\in L} \mathbb{N}^{\mathbb{N}^{a(F)}} \times \prod_{c\in L} \mathbb{N}$$

Then Mod_L is a Polish space on which S_∞ acts by

$$\begin{array}{rcl} g\cdot R^M & = & \{g\cdot(x_1,\cdots,x_{a(R)})\mid(x_1,\cdots,x_{a(R)})\in R^M\} \\ & = & \{(g\cdot x_1,\cdots,g\cdot x_{a(R)})\mid(x_1,\cdots,x_{a(R)}\in R^M\} \end{array}$$

$$\begin{array}{rcl} g\cdot F^M & = & \{g\cdot(x_1,\cdots,x_{a(F)},y)\mid F^M(x_1,\cdots,x_{a(F)})=y\} \\ & = & \{(g\cdot x_1,\cdots,g\cdot x_{a(F)},g\cdot y)\mid F^M(x_1,\cdots,x_{a(F)})=y\} \end{array}$$

The orbit equivalence relation of the logic action $S_\infty \curvearrowright \mathrm{Mod}_L$ is exactly the isomorphism relation between countably infinite L-structures.

Theorem (Becker–Kechris)
Let L be a countable signature with relation symbols of arbitrarily high arity. Then for any Borel action of S_∞ on a standard Borel space X there is an equivariant Borel injection

$$\theta : X \to \mathrm{Mod}_L$$

i.e. for all $x \in X$ and $g \in S_\infty$,

$$\theta(g \cdot x) = g \cdot \theta(x)$$

Theorem (model-theoretic folklore)
Let L be a countable signature. There is a Borel map

$$\theta : \mathrm{Mod}_L \to \mathcal{DG}$$

such that for all $M, M' \in \mathrm{Mod}_L$,

$$M \cong M' \iff \theta(M) \cong \theta(M')$$

This phenomenon is called interpretation in model theory.

Definition (Friedman–Stanley)
A class $\mathcal{C}$ of countable structures is called Borel complete if for any countable signature L, there is a Borel map

$$\theta : \mathrm{Mod}_L \to \mathcal{C}$$

such that for all $M, M' \in \mathrm{Mod}_L$,

$$M \cong M' \iff \theta(M) \cong \theta(M')$$

Fact
A class $\mathcal{C}$ is Borel complete iff for all Borel action of S_∞ on a standard Borel space X, there is a Borel map

$$\theta : X \to \mathcal{C}$$

such that for all $x, y \in X$,

$$x E^X_{S_\infty} y \iff \theta(x) \cong \theta(y).$$

Theorem

The following classes of countable structures are Borel complete:

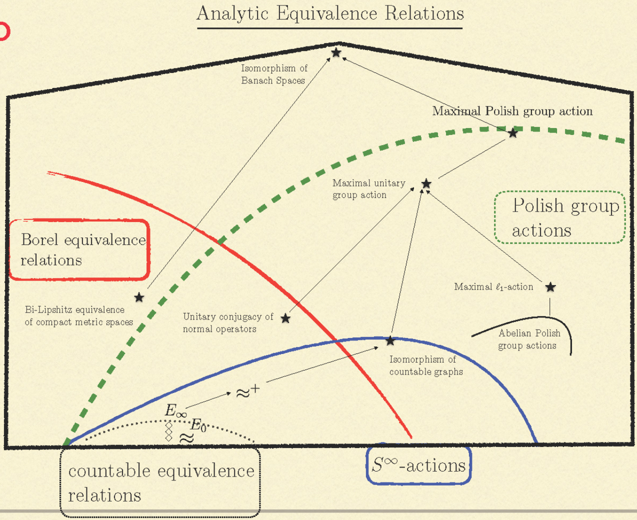
- (Friedman–Stanley) Countably infinite graphs
- (Friedman–Stanley) Countably infinite (rooted) trees
- (Friedman–Stanley) Countably infinite linear orders
- (Mekler) Countably infinite groups ; in fact countably infinite nilpotent groups of class 2
- (Friedman–Stanley) For any prime p, or $p = 0$, countably infinite fields of characteristic p
- (Camerlo–G.) Countably infinite Boolean algebras
- (Paolini–Shelah) Countable torsion-free abelian groups

Some synonyms:

- $\mathcal{C}$ is Borel complete
- $\cong_{\mathcal{C}}$ is S_∞-universal
- $\cong_{\mathcal{C}}$ is Borel bireducible with countable graph isomorphism

Theorem (Friedman–Stanley)
If $\mathcal{C}$ is Borel complete, then $\cong_{\mathcal{C}}$ is complete analytic as a subset of $\mathcal{C} \times \mathcal{C}$; in particular it is not Borel.

The Zoo



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A non-Borel-complete class of countable structures with a non-Borel isomorphism relation

Theorem (Friedman–Stanley)
Let $\mathcal{AT}$ be the class of all countably infinite abelian torsion groups. Then

- $\mathcal{AT}$ is not Borel complete
- $=^+$ is not Borel reducible to $\mathcal{AT}$
- $\cong_{\mathcal{AT}}$ is complete analytic, in particular not Borel

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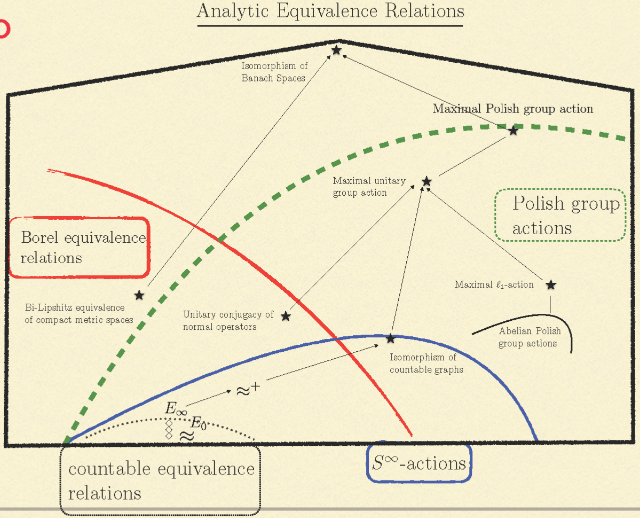
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Example: Classification by Countable Structures

$\mathrm{Hom}^+[0,1]$: the set of all autohomeomorphisms h of $[0,1]$ with $h(0)=0$ and $h(1)=1$

$\mathrm{Hom}^+[0,1] \curvearrowright \mathrm{Hom}^+[0,1]$ by conjugacy:

$$\gamma \cdot h = \gamma \circ h \circ \gamma^{-1}$$

The orbit equivalence relation is the conjugacy equivalence relation

$$h_1 \approx h_2 \iff \exists \gamma \ (\gamma \circ h_1 \circ \gamma^{-1} = h_2)$$

Let

$$L = \{<, S_+, S_-, S_=\}$$

To each $h \in \mathrm{Hom}^+[0,1]$, we associate a countable L-structure

$$M_h = (X_h, <^{M_h}, S_+^{M_h}, S_-^{M_h}, S_=^{M_h})$$

where

- $S_+^{M_h}$ is the set of all open intervals (a, b) where $h(a) = a$, $h(b) = b$, and $h(x) > x$ for $x \in (a, b)$
- $S_-^{M_h}$ is the set of all open intervals (a, b) where $h(a) = a$, $h(b) = b$, and $h(x) < x$ for $x \in (a, b)$
- $S_=^{M_h}$ is the set of all maximal open intervals (a, b) where $h(x) = x$ for all $x \in (a, b)$

One readily verifies that

$$h_1 \approx h_2 \iff M_{h_1} \cong M_{h_2}$$

Conversely, using the Cantor set construction one can associate to each linear ordering R of $\mathbb{N}$ a homeomorphism h_R so that

$$R \cong R' \iff h_R \approx h_{R'}$$

Theorem (folklore)
The conjuagcy equivalence relation of $\mathrm{Hom}^+[0,1]$ is Borel bireducible with countable graph isomorphism.

Cantor Systems

Let X be a Cantor set, i.e., a 0-dimensional compact metrizable space without isolated points. Let T be an autohomeomorphism of X. A Cantor system is the pair (X, T).

Consider the topological conjugacy relation

$$(X,T) \approx (Y,S) \iff \exists \gamma \in \mathrm{Hom}(X,Y)\ (\gamma \circ T \circ \gamma^{-1} = S)$$

Let $\mathrm{Hom}(2^{\mathbb{N}})$ be the set of all autohomeomorphisms of the Cantor set $2^{\mathbb{N}}$.

$\mathrm{Hom}(2^{\mathbb{N}}) \curvearrowright \mathrm{Hom}(2^{\mathbb{N}})$ by conjugacy. The orbit equivalence relation $\approx$ is the topological conjugacy relation between Cantor systems.

Theorem (Camerlo–G.)
The topological conjuagacy relation between Cantor systems is Borel bireducible with countable graph isomorphism.

Fact
Hom($2^{\mathbb{N}}$) is a non-Archimedean Polish group, i.e., G admits a nbhd base of the identity consisting of clopen subgroups.

Theorem (Becker–Kechris)
Let G be a Polish group. The following are equivalent:

1. G is isomorphic to a closed subgroup of S_∞
2. G is non-Archimedean,
3. G admits a compatible left-invariant ultrametric

Theorem (Mackey, Hjorth)
Let H be a Polish group and G be a closed subgroup of H. Suppose $a : G \curvearrowright X$ is a continuous (Borel) action of G on a Polish (standard Borel) space X. Then there is a Polish (standard Borel) space Y and a continuous (Borel) action $b : H \curvearrowright Y$ such that

- X is a closed subset of Y,
- for all $x \in X$ and $g \in G$, $b(g, x) = a(g, x)$
- every H-orbit in Y contains exactly one G-orbit in X

Corollary
For all $x, x' \in X$,

$$[x]_G = [x']_G \iff [x]_H = [x']_H$$

Corollary

The topological conjugacy between Cantor systems is Borel reducible to countable graph isomorphism.

Theorem (Camerlo–G.)

The homeomorphism relation between zero-dimensional compact metrizable spaces is Borel bireducible with countable graph isomorphism.

Reducing homeomorphism between zero-dimensional compact metrizable spaces to topological conjugacy between Cantor systems:

Pointed Cantor Minimal Systems

A pointed Cantor minimal system (pCMS) is a triple (X, T, x) where X is a Cantor set, T a minimal autohomeomorphism of X, and $x \in X$.

CMSs have been the objects of numerous studies since the 1980s.

Open Problem: Is the topological conjugacy relation for CMSs a Borel equivalence relation?

Theorem (Giordano–Putnam–Skau)
Two CMSs are topologically orbit equivalent iff they have isomorphic unital dimension groups.

Theorem (Giordano–Putnam–Skau)
Two CMSs are flip conjugate iff they have isomorphic topological full groups (as abstract groups).

Theorem (Vershik, Putnam, etc.)
Given a pCMS (X, T, x) one can associate a simple ordered Bratteli diagram $(V, E, \geq)$. Let (Y, S, y) be the Vershik map constructed from $(V, E, \geq)$. Then (X, T, x) and (Y, S, y) are topologically conjugate.

Theorem (Kaya)
The topological conjugacy for pCMS is Borel bireducible with $=^+$.

For clopen $U \subseteq X$ and $a \in X$, define

$$\mathrm{Ret}(a, U) = \{n \in \mathbb{Z} \mid T^n a \in U\}$$

Let

$$\mathcal{R}_X(a) = \{\mathrm{Ret}(x, U) \mid U \subseteq X \text{ clopen}\}$$

$$(X, T, a) \approx (Y, S, b) \iff \mathcal{R}_X(a) = \mathcal{R}_Y(b)$$

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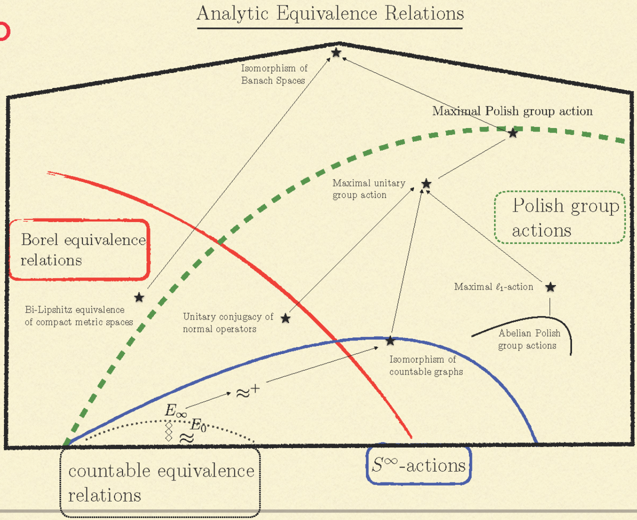
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Borel S_∞-Orbit Equivalence Relations

For a Borel equivalence relation E on a Polish space X, define the jump of E as an equivalence relation on $X^{\mathbb{N}}$:

$$(x_n)E^+(y_n) \iff \{[x_n]_E \mid n \in \mathbb{N}\} = \{[y_n]_E \mid n \in \mathbb{N}\}$$

Theorem (Friedman–Stanley)
If E has more than one class, then $E <_B E^+$.

Theorem (essentially Scott)
If E is a Borel S_∞-orbit equivalence relation, then there is $\alpha < \omega_1$ such that $E \leq_B =^{\alpha+}$.

Theorem (Hjorth–Kechris–Louveau)
For any countable ordinal $\alpha < \omega_1$, there is a countable ordinal $\beta = \beta(\alpha) < \omega_1$ such that for any Polish space X and equivalence relation E on X, $E \leq_B =^{\alpha+}$ iff there is a Polish topology τ on X such that E is a $\mathbf{\Pi}^0_\beta$ subset of (X^2, τ^2).
Here

$$\beta(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ \alpha + 2, & \text{if } 0 < \alpha < \omega \\ \alpha, & \text{if } \alpha \text{ is an infinite limit ordinal} \\ \alpha + 1, & \text{if } \alpha \text{ is an infinite successor ordinal} \end{cases}$$

Theorem (Ding–G., based on work of Solecki)
If G is a non-Archimedean abelian Polish group and G does not involve either $\mathbb{Z}^{\mathbb{N}}$ or $(\bigoplus_{\mathbb{N}} \mathbb{Z}(p))^{\mathbb{N}}$, then any orbit equivalence relation of G is Borel reducible to $(E_0^{\mathbb{N}})^{3+}$.

Subshifts and Subflows

The Bernoulli shift on a finite alphabet Σ is the Cantor system $(\Sigma^{\mathbb{Z}}, T)$ where

$$(Tx)(n) = x(n+1)$$

is the shift map.

A subshift is a Cantor system (X, T), where $X \subseteq \Sigma^{\mathbb{Z}}$ is a closed invariant subset and T is the shift map on X:

$$(Tx)(n) = x(n+1)$$

Let Γ be a countably infinite group.

The Bernoulli flow on a finite alphabet Σ is a pair (Σ^Γ, Γ) where $\Gamma \curvearrowright \Sigma^\Gamma$ by

$$(\gamma \cdot x)(g) = x(\gamma^{-1} g)$$

A subflow is a pair (X, Γ), where $X \subseteq \Sigma^{\mathbb{Z}}$ is a closed invariant subset and $\Gamma \curvearrowright X$ by

$$(\gamma \cdot x)(g) = x(\gamma^{-1} g)$$

Theorem (folklore)
The topological conjugacy relation between subshifts (subflows) is Borel reducible to E_∞, the universal countable Borel equivalence relation.

Theorem (Clemens)
The topological conjugacy relation for all subshifts on $2 = \{0, 1\}$ is Borel bireducible to E_∞.

Theorem (G.–Hill)
The topological conjugacy relation of all minimal rank-one subshifts is Borel bireducible with E_0.

Theorem (Thomas)
The topological conjugacy relation of all (minimal) Toeplitz subshifts Borel reduces E_0.

Theorem (G.–Jackson–Seward)
For any countably infinite group Γ, the following hold:

- The topological conjugacy of all Γ-subflows Borel reduces E_0.
- If Γ is locally finite, then the topological conjugacy of all Γ-subflows is Borel bireducible to E_0.

Hjorth's Turbulence Theory: Obstacles to Reduction to S_∞-Orbit Equivalence Relations

Let $G \curvearrowright X$ be a continuous action of a Polish group G on a Polish space X. For open $U \subseteq X$ with $x \in U$ and open $V \subseteq G$ with $1_G \in V$, the local U-V-orbit of x, denoted $\mathcal{O}(x, U, V)$, is the set of all $y \in U$ for which there exist $\ell \in \mathbb{N}$,

$$x = x_0, x_1, \cdots, x_\ell = y \in U$$

and

$$g_0, g_1, \cdots, g_{\ell-1} \in V$$

such that

$$x_{i+1} = g_i \cdot x_i$$

for all $i < \ell$.

Let G be a Polish group acting continuously on a Polish space X. The action $G \curvearrowright X$ is turbulent if

(T1) every orbit is meager,

(T2) every orbit is dense, and

(T3) every local orbit is somewhere dense, i.e., for any open $U \subseteq G$ with $x \in U$ and for any open $V \subseteq G$ with $1_G \in V$, $\mathcal{O}(x, U, V)$ is somewhere dense.

$G \curvearrowright X$ is preturbulent if for all $x, y \in X$, $U \subseteq X$ open with $x \in U$ and open $V \in G$ with $1_G \in V$,

$$\overline{\mathcal{O}(x, U, V)} \cap [y]_G \neq \varnothing.$$

Theorem (Hjorth)
Let G be a Polish group acting continuously on a Polish space X. Let S_∞ act continously on a Polish space Y. If $G \curvearrowright X$ is preturbulent, then E_G^X is not Borel reducible to $E_{S_\infty}^Y$.

Theorem (Hjorth)
The conjugacy relation of $H([0,1]^2)$ is not Borel reducible to any S_∞-orbit equivalence relation.

Anti-Idealisticity: Obstacles to Reduction to Orbit Equivalence Relations

Let E be an equivalence relation on a Polish space X. E is idealistic if there is an assignment $C \mapsto I_C$ that associates with each E-orbit C a σ-ideal I_C of subsets of C such that

- $C \notin I_C$
- for each Borel set $A \subseteq X^2$ the set

$$A_I = \{x \in X \mid \{y \in [x]_E \mid (x, y) \in A\} \in I_{[x]_E}\}$$

  is Borel

Fact

An orbit equivalence relation is idealistic.

On $\mathbb{R}^{\mathbb{N}}$ define

$$(x_n) E_1 (y_n) \iff \exists N \, \forall n \geq N \, (x_n = y_n)$$

Theorem (Kechris–Louveau)
E_1 is not Borel reducible to any idealistic equivalence relations. In particular, $E_1 \not\leq_B E_G^X$ for any continuous action of Polish group G on a Polish space X.

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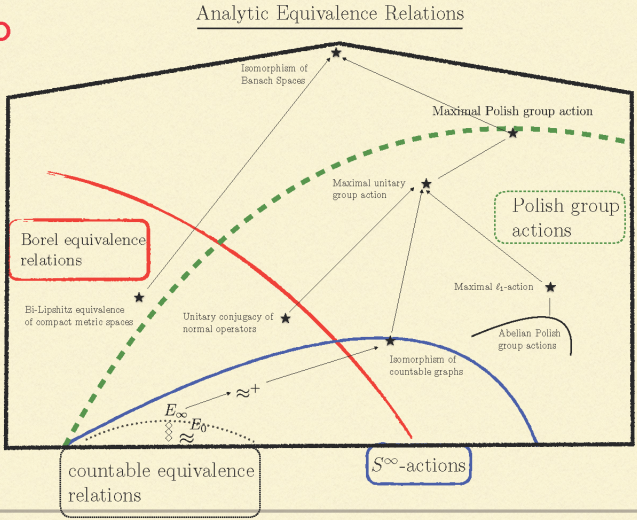

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