Descriptive Set Theory

The category we are in:

- Polish spaces (= separable completely metrizable spaces)
- continuous maps between Polish spaces

Y a Polish space
$A \subseteq Y$

A is *analytic* if $A = f(X)$
for some continuous $f : X \to Y$

A is *Borel* if $A = f(X)$
for some continuous *injective*
$f : X \to Y$

Coanalytic if it is the
complement of
an analytic set

Comparing Borel and analytic
(Lebesgue's mistake)

Structure of sets

analytic are close to Borel

Classification problems

analytic non-Borel are far from Borel

Some examples

A Polish space:

$\mathcal{C}([0,1],\mathbb{R})$ = all continuous functions $[0,1] \longrightarrow \mathbb{R}$ with the unif. conv. topology

$$\rotatebox{90}{$\subseteq$}$$

A Borel set:

$\mathrm{Homeo}([0,1])$ = all homeomorphisms $[0,1] \longrightarrow [0,1]$

$$d(f,g) = \mathrm{unif}(f,g) + \mathrm{unif}(f^{-1},g^{-1})$$

An analytic non-Borel set:

$$\underbrace{\{ f \in \mathcal{C}([0,1],\mathbb{R}) \mid f'(x) \text{ exists at some } x \in [0,1] \}}$$

Basics on Polish spaces

① Each Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$

② Each Polish space is a cont. injective image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

Combinatorics of $\mathbb{N}^{\mathbb{N}}$ (ω^ω)

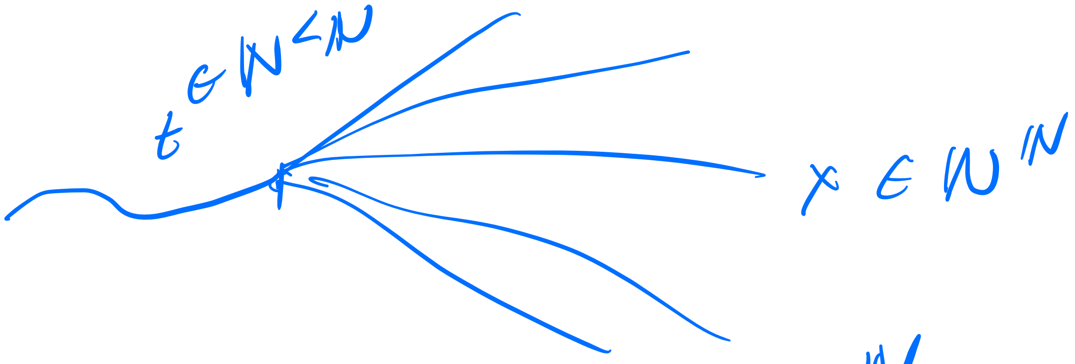
Baire space

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & & \mathbb{N}^{<\mathbb{N}} \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ F & & T_F \\ \text{closed} & & \text{a tree} \end{array}$$

$\mathbb{N}^{<\mathbb{N}}$ = finite sequences

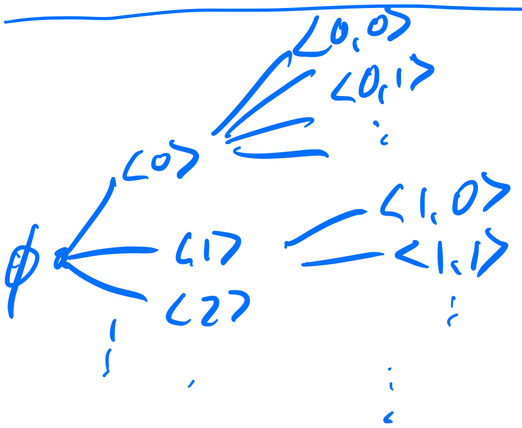
$$t : \underbrace{\{0, \dots, n-1\}}_{n} \to \mathbb{N}$$

$$N_t = \{x \in \mathbb{N}^{\mathbb{N}} \mid t \subseteq x\}$$



N_t is clopen in $\mathbb{N}^{\mathbb{N}}$

$\{N_t \mid t \in \mathbb{N}^{<\mathbb{N}}\}$ is a basis



$\mathbb{N}^{<\mathbb{N}} \ni s, t \quad \rightsquigarrow \quad s \subseteq t$

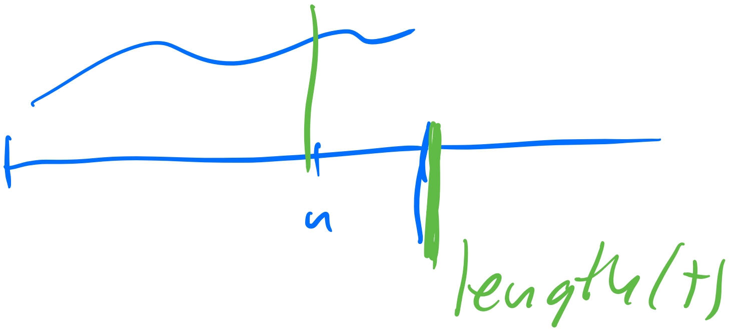
$N_s \supseteq N_t$

$\underline{F \subseteq \mathbb{N}^{\mathbb{N}}}$ closed

$T \subseteq \mathbb{N}^{<\mathbb{N}}$ a tree

$t \in T \Longrightarrow t \upharpoonright n \in T$ if

$n \leq \text{length}(t)$



$F \subseteq \mathbb{N}^{\mathbb{N}}$ closed

iff

$\exists T \subseteq \mathbb{N}^{<\mathbb{N}}$ a tree $\quad F = [T]$

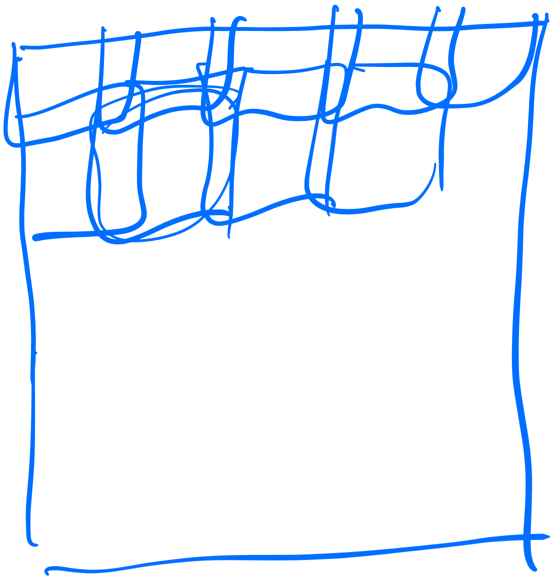
$$[T] = \{x \in \mathbb{N}^{\mathbb{N}} \mid \forall n \; x \upharpoonright n \in T\}$$

$$F \rightsquigarrow T$$

$$T = \{t \in \mathbb{N}^{<\mathbb{N}} \mid N_t \cap F \neq \emptyset\}$$

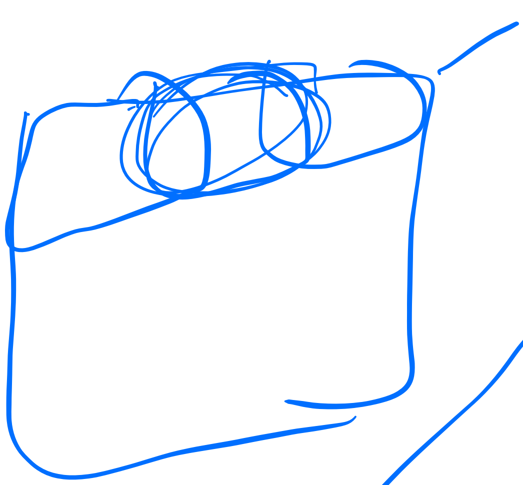
a tree

$$[T] = F$$



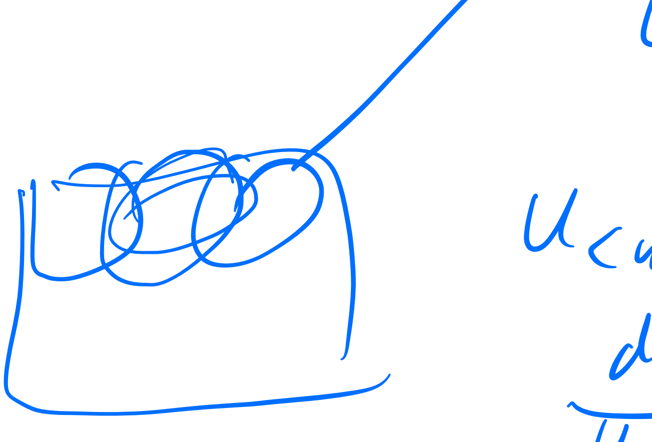
d a *complete* metric

$U_{\langle n\rangle}$
diam ≤ 1



$U_{\langle n,m\rangle}$
diam $\leq \frac{1}{2}$

$\overline{U_{\langle n,m\rangle}} \subseteq U_{\langle n\rangle}$
$\forall m$



$U_{\langle n,m,\ell\rangle}$
diam $\leq \frac{1}{3}$

$\overline{U_{\langle n,m,1\rangle}} \subseteq U_{\langle n,m\rangle}$

$f : \mathbb{N}^{\mathbb{N}} \to X$

$f(x) =$ the unique point in $\bigcap_n U_{x|n}$

surj, cont.

Thm ① Polish spaces are closed under taking ctble products

② Polish spaces are closed under taking closed subsets and open subsets.

③ Polish spaces are closed under taking countable discrete unions.

④ Let τ_n be Polish topologies on a set X, $n \in \mathbb{N}$, with all of them containing a fixed Hausdorff topology τ. Then the topology generated by $\{\tau_n \mid n \in \mathbb{N}\}$ is Polish.

$$X \xhookrightarrow{\iota} \prod_n (X, \tau_n)$$

Polish

$$x \longrightarrow (x, x, \dots)$$

Polish $\rightarrow$ $\iota(X)$ $\underset{2}{\text{closed}}$ in $\prod_n (X, \tau_n)$

Books:

Kechris "Classical Descriptive Set Theory"

Srivastava "Borel Sets"

Moschovakis "Descriptive Set Theory"

Basic structure of Borel and analytic and the relationship between the two

- Can Borelness and analyticity of A be detected within the Polish space of which A is a subset?

- How does analycity relate to Borelness?

$\mathcal{B}(X)$ = the σ-algebra generated by open subsets of X

Thm (Luzin)

$A, B \subseteq X$ analytic disjoint

There exists $C \in \mathcal{B}(X)$ s.t.

$A \subseteq C$, $C \cap B = \emptyset$

Proof. Fix $f : \mathbb{N}^{\mathbb{N}} \to A$, $g : \mathbb{N}^{\mathbb{N}} \to B$ cont. surjections

assume A & B not separated

$\forall n \ \exists s_n, t_n \in \mathbb{N}^{<\mathbb{N}}$ $|s_n| = |t_n| = n$

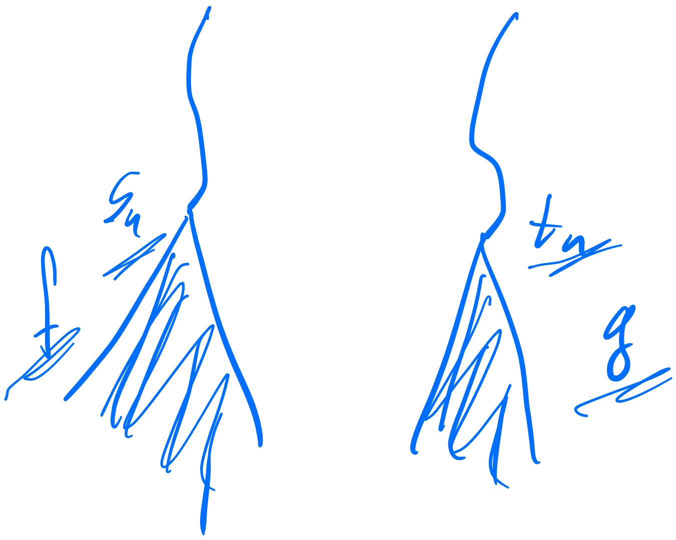
$f(N_{s_n})$

$g(N_{t_n})$

are not separated

$s_n \subsetneq s_{n+1}$

$t_n \subsetneq t_{n+1}$



$\exists x \supseteq s_n$ all n

$\rotatebox{90}{$\in$}\ \mathbb{N}^{\mathbb{N}}$

$\exists y \supseteq t_n$ all n

$\rotatebox{90}{$\in$}\ \mathbb{N}^{\mathbb{N}}$

$x, y \in \mathbb{N}^{\mathbb{N}}$

$f(x) \in A \qquad g(y) \in B$

$$\begin{array}{cc} f(x) \notin & g(y) \\ \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} \\ \boxed{U} & \boxed{V} \\ \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} \\ f(N_{s_n}) & g(N_{t_n}) \end{array}$$

open disjoint

contradiction

---

s, t s.t. $f(N_s), g(N_t)$ not separated

$$\begin{array}{cc} f(N_{s^\frown i}) & g(N_{t^\frown j}) \\ \rotatebox{-90}{$\in$} & \\ C_{ij} \in \mathcal{B}(X) & \end{array}$$

s.t. $C_{ij} \cap g(N_{t^\frown j}) = \emptyset$

$\underline{\bigcap_j C_{ij}}$ separates $\underline{f(N_{s,i})}$

from $\bigcup_j g(N_{t,j})$

$= f(N_s)$

$\bigcup_i \bigcap_j C_{ij}$ separates $\bigcup_i f(\underline{\underline{N_{s,i}}})$

from $\bigcup_j g(N_{t,j})$

$\overset{\cap}{\mathcal{B}(X)}$

$\overset{\|}{g(N_t)}$

$\square$

Corollary Borel subsets of X

$$\rotatebox{90}{$\subseteq$}$$

$$B(X)$$

Proof A Borel

$A = f(F)$ $\quad f : F \to A$ cont. inj, $\quad F \subseteq \mathcal{N}^{\mathcal{N}}$, $\quad F = [T]$

$$P_\emptyset = f(N_\emptyset)$$

$$f(N_{t^\frown i}) \subseteq P_{t^\frown i} \subseteq \overline{f(N_{t^\frown i})}$$

$$P_{t^\frown i} \subseteq P_t$$

$$P_{t^\frown i} \cap P_{t^\frown j} = \emptyset \quad \text{if} \quad i \neq j$$

$$N_t^T = N_t \cap [T] \qquad t \in T$$

$$\| N_t$$

$$P_t \in B(X)$$

$$f(F) = \bigcap_n \bigcup_{\substack{t \\ |t| = n}} \underbrace{\bigcirc\!\!\!\!\!P_t}_{} \; B(X)$$

$$x \in \bigcap_{n} \bigcup_{\substack{t \\ |t|=n}} P_t$$

$$\forall n \ \exists t_n \ x \in \underline{P_t}$$
$$|t_n| = n$$

$$t_n \subseteq t_{n+1} \longrightarrow \alpha \in F$$

$$x = f(\alpha). \quad \square$$

Thm (Kuratowski)

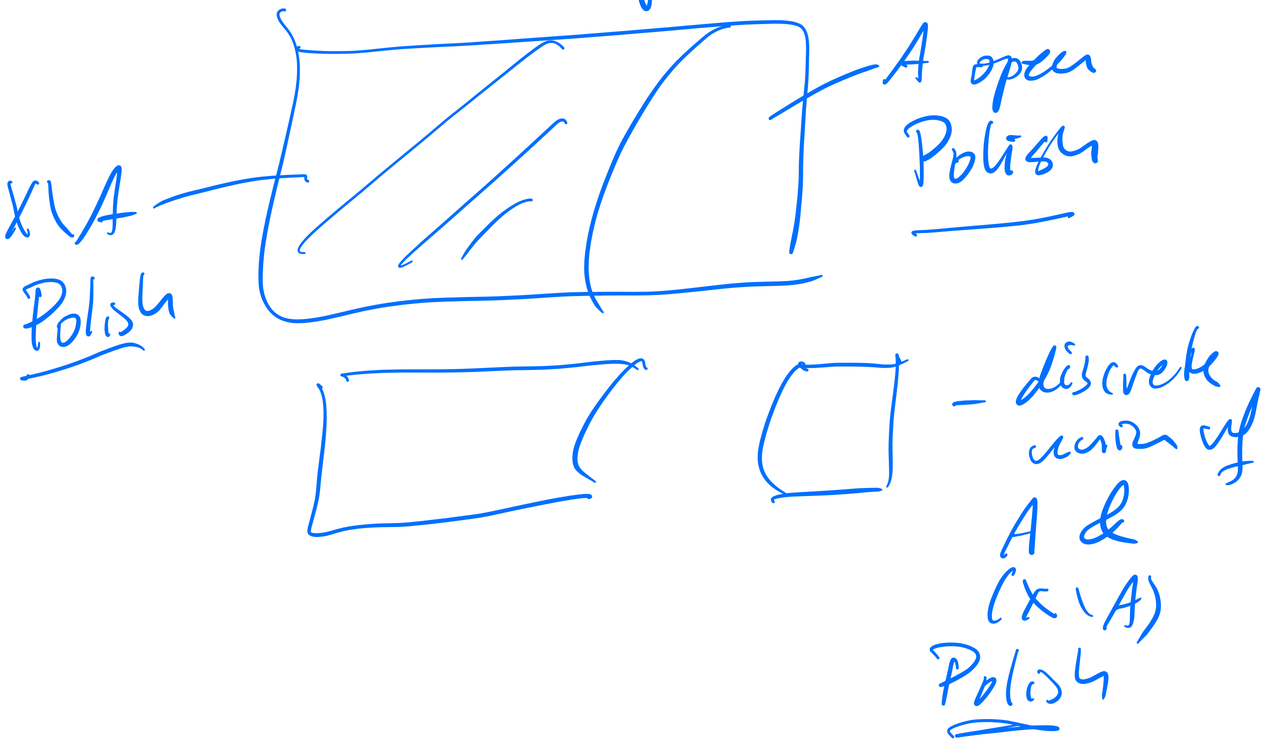
$A \subseteq X$ in $\mathcal{B}(X)$.

Then there exists a Polish top.

$\circledast$ $\{$ τ on X extending the original topology s.t. A is τ-clopen.

Proof. Consider the class $\mathcal{A}$ of subsets of X with $\circledast$.

$\mathcal{A}$ contains open sets



$\mathcal{A}$ is closed under complements

$\mathcal{A}$ closed under ctble unions

A_n — $\tau_n \supseteq \sigma$

the init. top.

A_n is τ_n-clopen

τ = the top generated by $\{\tau_n\}_{n \in \mathbb{N}}$

Polish

So each A_n is clopen in τ

So $\bigcup_n A_n$ is open in τ $\boxed{\Pi}$

τ'

So that $\bigcup_n A_n$ is clopen in τ'!

So $\mathcal{A}$ is a σ-alg. containing open sets. So $\mathcal{B}(X) \subseteq \mathcal{A}$. $\square$

Corollary $\mathcal{B}(X) \subseteq$ Borel subsets of X

Proof. $A \in \mathcal{B}(X)$ $\quad X, \sigma$ – Polish

$$\rightsquigarrow$$

τ Polish on X

s.t. A is τ-clopen

$\tau \supseteq \sigma$

Consider $\underbrace{A \text{ with } \tau\restriction A}_{\text{Polish}}$.

$$id : \underset{\text{Polish}}{(A, \tau\restriction A)} \xrightarrow[\text{cont}]{1\text{-to-}1} \underset{(X, \sigma)}{(A, \sigma\restriction A)}$$

$\square$

Main Corollaries

Corollary Borel subsets of X

||

the σ-algebra generated by open subsets of X

||

the smallest family of subsets of X containing open sets and closed under ctble unions and intersections

Corollary 1. (Luzin) Disjoint analytic sets can be separated by Borel sets

2. (Suslin) A set is Borel iff it and its complement are analytic (i.e. the set is analytic and coanalytic)

$\Leftarrow$

$$\begin{array}{ccc} A & & X \setminus A \\ \cap\!| & & C \cap (X \setminus A) = \emptyset \\ C & & \\ \text{Borel} & & \\ \hline \end{array}$$

$$A = C$$

Suslin's operation $\mathcal{A}$

$F_s \subseteq X \qquad s \in \mathbb{N}^{<\mathbb{N}}$

$x \in \mathcal{A}_s\{F_s\}$ iff $\exists \alpha \in \mathbb{N}^{\mathbb{N}}$ $x \in \bigcap_n F_{\alpha \restriction n}$.

$x \in \bigcup_n A_n$ iff $\exists n$ $x \in A_n$

Thm A set $A \subseteq X$ is analytic iff $A = \mathcal{A}_s F_s$ for some $F_s \subseteq X$ closed, $s \in \mathbb{N}^{<\mathbb{N}}$

($\Downarrow$) $f : \mathbb{N}^{\mathbb{N}} \longrightarrow A$ cont surj

$F_s = \overline{f(N_s)}, \quad s \in \mathbb{N}^{<\mathbb{N}}$

$\mathcal{A}_s F_s = A$

$(\Uparrow)$ $H \subseteq X \times \mathbb{N}^{\mathbb{N}}$

$$(x,\alpha) \in \underset{\text{closed}}{H} \text{ iff } \forall n \underbrace{\left(x \in F_{\alpha \restriction n}\right)}_{(x,\alpha,n)}$$

$$\underbrace{\bigcap_n \qquad\qquad\qquad\qquad}_{\text{closed}}$$

Check: $\underset{\text{analytic.}}{A} = \mathrm{proj}_X \underset{\text{Polish.}}{H}$

$\square$

- Definitions of Borel and analytic
- Representations within the space (σ-algebra, operation $\mathcal{A}$)
- Relationship between Borel and analytic
- Closure and regularity properties

Thm 1. (Saks) Analytic sets are measurable

2. (Nikodym) Analytic sets have the Baire property.

|| measurable sets are closed under operation $\mathcal{A}$

|| Baire property - - -

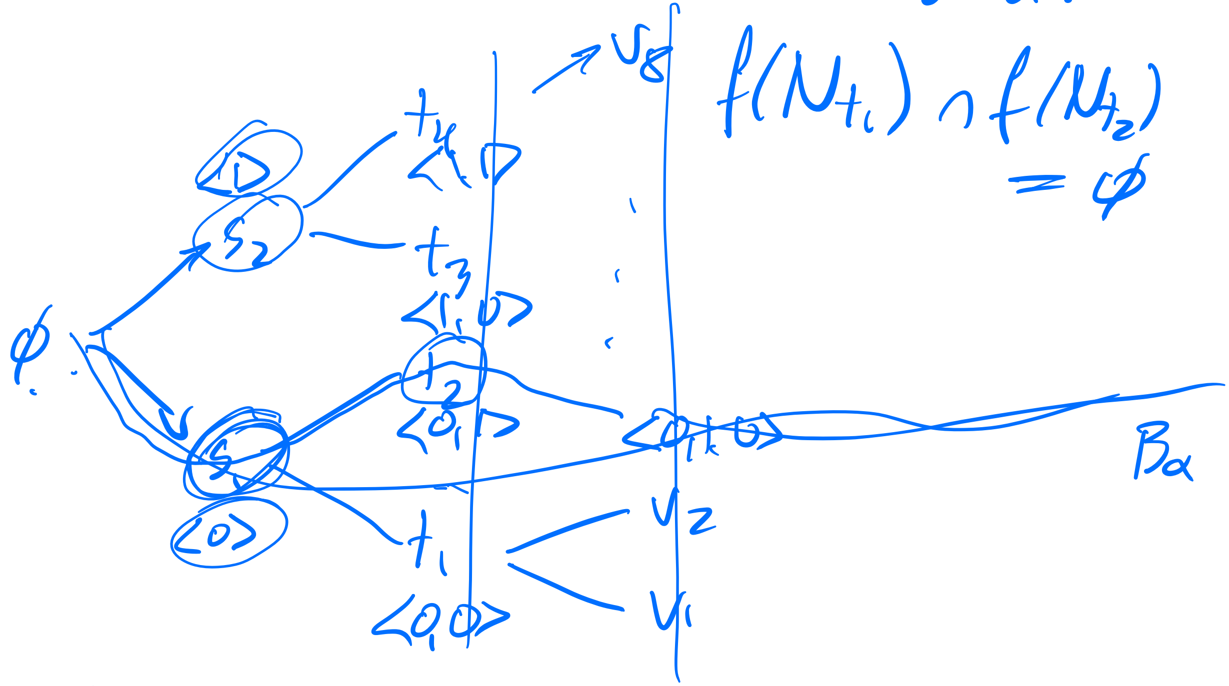
Thm (Hausdorff, Suslin)
Each analytic uncountable set contains a copy of the Cantor set $\{0,1\}^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$

Proof $f : \mathbb{N}^{\mathbb{N}} \longrightarrow A$

$\forall s \in \mathbb{N}^{<\mathbb{N}}$ $\quad \exists t_1, t_2 \supsetneq s$

$f(N_s)$ uncountable $\quad f(N_{t_1}), f(N_{t_2})$ uncountable

$f(N_{t_1}) \cap f(N_{t_2}) = \emptyset$



$\alpha = \langle 0, 1, 0$

$$2^{<\mathbb{N}} \ni s \longrightarrow t_s \in \mathbb{N}^{<\mathbb{N}}$$

$$\boxed{f(N_{t_s}) \cap f(N_{t_{s'}}) = \emptyset}$$

if $|s| = |s'|$

$s \neq s'$

$$2^{\mathbb{N}} \ni \alpha \longrightarrow \beta_\alpha \in \mathbb{N}^{\mathbb{N}}$$

$\shortparallel$

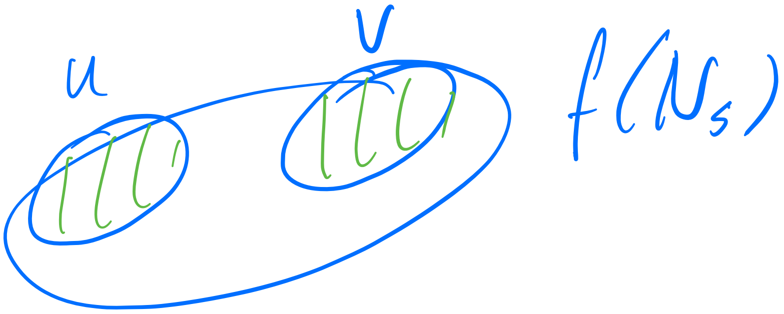
the common extension of $t_{\alpha|n}$, $n = 0, 1, \ldots$

$$\underline{t_{\alpha|0} \subsetneqq t_{\alpha|1} \subset t_{\alpha|2} \subset \cdots} \longrightarrow \beta_\alpha$$

$$2^{\mathbb{N}} \ni \alpha \xrightarrow[\substack{\text{injective}\\ \text{continuous}}]{} f(\beta_\alpha) \in A$$

$$2^{\mathbb{N}} \underset{\substack{\text{homeo}\\ \text{embedding!}}}{\hookrightarrow} A$$

$f(N_s)$ unctble



$U \cap f(N_s)$, $V \cap f(N_s)$ unctbl

$t_1 \in S$ $U \supseteq f(N_{t_1})$

$(U \cap V = \emptyset$

$\hookrightarrow f(N_{t_2})$ $t_2 \in S$

D

$$\begin{tikzcd} \mathbb{N}^{\mathbb{N}} \supseteq & 2^{\mathbb{N}} \arrow[r, hook] & A \\ & \mathbb{N}^{\mathbb{N}} \arrow[u, "\text{homeo}"'] & \end{tikzcd}$$

<u>unctble</u>

Borel functions

$$\begin{tikzcd} f\colon B \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & C \arrow[d, phantom, sloped, "\subseteq"] \\ X & Y \end{tikzcd}$$

B, C Borel

f is **Borel** if preimages of Borel sets are Borel.

Thm f is Borel $(B \to C)$ iff graph(f) is a Borel set $(B \times C)$

Proof. $(\Downarrow)$ $(U_n)_{n \in \mathbb{N}}$ a top basis of C

$$\mathrm{graph}(f) = \bigcap_n \left(f^{-1}(U_n) \times U_n\right) \cup \left((X \setminus f^{-1}(U_n)) \times Y\right)$$

(ii)

$f : B \to C$

$\phantom{f : B \to {}} \overset{\cup|}{D}$ Borel

$f^{-1}(D) = \{x \mid f(x) \in D\}$

$E = \mathrm{graph}(f) \cap (X \times D)$ — Borel

$x \in f^{-1}(D)$ iff $\exists y \ (x,y) \in E$

$$\| \ \underbrace{\mathrm{proj}_{\mathrm{on}\,X} E}_{\text{analytic}}$$

$f^{-1}(D)$ analytic

$f^{-1}(C \setminus D)$ analytic $\Longrightarrow$ $f^{-1}(D)$

$\|$ Borel $\qquad\qquad$ is Borel.

$B \setminus f^{-1}(D)$

$\square$

Thm (Kuratowski)
X, Y Polish spaces, <u>unctble</u>

$$\Downarrow$$

$$\exists\ f: X \to Y \quad \text{a Borel bijection}$$

$$\mathbb{N}^{\mathbb{N}} \supseteq F \xleftarrow[\text{1-1}]{} \underset{\text{Borel}}{\underline{X}}$$

$$\mathbb{N}^{\mathbb{N}} \xrightarrow[\substack{\text{1-1}\\ \text{cont}}]{} \underset{(\text{Borel})}{\underline{X}}$$

Borel hierarchy

ω_1 = the least unctble ordinal

Σ^0_1 = open sets

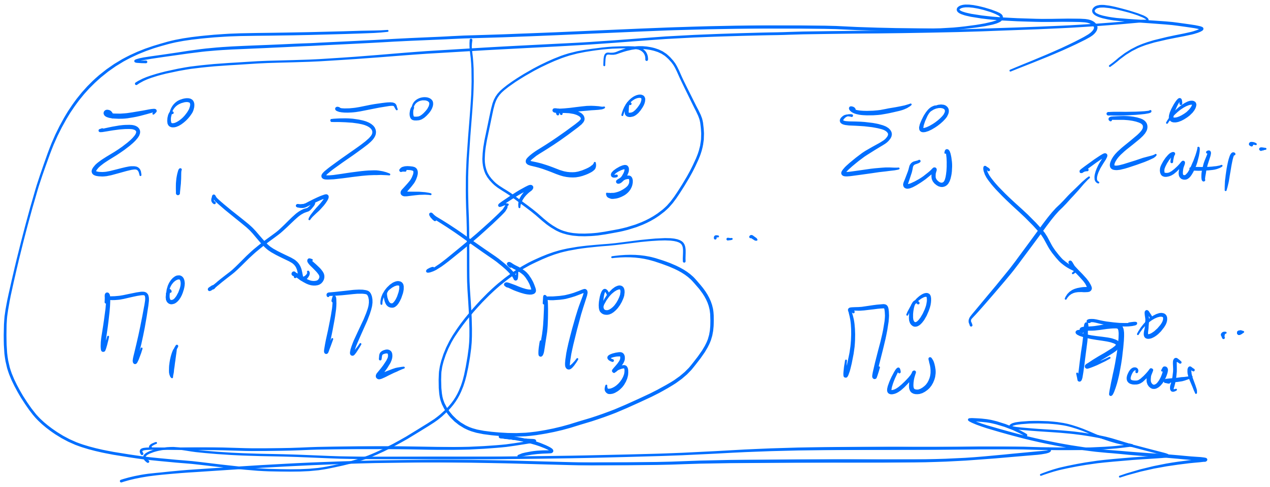
Π^0_1 = closed sets

$\Sigma^0_{\alpha+1}$ = ctble unions of sets in Π^0_α

$\Pi^0_{\alpha+1}$ = complements of sets in $\Sigma^0_{\alpha+1}$
= ctble intersections of sets in Σ^0_α

Σ^0_λ = ctble unions of sets in $\bigcup_{\alpha<\lambda} \Pi^0_\alpha$

Π^0_λ = complements of sets in Π^0_λ
= ctble ints of sets in $\bigcup_{\alpha<\lambda} \Sigma^0_\alpha$



$$\text{Borel sets} = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$$

$A_0 \quad A_1 \quad \dots \qquad \bigcup_n A_n$

$\Sigma^0_{\alpha_0} \quad \Sigma^0_{\alpha_1} \quad \dots$

$\cap$

Σ^0_α

Examples 0. $\mathbb{Q} \subseteq \mathbb{R}$ Σ^0_2, not Π^0_2

1. $C^n([0,1],\mathbb{R})$ = functions n-times continuously differentiable $n \geq 1$

Π^0_3, not Σ^0_3 in $C([0,1],\mathbb{R})$

2. $\mathrm{Homeo}([0,1])$ Π^0_2, not Σ^0_2 in $\underline{C([0,1],\mathbb{R})}$

3. $\{x \in 2^{\mathbb{N}\times\mathbb{N}} \mid \underline{\exists^\infty_m \exists^\infty_n}\ x(m,n)=0\}$

Π^0_4, not Σ^0_4

$$\underbrace{\forall m' \underbrace{\exists m>m' \underbrace{\forall n' \underbrace{\exists n>n'\ x(m,n)=0}_{\Sigma^0_1}}_{\Pi^0_2}}_{\Sigma^0_3}}_{\Pi^0_4}$$

analytic sets = continuous images of Polish spaces = results of applications of $\mathcal{A}$ to closed sets

⇕

Borel sets = continuous <u>injective</u> images of Polish spaces = results of applications of ctble unions, ctble intersections, complements to open sets (σ-algebra)

Analytic and Borel are distinct

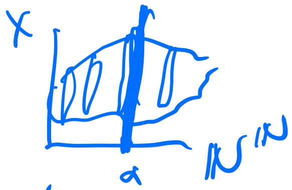
Universal sets

X a Polish space

$\{U_n \mid n \in \mathbb{N}\}$ a top. basis for X including the empty set

$$\mathbb{N}^{\mathbb{N}} \ni \alpha \longrightarrow \bigcup_n U_{\alpha(n)}$$

$$\underset{\underset{\mathcal{U} \text{ open}}{\shortparallel}}{\{(\underline{\alpha}, \underline{x}) \mid x \in \bigcup_n U_{\alpha(n)}\}} \subseteq \mathbb{N}^{\mathbb{N}} \times X$$



$U \subseteq X$ open iff $\exists \alpha \in \mathbb{N}^{\mathbb{N}}$ $U = \mathcal{U}_\alpha$

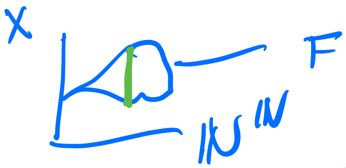
$\boxed{\mathcal{U} \text{ universal for open sets}}$

$$\underline{\underline{\mathcal{F}}} = (\mathbb{N}^{\mathbb{N}} \times \underline{X}) \setminus \mathcal{U} \subseteq \mathbb{N}^{\mathbb{N}} \times X$$

univ. for closed

$F \subseteq X$ is closed iff $\exists \alpha \in \mathbb{N}^{\mathbb{N}}$

$\underline{F} = \mathcal{F}_\alpha$



$$X = \underbrace{Y \times \mathbb{N}^{\mathbb{N}}}_{} \rightsquigarrow \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \times (Y \times \mathbb{N}^{\mathbb{N}})$$

$$\mathcal{A} \overset{\subseteq\, \mathbb{N}^{\mathbb{N}} \times Y}{=} \underline{\mathrm{proj}_{\text{onto}\,(\mathbb{N}^{\mathbb{N}} \times Y)}}\, \mathcal{F}$$

analytic

$$= \underline{\{(\alpha, y) \mid \exists \beta \ (\alpha, y, \beta) \in \mathcal{F}\}}$$

$\mathcal{A}$ univ. for analytic subsets of Y

$A \subseteq Y$ analytic iff $\underline{A = \mathcal{A}_\alpha}$
some $\alpha \in \mathbb{N}^{\mathbb{N}}$

Thm (Suslin)
Each unctble Polish space contains an analytic set that is not Borel.

Let's show only that there exists an analytic non-Borel set.

$$\mathcal{A} \subseteq \underbrace{\mathbb{N}^{\mathbb{N}} \times \overset{Y}{\widetilde{\mathbb{N}^{\mathbb{N}}}}}_{\text{all } \mathbb{N}^{\mathbb{N}}} \quad \text{univ. for analytic subsets of } Y$$

$\mathcal{A}$ is not Borel.

$\mathcal{A}$ is Borel $\Longrightarrow$

$C = \{\alpha \in \mathbb{N}^{\mathbb{N}} \mid (\alpha,\alpha) \notin \mathcal{A}\}$ is Borel so analytic

There is α_0 s.t. $C = \mathcal{A}_{\alpha_0}$

$\alpha_0 \in C$ iff $\alpha_0 \in \mathcal{A}_{\alpha_0}$ iff $(\alpha_0, \alpha_0) \in \mathcal{A}$
iff $\alpha_0 \notin C$ contradi.

Borel automata

X = a Polish space — fixed

S = the set of states (ctble)

$s_0 \in S$

Φ = a set of statements about a variable x, say "$x \in U$", $U \subseteq X$ open

program instructions:

s: all A $\qquad A \subseteq S$

s: some A

s: not t $\qquad t \in S$

s: if φ then t else u $\qquad t, u \in S$, $\varphi \in \Phi$

A Borel automaton is a function Δ on S with values

$\Delta(s) = (\text{all}, A)$

$\Delta(s) = (\text{some}, A)$

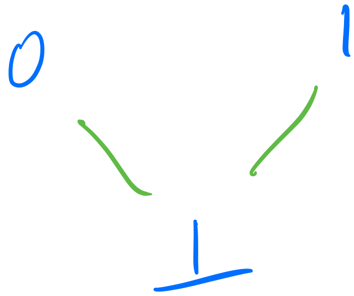
$\Delta(s) = (\text{not}, t$

$\Delta(s) = (\varphi, t, u)$

$\{0, 1, \perp\}$

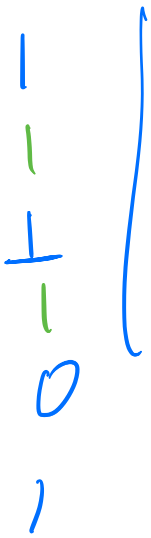
reject accept undecided

The Scott order on the set above



$\perp \sqsubseteq 0, \quad \perp \sqsubseteq 1$

The Łukasiewicz order



$0 \leq \perp \leq 1$

Running an automaton

Input $a \in X$ $\qquad$ $L_1 \sqsubseteq L_2$

For $L : S \to \{0, 1, \bot\}$

$$T_a(L)(s) = \begin{cases} \bigwedge_{t \in A} L(t) & \text{if } \Delta(s) = (\text{all}, A) \\ \bigvee_{t \in A} L(t) & \text{if } \Delta(s) = (\text{some}, A) \\ \neg L(t) & \text{if } \Delta(s) = (\text{not}, t) \\ L(t) & \text{if } \Delta(s) = (\varphi, t, u) \text{ and } \varphi(a) \text{ holds in } X \\ L(u) & \text{if } \Delta(s) = (\varphi, t, u) \text{ and } \varphi(a) \text{ fails in } X \end{cases}$$

with respect to the Łukasiewicz order

There exists $\sqsubseteq$-least $L = L_a$ s.t. $T_a(L) = L$. $\qquad$ $L_0 \equiv \bot$

$a \in X$ accepted by Δ if $\boxed{L_a}(s_0) = 1$

a rejected by Δ if $\boxed{L_a}(s_0) = 0$

Δ halts at a if it either accepts a or rejects a

$$\boxed{L(\Delta) = \text{the set of all } a \in X \text{ accepted by } \Delta}$$

Thm (Kozen-Mota) $A \subseteq X$

1. A is coanalytic iff $A = L(\Delta)$ for some Δ
   $X \setminus A$ analytic
2. A is Borel iff $A = L(\Delta)$ for some Δ that hats on all inputs.

Spaces of Trees

$T \subseteq \mathbb{N}^{<\mathbb{N}}$

$T \in 2^{\underbrace{\mathbb{N}^{<\mathbb{N}}}_{}}$

$Tr = \{T \in 2^{\mathbb{N}^{<\mathbb{N}}} \mid T \text{ is a tree}\}$

closed $\subseteq \boxed{2^{\mathbb{N}^{<\mathbb{N}}}}$

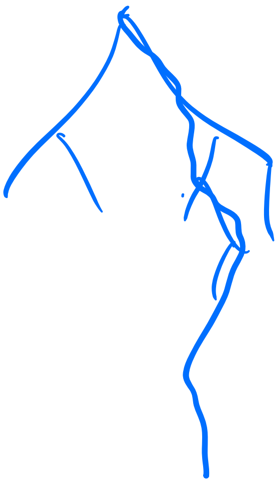
$\{T \in Tr \mid T \text{ is } \underbrace{\text{illfounded}}\}$

<u>analytic</u>

<u>not Borel</u>

$\exists \alpha \in \mathbb{N}^{\mathbb{N}} \; \forall n$

$\alpha \restriction n \in T$



X a set, $\mathcal{S}$ a σ-algebra on X
$A \subseteq X$

An <u>$\mathcal{S}$-cover</u> of A is $\hat{A} \in \mathcal{S}$
s.t.
(1) $A \subseteq \hat{A}$
(2) if $A \subseteq B \in \mathcal{S}$, then
all subsets of $\hat{A} \setminus B$ are in $\mathcal{S}$

$\mathcal{S}$ <u>admits covers</u> if all sets have $\mathcal{S}$-covers

<u>Thm</u> (Szpilrajn-Marczewski)
If $\mathcal{S}$ admits covers, then $\mathcal{S}$ is closed under operation $\mathcal{A}$.