On Some Open Questions Related to the Log-Approximate-Rank Conjecture

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Background: The Log-Approximate-Rank Conjecture

$$\mathrm{R}^{\mathrm{cc}}_{\epsilon}(F)$$

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There is a communication protocol for F of cost $O(\mathrm{rank}_\epsilon(F))$. [Gál and Syed '19]

The Log-Approximate-Rank Conjecture

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Refuting The Log-Approximate-Rank Conjecture

[Chattopadhyay Mande S ‘19]

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Refuting The Log-Approximate-Rank Conjecture

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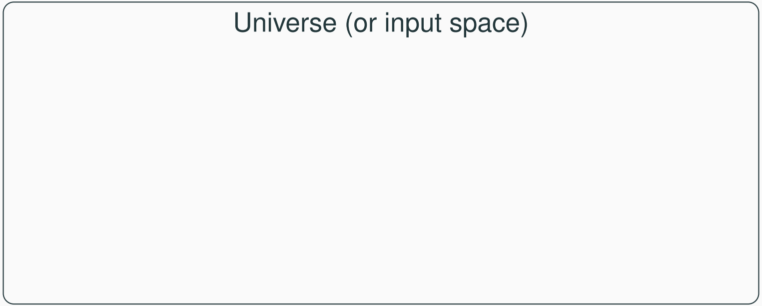
2. Can we refute $\mathrm{R}^{\mathrm{cc}}_\epsilon(F) \leq \log\left(\max\{\operatorname{rank}^+_\epsilon(F), \operatorname{rank}^+_\epsilon(\neg F)\}\right)^{O(1)}$?
[Kol Moran Shpilka Yehudayoff '14]

Functions with small Approximate Rank



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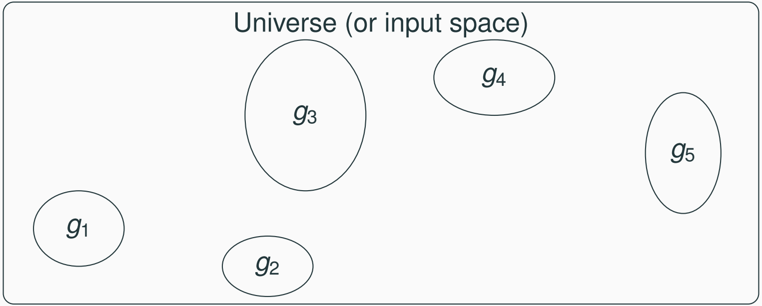

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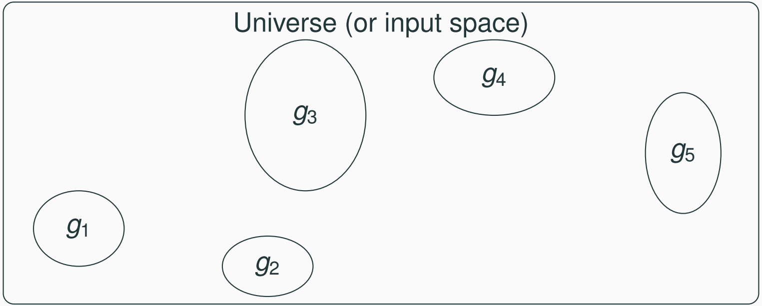

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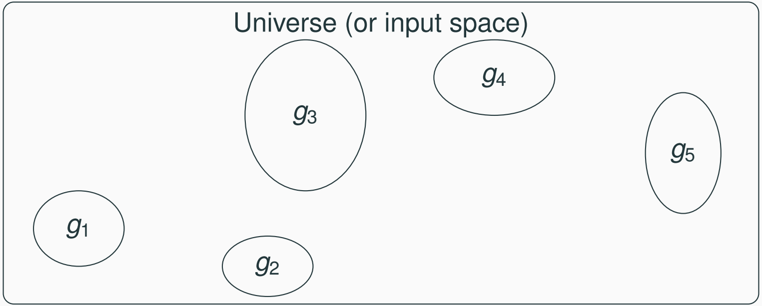

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Log-Approximate-Rank Conjecture implies cost of computing f is $\mathrm{poly}(\log t,$ cost of computing $g)$.

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Can't we be forced to compute each g_i, resulting in $\Omega(t)$ cost?

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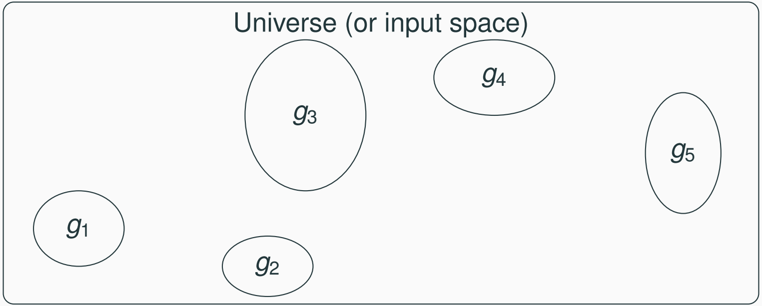

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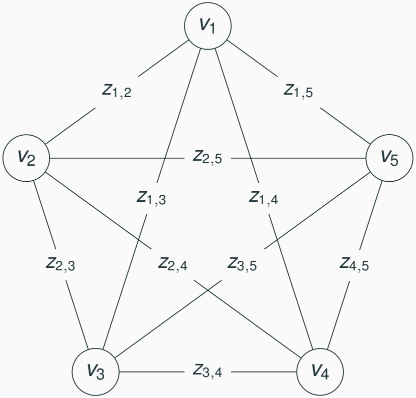
2. Sanity check: Show that these are hard for randomized parity decision trees. (Can make linear queries of the form $\langle v, x\rangle$ thinking of $x \in \mathbb{F}_2^n$.)
3. Show that $f \circ \mathrm{XOR}$ is hard for randomized communication protocols.

The counterexample

$$\text{SINK} : \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$



The input bits of SINK orient the edges of the complete graph.

$\text{SINK}(z) = 1$ iff there is a sink in the directed graph G_z.

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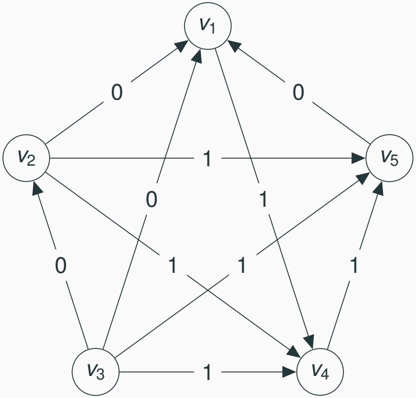

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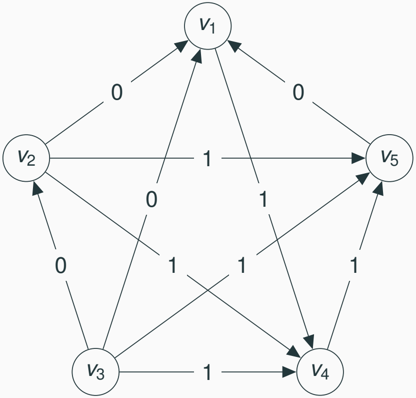

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SINK $= \sum_{i \in [n]}$ SINK$_i$. Each SINK$_i$ is a subcube.

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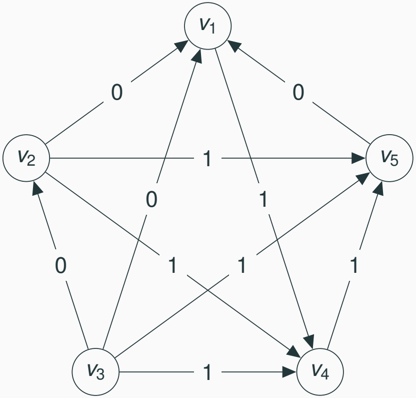

A subspace/rectangle A that is biased against v_i being a sink must have a slightly small $A|_i$.

A subspace/rectangle A that is biased against inputs with sinks must be slightly small for many $A|_i$s.

A subspace/rectangle A that is biased against inputs with sinks must be very small. (Shearer's Lemma)

Doing Better Than $\sqrt[4]{\mathrm{rank}_\epsilon(F)}$

with Arkadev Chattopadhyay and Ankit Garg

Dual Subspace Designs

Subspace Designs [Guruswami Xing ‘12]:

A set of subspaces such that any small subspace intersects only a few of them non-trivially.

$S_1, S_2, \ldots, S_k \subset \mathbb{F}_2^n$ such that
$\forall W \subset \mathbb{F}_2^n$ with $\dim(W) < a$,
$W \cap S_i \neq \{0\}$ for only h values of i.

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Randomized Parity Decision Tree lower bound follows immediately.

Open Question 1

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Do subspace designs have an analog of Shearer's lemma?

Open Question 1.1

Our Conjecture

If a distribution A over $\{0,1\}^n$ satisfies $H(A|_T) \leq \mathrm{codim}(T) - \Omega(1)$
for many subspaces T from a subspace design,
then $H(A) < n - \Omega(n)$.

A concrete problem: CyclicShift

$$f : \{0,1\}^{n+n} \to \{0,1\}$$

$$f : (x, y) \mapsto 1 \text{ iff } \exists i \in [n] \text{ such that } x_{\to i} = y$$

$$\left((a_1 a_2 a_3 a_4 a_5)_{\to 2} = (a_4 a_5 a_1 a_2 a_3) \right)$$

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(Function inspired by a function from 'String Matching: Communication, Circuits, and Learning', by Golovnev, Göös, Reichman and Shinkar '19)

Approximate Nonnegative Rank

with Arkadev Chattopadhyay

The issue

The function SINK was a sum of simple functions.

¬SINK was not.

Rectifying it in 3 simple steps

- Partition $\{0,1\}^n$ into two sets so that both parts are unions of a few disjoint subcubes.

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If $f^{-1}(0)$ and $f^{-1}(1)$ can be covered by c monochromatic subcubes, there is a size-$2^{\mathrm{polylog}(c,n)}$ decision tree computing f. [Ehrenfeucht and Haussler ‘89]

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Conjecture:
For any partition of $\{0,1\}^n$ into subspaces, there is a tree-like partition of $\{0,1\}^n$ that refines it without having too many more parts.

Thank you. ~~I am now open to questions.~~ The questions are now open to you.