On the Approximation by Conjugation method

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Interactions between Descriptive Set Theory and Smooth Dynamics
March 31, 2022

Anti-classification results in Smooth Ergodic Theory

Smooth Ergodic Theory

Another important question dating back to the foundational paper of von Neumann (1932):

ZUR OPERATORENMETHODE IN DER KLASSISCHEN MECHANIK[1].

VON J. V. NEUMANN, PRINCETON.

590 J. v. NEUMANN.

morphieinvarianten Eigenschaften. Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden[13], vielleicht sogar eine stetig-differentiierbare, oder gar eine mechanische. Dies mag es rechtfertigen, daß hier an Stelle der eigentlich interessanten mechanischen Strömungen alle allgemeinen untersucht werden.

[13] Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben.

Smooth realization problem

FIVE MOST RESISTANT PROBLEMS
IN DYNAMICS

A. Katok

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Are there smooth versions to the objects and concepts of abstract ergodic theory?

By a smooth version we mean a C^∞-diffeomorphism of a compact manifold preserving a C^∞-measure equivalent to the volume element that is measure-isomorphic to a given measure-preserving transformation.

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- Existence of volume-preserving diffeomorphisms with ergodic properties?
- What ergodic properties, if any, are imposed upon a dynamical system by the fact that it should be smooth?

Smooth realization problem

Known restrictions:

- M smooth compact manifold, $T \in \mathrm{Diff}^\infty(M, \mu)$. Then: $h_\mu(T) < \infty$. (Kushnirenko 1965)
- In case of $M = \mathbb{S}^1$: Any diffeomorphism with invariant smooth measure is conjugated to a rotation
- In dimension $d = 2$: Weakly mixing diffeomorphisms of positive measure entropy are Bernoulli (Pesin 1977)
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Realization of ergodic properties on a case-by-case basis

- area-preserving ergodic C^{∞}-diffeomorphisms of $\mathbb{D}^2$ (Anosov-Katok 1970)

Anti-classification result for C^∞-diffeos

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In a recent series of papers Foreman and Weiss extended their anti-classification result to the C^∞-setting:

Theorem (Foreman-Weiss)

Let M be either the torus $\mathbb{T}^2$, the disk $\mathbb{D}^2$ or the annulus $\mathbb{S}^1 \times [0, 1]$. Then the measure isomorphism relation among pairs (S, T) of area-preserving ergodic C^∞-diffeomorphisms of M is complete analytic and hence not Borel.

von Neumann's classification problem is impossible even when restricting to smooth diffeomorphisms

Odometer-based systems

The ergodic transformations constructed in Foreman-Rudolph-Weiss and Gerber-K are so-called *odometer-based systems*.

Definition: Odometer-based systems

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers $k_n \geq 2$. Let $(\mathtt{W}_n)_{n\in\mathbb{N}}$ be a uniquely readable construction sequence with $\mathtt{W}_0 = \Sigma$ and $\mathtt{W}_{n+1} \subseteq (\mathtt{W}_n)^{k_n}$ for every $n \in \mathbb{N}$. The associated symbolic shift will be called an *odometer-based system*.

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Mathematically:

- Let $\mathcal{O} = \prod_{n\in\mathbb{N}} \mathbb{Z}/k_n\mathbb{Z}$
- Then $\mathcal{O}$ has a natural product measure that is preserved by "adding one and carrying right"

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A measure-preserving transformation has an odometer factor if and only if it is isomorphic to an odometer-based symbolic system.

The odometer obstacle

Smooth realization of transformations with a non-trivial odometer factor is an open problem.

46 BASSAM FAYAD, ANATOLE KATOK

PROBLEM 7.10. Find a smooth realization of:

(1) a Gaussian dynamical system with simple (Kronecker) spectrum;
(2) a dense G_δ set of minimal interval exchange transformations;
✗ (3) an adding machine;
(4) the time-one map of the horocycle flow 2.3.1 on the modular surface $SO(2)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ (which is not compact, so the standard realization cannot be used).

B. Fayad, A. Katok
Constructions in elliptic dynamics
ETDS 24 (2004), 1477-1520.

Approximation by Conjugation-method: Setting

Let M be a smooth compact connected manifold of dimension $d \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ preserving a smooth volume μ,
e.g. torus $\mathbb{T}^2$, annulus $\mathbb{S}^1 \times [0,1]$ or disc $\mathbb{D}^2$ with standard circle action comprising of the diffeomorphisms $S_t(\theta, r) = (\theta + t, r)$.

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- We construct a sequence of measure-preserving diffeomorphisms

$$T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1},$$

where
$\alpha_n = \frac{p_n}{q_n} \in \mathbb{Q}$ with p_n, q_n relatively prime,
$H_n = h_1 \circ h_2 \circ ... \circ h_n$ with h_i measure-preserving diffeomorphism of M.

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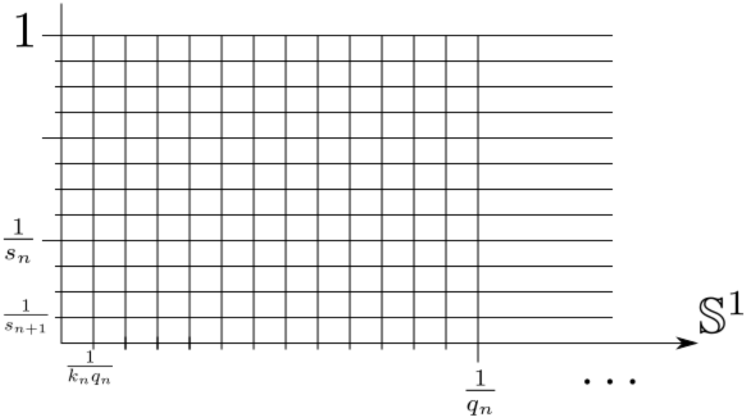
  $H_n = h_1 \circ h_2 \circ ... \circ h_n$ with h_i measure-preserving diffeomorphism of M.
- We need a criterion for the aimed property expressed on the level of the maps T_n and appropriate partitions of the manifold.

Combinatorial picture for h_{n+1}



Permutation of rectangles $\rightsquigarrow$ Realization as area-preserving diffeomorphism

Scheme

Construction of $T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1}$:

- Initial step: Choose $\alpha_0 = \frac{p_0}{q_0}$ arbitrary, $T_0 = S_{\alpha_0}$.

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  The conjugation map h_{n+1} and the parameter k_n are chosen such that $h_{n+1} \circ S_{\alpha_n} = S_{\alpha_n} \circ h_{n+1}$ and T_{n+1} imitates the desired property with a certain precision.

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  Then the parameter l_n is chosen large enough to guarantee closeness of T_{n+1} to T_n in the C^∞-topology:

$$\begin{aligned} T_{n+1} &= H_{n+1} \circ S_{\alpha_{n+1}} \circ H_{n+1}^{-1} \\ &= H_n \circ h_{n+1} \circ S_{\alpha_n} \circ S_{\frac{1}{l_n \cdot k_n \cdot q_n^2}} \circ h_{n+1}^{-1} \circ H_n^{-1} \\ &= H_n \circ S_{\alpha_n} \circ h_{n+1} \circ S_{\frac{1}{l_n \cdot k_n \cdot q_n^2}} \circ h_{n+1}^{-1} \circ H_n^{-1} \approx H_n \circ S_{\alpha_n} \circ H_n^{-1} = T_n \end{aligned}$$

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$\Longrightarrow$ Convergence of the sequence $(T_n)_{n \in \mathbb{N}}$ to a limit diffeomorphism with the aimed properties

Some C^∞ realization results

- Nonstandard smooth realizations: There exists an ergodic $f \in \text{Diff}^\infty(M, \mu)$ measure-theoretically isomorphic to a circle rotation (Anosov-Katok 1970)
- Minimal but not uniquely ergodic diffeomorphisms (Windsor 2001)
- Weakly mixing diffeomorphisms of the disc with prescribed Liouville rotation number on the boundary (Fayad-Saprykina 2005)
- Volume-preserving diffeomorphisms with ergodic derivative extension (K2020)

Circular systems

Symbolic representation of untwisted AbC-diffeomorphisms: circular systems.

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A *circular coefficient sequence* is a sequence of pairs of integers $(k_n, l_n)_{n\in\mathbb{N}}$ such that $k_n \geq 2$ and $\sum_{n\in\mathbb{N}} \frac{1}{l_n} < \infty$.

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Symbolic representation of untwisted AbC-diffeomorphisms: circular systems.
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- Set $\mathcal{W}_0 = \Sigma$.
- Having built $\mathcal{W}_n$ we choose a set $P_{n+1} \subseteq (\mathcal{W}_n)^{k_n}$ of so-called *prewords* and form $\mathcal{W}_{n+1}$ by taking all words of the form

$$\mathcal{C}_n\left(w_0, w_1, \dots, w_{k_n-1}\right) = \prod_{i=0}^{q_n-1} \prod_{j=0}^{k_n-1} \left(b^{q_n-j_i} w_j^{l_n-1} e^{j_i}\right)$$

  with $w_0 \dots w_{k_n-1} \in P_{n+1}$. If $n = 0$ we take $j_0 = 0$, and for $n > 0$ we let $j_i \in \{0, \dots, q_n - 1\}$ be such that

$$j_i \equiv (p_n)^{-1}\, i \mod q_n.$$

  We note that each word in $\mathcal{W}_{n+1}$ has length $q_{n+1} = k_n l_n q_n^2$.

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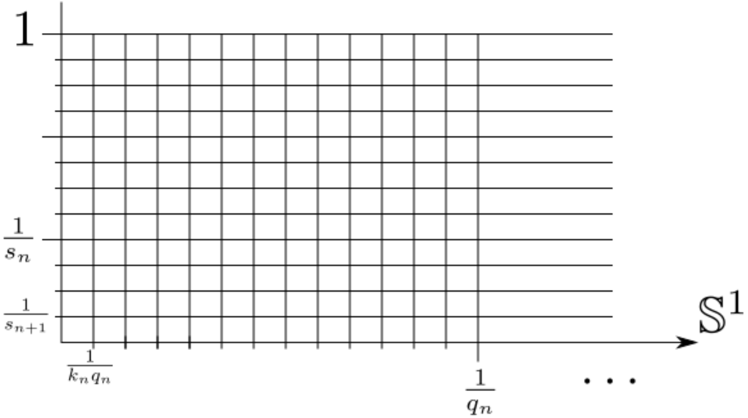


Recall $T_{n+1} = H_n \circ h_{n+1} \circ S_{\alpha_{n+1}} \circ h_{n+1}^{-1} \circ H_n^{-1}$ with $\alpha_{n+1} = \alpha_n + \frac{1}{k_n l_n q_n^2}$.

Circular systems

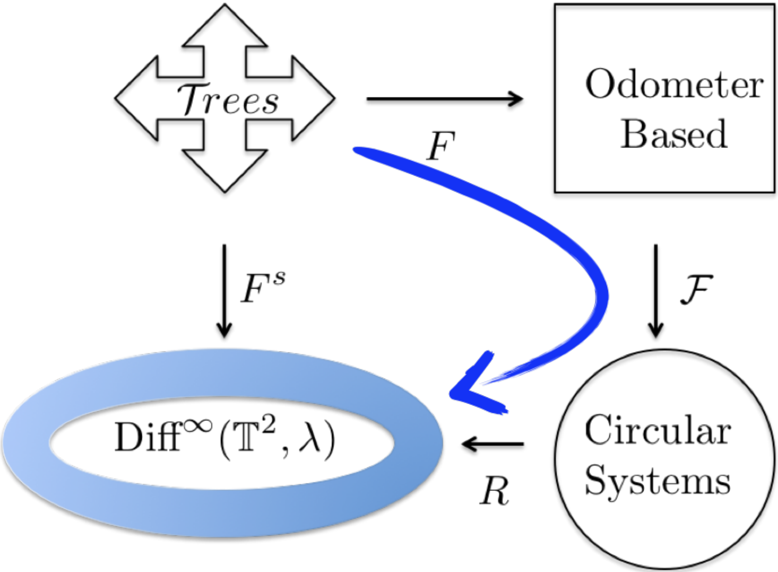
A construction sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ will be called *circular* if it is built in this manner using the $\mathcal{C}$-operators, a circular coefficient sequence and each P_{n+1} is uniquely readable in the alphabet with the words from $\mathcal{W}_n$ as letters.

Circular system

A symbolic shift $\mathbb{K}^c$ built from a circular construction sequence is called a *circular system*.

realizable as smooth diffeomorphisms using the untwisted AbC method

Overview


Trees
Odometer Based
F
F^s
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$\text{Diff}^\infty(\mathbb{T}^2, \lambda)$
R
Circular Systems

Functor between $\mathcal{OB}$ and $\mathcal{CB}$

Let Σ be an alphabet and $(\mathtt{W}_n)_{n\in\mathbb{N}}$ be a construction sequence for an odometer-based system with coefficients $(k_n)_{n\in\mathbb{N}}$. Then we define a circular construction sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ and bijections $c_n : \mathtt{W}_n \to \mathcal{W}_n$ by induction:

- Let $\mathcal{W}_0 = \Sigma$ and c_0 be the identity map.
- Suppose that $\mathtt{W}_n$, $\mathcal{W}_n$ and c_n have already been defined. Then we define

  $$\mathcal{W}_{n+1} = \{\mathcal{C}_n\left(c_n\left(\mathtt{w}_0\right), c_n\left(\mathtt{w}_1\right), \dots, c_n\left(\mathtt{w}_{k_n-1}\right)\right) \; : \; \mathtt{w}_0\mathtt{w}_1\dots\mathtt{w}_{k_n-1} \in \mathtt{W}_{n+1}\}$$

  and the map c_{n+1} by setting

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  In particular, the prewords are

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Functor $\mathcal{F}$

Suppose that $\mathbb{K}$ is built from a construction sequence $(\mathtt{W}_n)_{n\in\mathbb{N}}$ and $\mathbb{K}^c$ has the circular construction sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ as constructed above. Then we define a map $\mathcal{F}$ by

$$\mathcal{F}\left(\mathbb{K}\right) = \mathbb{K}^c.$$

Properties of the functor

Theorem (Foreman-Weiss 2019)

The functor $\mathcal{F}$ preserves

- weakly mixing extensions,
- compact extensions,
- factor maps,
- certain types of isomorphisms,
- the rank-one property,
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M. Foreman and B. Weiss
From Odometers to Circular Systems: A Global Structure Theorem.
Journal of Modern Dynamics, 15: 345–423, 2019.

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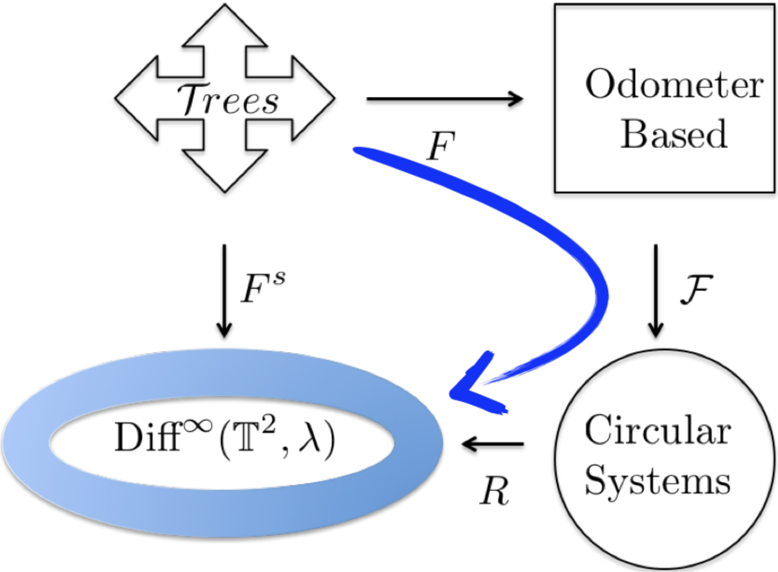
Warning (Gerber-K 2022)

The functor $\mathcal{F}$ does NOT preserve Kakutani equivalence.

Overview of the proof



Real-analytic topology

Real-analytic diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity have a lift of type

$$F(x_1, x_2) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1, x_2)),$$

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For any $\rho > 0$ we consider the set of real-analytic $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$, that can be extended to a holomorphic function on

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- $\mathrm{Diff}^\omega_\rho(\mathbb{T}^2, \mu)$: set of volume-preserving diffeomorphisms homotopic to the identity, whose lift satisfies $f_i \in C^\omega_\rho(\mathbb{T}^2)$ for $i = 1, 2$.

Anti-classification result for real-analytic diffeos

Theorem (Banerjee-K)

For every $\rho > 0$ the measure-isomorphism relation among pairs (S, T) of ergodic $\text{Diff}^{\omega}_{\rho}\left(\mathbb{T}^2, \mu\right)$-diffeomorphisms is a complete analytic set and hence not Borel.

von Neumann's classification problem is impossible even when restricting to real-analytic diffeomorphisms of the torus

(Anti-)classification results for circle maps

Maps of the circle

Some notation:

- Unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$
- Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the map $x \mapsto [x]$, where $[x]$ is the positive fractional part of x

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- For $k \in \mathbb{N} \cup \{\infty, \omega\}$ let $\mathcal{H}^k$ be the collection of orientation-preserving C^k diffeomorphisms of $\mathbb{S}^1$
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- $\mathcal{H}^{k+\beta}$: orientation-preserving C^k diffeomorphisms of $\mathbb{S}^1$ with β-Hölder continuous k-th derivative
- A *lift* of $f \in \mathcal{H}$ is an increasing function $F : \mathbb{R} \to \mathbb{R}$ with $[F(x)] = f([x])$.

Topological Conjugacy

A well-studied equivalence relation on $\mathcal{H}$ is conjugacy by an orientation-preserving homeomorphism.

Definition (Topological Conjugacy)

Maps $f, g \in \mathcal{H}$ are *conjugate by an orientation-preserving homeomorphism* if there is $\varphi \in \mathcal{H}$ such that

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Smale proposed using Topological Conjugacy to study the *qualitative behavior* of dynamical systems.

Smale's program

Classify systems up to Topological Conjugacy.

An invariant: Rotation number

Definition (Rotation number)

Let $f \in \mathcal{H}$ and F be a lift of f. Define

$$\tau(F) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$

Then $\tau(f) := [\tau(F)]$ is called the *rotation number* of f.

Some properties:

- $\tau(f)$ exists and is independent of x.
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Is the rotation number a *complete numerical invariant* for Topological Conjugacy?

An invariant: Rotation number

Theorem (Poincaré 1885)

Let $f \in \mathcal{H}$ have an irrational rotation number. If f has a dense orbit, then $f \sim R_{\tau(f)}$.

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Theorem (Denjoy 1932)

If $f \in \mathcal{H}^2$ has an irrational rotation number, then f is transitive and hence $f \sim R_{\tau(f)}$.

Altogether: Irrational rotation numbers are complete invariants for C^2-diffeomorphisms up to Topological conjugacy.

And for smooth conjugacy?

Question

Are there complete numerical invariants for orientation-preserving diffeomorphisms of the circle up to conjugation by orientation-preserving diffeomorphisms?

Some positive results

A number α is called Diophantine of class $D(\nu)$ for $\nu \geq 0$, if there exists $C > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C}{q^{2+\nu}} \quad \text{for every } p \in \mathbb{Z} \text{ and } q \in \mathbb{N}.$$

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An irrational number α is called *Liouville* if it is not Diophantine, that is, for every $C > 0$ and every $n \in \mathbb{N}$ there are infinitely many pairs $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that

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Theorem (Herman 1979)

If $f \in \mathcal{H}^\infty$ (respectively, $f \in \mathcal{H}^\omega$) has a Diophantine rotation number α, then f is C^∞-conjugate (respectively, C^ω-conjugate) to R_α.

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Theorem (Yoccoz 1984, Katznelson-Ornstein 1989)

If $f \in \mathcal{H}^k$ has a rotation number α in $D(\nu)$ with $k > \nu + 2$, then f is $C^{k-1-\nu-\varepsilon}$-conjugate to R_α for every $\varepsilon > 0$.

Some negative results

Theorem (Arnold 1961)

There exists $f \in \mathcal{H}^\omega$ such that $f = \varphi \circ R_{\tau(f)} \circ \varphi^{-1}$ with a nondifferentiable homeomorphism φ.

Inductive construction within the family of circle diffeomorphisms induced by

$$F_\alpha(x) = x + \alpha + \mu \sin(2\pi x)$$

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Conjugacies of intermediate regularity (Hasselblatt-Katok 1995)

There are examples of $f \in \mathcal{H}^{\infty}$ conjugate to some irrational rotation R_{α} via a conjugacy φ with any one of the following properties:

- φ is singular
- φ is absolutely continuous, but not Lipschitz continuous
- φ is C^k, but not C^{k+1}, where $k \in \mathbb{N}$ is arbitrary.

Matsumoto 2011&2012, K 2018: Conjugacies of other intermediate regularity for prescribed Liouville rotation number.

No complete numerical invariants

Theorem (K)

Let $\mathcal{C}$ be the collection of circle homeomorphisms with regularity (D), where (D) could be any degree of regularity from Hölder to C^{∞}.
Then there is no complete numerical invariant for $\mathcal{C}$-conjugacy of orientation-preserving C^{∞} diffeomorphisms of the circle.

Reduction

The main tool is the idea of a reduction for equivalence relations.

Definition (Reduction)

Let X and Y be Polish spaces (i.e. separable completely metrizable topological spaces) and $E \subseteq X \times X$, $F \subseteq Y \times Y$ be equivalence relations.

A function $f : X \to Y$ *reduces* E to F
if and only if
for all $x_1, x_2 \in X$: $x_1 E x_2$ if and only if $f(x_1) F f(x_2)$.

Such a function f is called a Borel (respectively, continuous) reduction if f is a Borel (respectively, continuous) function.
We write $E \precsim_{\mathcal{B}} F$ (respectively, $E \precsim_{C} F$)

"F is at least as complicated as E"

Equality equivalence relation

For complete numerical invariants:

Equality equivalence relation

For a Polish space Y we let $=_Y \subseteq Y \times Y$ be the equality equivalence relation.

If Y is a Polish space, then there is a Borel injection $g : Y \to \mathbb{R} \setminus \mathbb{Q}$. Let f be a Borel reduction of any equivalence relation $E \subseteq X \times X$ to $(Y, =_Y)$. Then $g \circ f$ is a Borel reduction of (X, E) to $(\mathbb{R}, =_R)$.
Thus we can assume that Borel reductions to any $=_Y$ can be changed to Borel reductions to equality on the real numbers.

Equivalence relation E_0

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Let E_0 be the equivalence relation on $\{0,1\}^{\mathbb{N}}$ defined by setting

$$\mathbf{a}E_0\mathbf{b} \text{ if and only if there is } N \in \mathbb{N} \text{ such that } a_m = b_m \text{ for all } m > N.$$

for $\mathbf{a} = (a_n)_{n\in\mathbb{N}}, \mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$.

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We can use this to exclude the existence of complete numerical invariants:

Fact

Suppose that E is an equivalence relation on an uncountable Polish space X and $E_0 \precsim_{\mathcal{B}} E$. Then $E \not\precsim_{\mathcal{B}} =_{\mathbb{R}}$.

Ideas of proof

Let $\mathcal{H}_\alpha^\infty$ be the collection of orientation-preserving C^∞-diffeomorphisms with rotation number $\alpha \in \mathbb{S}^1$.

Proposition

Let $\alpha \in \mathbb{S}^1$ be a Liouville number. There is a continuous one-to-one map

$$\Psi : \{0,1\}^{\mathbb{N}} \to \mathcal{H}_\alpha^\infty$$

such that for any two sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$ the following properties hold:

1. If there is $N \in \mathbb{N}$ such that $a_n = b_n$ for every $n \geq N$, then the C^∞-diffeomorphisms $\Psi(\mathbf{a})$ and $\Psi(\mathbf{b})$ are C^∞-conjugate.
2. If there are infinitely many $n \in \mathbb{N}$ with $a_n \neq b_n$, then the C^∞-diffeomorphisms $\Psi(\mathbf{a})$ and $\Psi(\mathbf{b})$ are not Hölder-conjugate.

Ideas of proof

Using the notions from Descriptive Set Theory:

Corollary

Let $\mathcal{C}$ be the collection of circle homeomorphisms with regularity (D), where (D) could be any degree of regularity from Hölder to C^∞.
Then there is a continuous reduction from E_0 to the $\mathcal{C}$-conjugacy relation of orientation-preserving C^∞-diffeomorphisms of the circle.

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Let $\mathcal{C}$ be the collection of circle homeomorphisms with regularity (D), where (D) could be any degree of regularity from Hölder to C^{∞}.
Then there is a continuous reduction from E_0 to the $\mathcal{C}$-conjugacy relation of orientation-preserving C^{∞}-diffeomorphisms of the circle.

Hence, there is no complete numerical invariant for $\mathcal{C}$-conjugacy of orientation-preserving C^{∞} diffeomorphisms of the circle.

Ideas of proof: Building Ψ

Inductive construction of $T_{\mathbf{a}} := \Psi(\mathbf{a}) \in \mathcal{H}_{\alpha}^{\infty}$ via the AbC method:

$$T_{\mathbf{a},n} = H_{\mathbf{a},n} \circ R_{\alpha_{n+1}} \circ H_{\mathbf{a},n}^{-1}$$

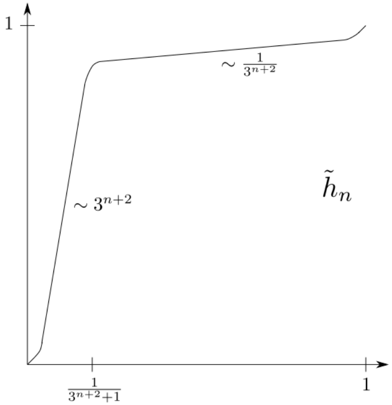
with conjugation maps

$$H_{\mathbf{a},n} = H_{\mathbf{a},n-1} \circ h_{\mathbf{a},n}$$

with C^{∞}-diffeomorphism $h_{\mathbf{a},n}$ satisfying

$$h_{\mathbf{a},n} \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ h_{\mathbf{a},n}.$$

Ideas of proof: Conjugation map $h_{\mathbf{a},n}$



Let h_{q_n} be the q_n-fold lift of $\tilde{h}_n$. Then

$$h_{\mathbf{a},n} = \begin{cases} h_{q_n} & \text{if } a_n = 0, \\ h_{q_n}^{-1} & \text{if } a_n = 1. \end{cases}$$

Ideas of proof: Convergence of $(T_{\mathbf{a},n})_n$

Note:

$$\left\| h_{\mathbf{a},n} \right\|_r \leq C_n q_n^r$$

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Then:

$$\begin{aligned}
& d_n(T_{\mathbf{a},n}, T_{\mathbf{a},n-1}) \\
=& \left\| H_{\mathbf{a},n} \circ R_{\alpha_{n+1}} \circ H_{\mathbf{a},n}^{-1} - H_{\mathbf{a},n-1} \circ R_{\alpha_n} \circ H_{\mathbf{a},n-1}^{-1} \right\|_n \\
=& \left\| H_{\mathbf{a},n} \circ R_{\alpha_{n+1}} \circ H_{\mathbf{a},n}^{-1} - H_{\mathbf{a},n-1} \circ R_{\alpha_n} \circ h_{\mathbf{a},n} \circ h_{\mathbf{a},n}^{-1} \circ H_{\mathbf{a},n-1}^{-1} \right\|_n \\
=& \left\| H_{\mathbf{a},n} \circ R_{\alpha_{n+1}} \circ H_{\mathbf{a},n}^{-1} - H_{\mathbf{a},n-1} \circ h_{\mathbf{a},n} \circ R_{\alpha_n} \circ h_{\mathbf{a},n}^{-1} \circ H_{\mathbf{a},n-1}^{-1} \right\|_n \\
=& \left\| H_{\mathbf{a},n} \circ R_{\alpha_{n+1}} \circ H_{\mathbf{a},n}^{-1} - H_{\mathbf{a},n} \circ R_{\alpha_n} \circ H_{\mathbf{a},n}^{-1} \right\|_n \\
\leq& C_n \cdot \|H_n\|_{n+1}^{n+1} \cdot \|R_{\alpha_{n+1}} - R_{\alpha_n}\|_n \\
\leq& C_n \cdot q_n^{(n+1)^2} \cdot |\alpha_{n+1} - \alpha_n| \\
\leq& C_n \cdot q_n^{(n+1)^2} \cdot 2 \cdot |\alpha - \alpha_n|,
\end{aligned}$$

which can be made small since α is Liouville.

Ideas of proof

We consider $T_{\mathbf{a},n} \to T_{\mathbf{a}} = H_{\mathbf{a}} \circ R_{\alpha} \circ H_{\mathbf{a}}^{-1}$ and $T_{\mathbf{b},n} \to T_{\mathbf{b}} = H_{\mathbf{b}} \circ R_{\alpha} \circ H_{\mathbf{b}}^{-1}$.

The conjugation maps $H_{\mathbf{b},n} H_{\mathbf{a},n}^{-1} \to H_{\mathbf{b}} H_{\mathbf{a}}^{-1}$ in $\mathcal{H}$.

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- If $\mathbf{a} E_0 \mathbf{b}$, then there is $N \in \mathbb{N}$ such that $a_n = b_n$ for all $n > N$. Hence:

$$\begin{aligned} H_{\mathbf{b},n} H_{\mathbf{a},n}^{-1} &= H_{\mathbf{b},N} \circ h_{\mathbf{b},N+1} \circ \cdots \circ h_{\mathbf{b},n} \circ h_{\mathbf{a},n}^{-1} \circ \cdots \circ h_{\mathbf{a},N+1}^{-1} \circ H_{\mathbf{a},N}^{-1} \\ &= H_{\mathbf{b},N} H_{\mathbf{a},N}^{-1} \in \mathcal{H}^\infty . \end{aligned}$$

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- Otherwise: $H_{\mathbf{b},n} H_{\mathbf{a},n}^{-1}$ does not converge in any Hölder space by construction of $\tilde{h}_n$.

Thank you very much for your attention!