Lianna Hambardzumyan

joint with Hamed and Pooya Hatami based on [HHH'21]

DIMENSION-FREE RELATIONS IN COMMUNICATION COMPLEXITY

McGill University

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2022

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- In contrast to the **Log-rank conjecture:**

$$\exists C \quad \log rank(M) \leq D(M) \leq \left(\log rank(M)\right)^{C}$$

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Dimension-free bounds $\iff$ Structural result

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- Prove dimension-free bounds of form $\alpha(M) \leq \mathbf{f}(\beta(M)) \quad \forall M$

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Conjecture template

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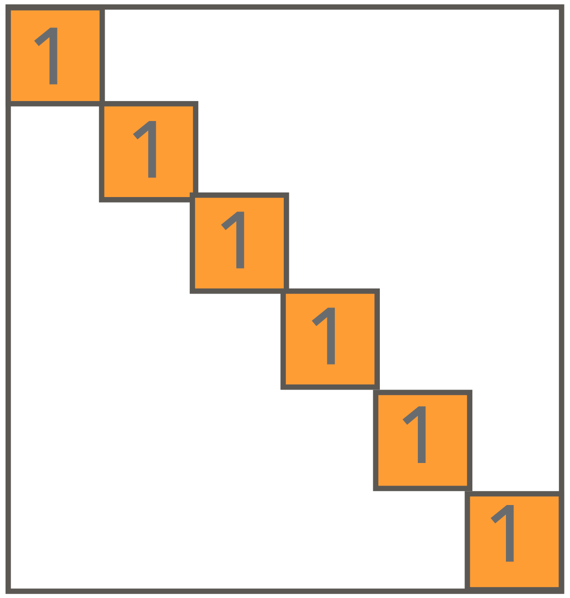

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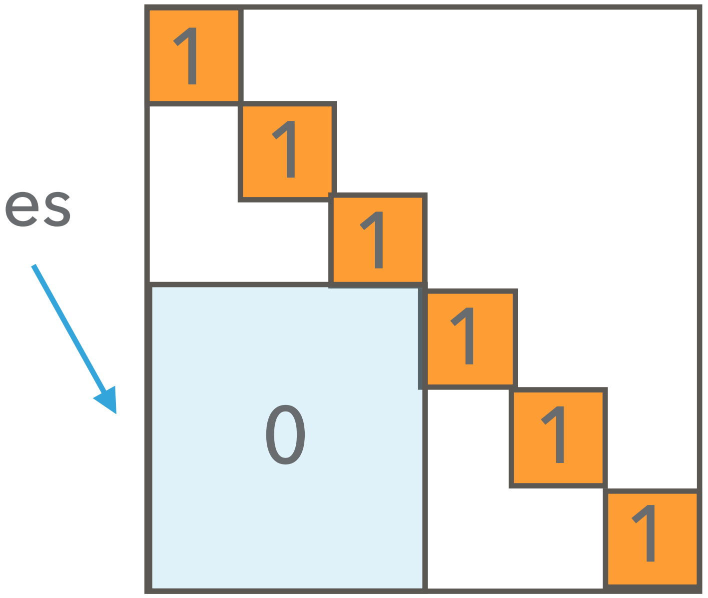

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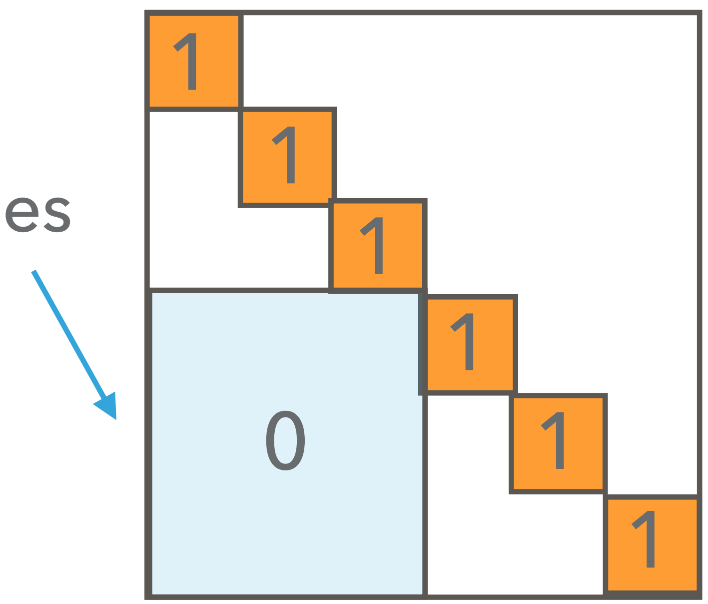

Question: $R(M) = O(1) \implies M$ has a large monochromatic rectangle?

CONJECTURE I

Conjecture I: For a Boolean matrix M of size $n \times n$, if $R_\epsilon(M) \leq c$ for some constant c, then M has a monochromatic rectangle of size $\delta_c n \times \delta_c n$, where δ_c is a constant depending on c.

CONJECTURE I: GENERAL

Conjecture [CLV19]: For a Boolean matrix M of size $n \times n$, if $R_\epsilon(M) \leq c(n)$ ~~for some constant c~~, then M has a monochromatic rectangle of size $\delta_c n \times \delta_c n$, where $\delta_c = 2^{-O(c(n))}$.

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  - If Conjecture I is false, then there is a separation between these classes.

CONJECTURE I: ONE-SIDED ERROR

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- This theorem covers all the matrices for which the randomized protocol with constant complexity uses *hashing technique*.
- In particular, this includes EQ, Hamming-Distance-d for constant d.

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Proof idea:

- Forbid a submatrix that is *hard* for one-sided error randomized protocol.

Recall: $R^1(EQ_k) = \Theta(k)$ (if the protocol doesn't make an error on 0's).

CONJECTURE I: ZERO ERROR

Theorem [**H**HH21]: Let M be a Boolean matrix of size $n \times n$. If $R_0(M) \leq c$, then M can be partitioned into δ_c monochromatic rectangles, where δ_c depends only on c.

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- Forbid a submatrix that is *hard* for two-sided error randomized protocol.

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Barrier theorem [HHH21]: For all sufficiently large n, there exists an $n \times n$ Boolean matrix M s.t.

(1) Every $n^{1/4} \times n^{1/4}$ submatrix F of M has $R_\epsilon(F) = O(1)$.

(2) M doesn't contain a monochromatic rectangle of size $n^{0.99} \times n^{0.99}$.

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- *[**H**HH21]*: Barrier theorem refuted the Probabilistic Universal Graph Conjecture of Harms, Wild, and Zamaraev [HWZ21].

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- *[**H**HH21]:* Barrier theorem refuted the Probabilistic Universal Graph Conjecture of Harms, Wild, and Zamaraev [HWZ21].
- *[HH21]:* Barrier theorem + counting argument refuted the Implicit Graph Conjecture [HWZ21].

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  - $\|M\|_{\circ,\epsilon} = \min_{M'}\{\|M'\|_{\circ} : \forall (x,y) \quad |M(x,y) - M'(x,y)| \le \epsilon$ and M' is real-valued$\}$

CONJECTURE I: TRACE NORM

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CONJECTURE I: TRACE NORM

$$R_\epsilon(M) \geq \log \frac{\|M\|_{tr,\epsilon}}{n}$$

M is $n \times n$-sized

Strong **Conjecture I:** $\frac{\|M\|_{tr,\epsilon}}{n} \leq c \implies M$ has a mon. rec. of size $\delta_c n \times \delta_c n$

Conjecture II: $\frac{\|M\|_{tr}}{n} \leq c \implies M$ has a mon. rec. of size $\delta_c n \times \delta_c n$

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Conjecture II: $\frac{\|M\|_{tr}}{n} \leq c \implies M$ has a mon. rec. of size $\delta_c n \times \delta_c n$

Theorem [**H**HH21]: Conjecture II holds for matrices of form $F(x, y) = f(y^{-1}x)$,

where $f : G \to \{0,1\}$ and G is any *finite* group.

CONJECTURE II: GRAPH THEORY

Conjecture II: $\frac{\|M\|_{tr}}{n} \leq c \implies M$ has a mon. rec. of size $\delta_c n \times \delta_c n$

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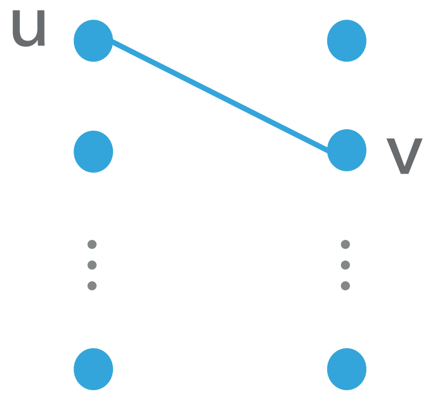
Matrix

Bipartite graph



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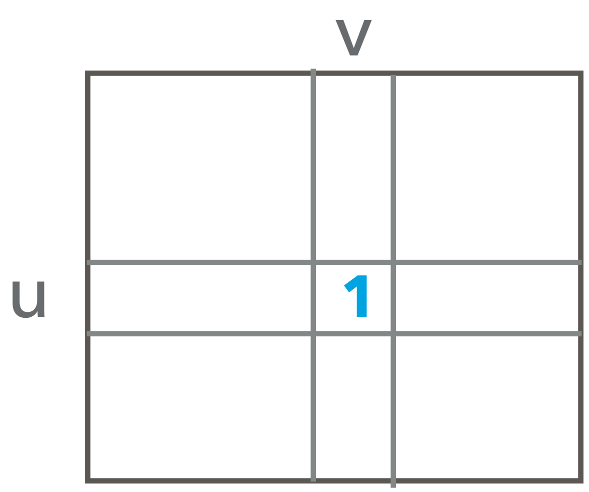

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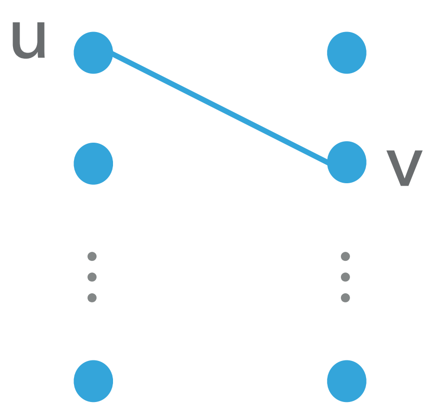

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Matrix

Bipartite graph

Trace norm

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Matrix		**Bipartite graph**
Trace norm	=	**Graph energy**

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Matrix		Bipartite graph
Trace norm		**Graph energy**
Monochromatic rectangle	=	**Complete bipartite subgraph***
has a large mon. rect.		

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Matrix		**Bipartite graph**
Trace norm		**Graph energy**
Monochromatic rectangle	=	**Complete bipartite subgraph***
has a large mon. rect.		satisfies **Strong Erdős-Hajnal property**

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Matrix		Bipartite graph
Trace norm		**Graph energy**
Monochromatic rectangle	=	**Complete bipartite subgraph***
has a large mon. rect.		satisfies **Strong Erdős-Hajnal property**

Conjecture II (graph theoretic)**:** If a bipartite graph has small graph energy, then it satisfies the Strong Erdős-Hajnal property.

CONJECTURE I: μ-NORM

Recall: $\|M\|_\mu = \min\left\{ \sum_i |\alpha_i| : M = \sum_i \alpha_i R_i \right\}$,

where R_i are rank-1 matrices and $\alpha_i \in \mathbb{R}$.

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$$R(\cdot) \longleftrightarrow \|\cdot\|_{\mu,\epsilon}$$

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Conjecture I (Equivalent): $\|M\|_{\mu,\not{\epsilon}} \leq c \implies M$ has a mon. rec. of size $\delta_c n \times \delta_c n$.

CONJECTURE III

Conjecture III [HHH21]: If $\|M\|_\mu \leq c \implies M$ can be written as a linear combination of k_c blocky matrices.

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Blow-up of identity matrix

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1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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Equivalent versions: $\|M\|_\mu = O(1) \overset{?}{\implies} br(M) = O(1)$

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Equivalent versions:

$$\|M\|_\mu = O(1)$$

$$\|M\|_\nu = O(1) \quad \stackrel{?}{\implies} \quad br(M) = O(1)$$

$$\|M\|_{\gamma_2} = O(1)$$

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
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0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

- Graph theory - *equivalence graphs*

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

- Graph theory - *equivalence graphs*
- Complexity theory - *fat matchings*

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

- Graph theory - *equivalence graphs*
- Complexity theory - *fat matchings*
- Operator theory - *contractive idempotents*

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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Answer [**H**HH21, HWZ21]**:** **Hypercube** [HHH21, HWZ21]

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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Answer [HHH21, HWZ21]: Hypercube [HHH21, HWZ21]

$$1/2 \log_2 br(M) \leq D^{EQ}(M) \leq br(M)$$

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

- Graph theory - *equivalence graphs*
- Complexity theory - *fat matchings*
- Operator theory - *contractive idempotents*

Blocky rank:

$br(M) = \min\left\{r : M = \sum_{i=1}^{r} \alpha_i B_i\right\}$, where B_i is a blocky matrix

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$$1/2 \log_2 br(M) \leq D^{EQ}(M) \leq br(M)$$

$$D^{EQ}(\,\cdot\,) \longleftrightarrow br(\,\cdot\,)$$

CONJECTURE III

Conjecture III [HHH21]: If $\|M\|_\mu \leq c \implies M$ can be written as a linear combination of k_c blocky matrices.

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Equivalent versions:

$$\|M\|_\mu = O(1)$$

$$\|M\|_\nu = O(1) \overset{?}{\implies} br(M) = O(1)$$

$$D^{EQ}(M) = O(1)$$

$$\|M\|_{\gamma_2} = O(1)$$

CONJECTURE III: BLOCKY MATRICES

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
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1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

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$$\Omega(\log_2 \|M\|_\mu) \leq br(M)$$

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Equivalent versions:

$$\|M\|_{\mu} = O(1)$$

$$\|M\|_{\nu} = O(1) \quad \overset{?}{\Longrightarrow} \quad br(M) = O(1)$$

$$\Longleftarrow \checkmark \quad D^{EQ}(M) = O(1)$$

$$\|M\|_{\gamma_2} = O(1)$$

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Equivalent versions:

$\|\cdot\|_\mu$

$\|\cdot\|_\nu$

$\|\cdot\|_{\gamma_2}$

$\overset{?}{\longleftrightarrow}$

$br(\cdot)$

$D^{EQ}(\cdot)$

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$\|\cdot\|_{\gamma_2}$

Theorem [HHH21]: Conjecture III holds for matrices of form $F(x, y) = f(y^{-1}x)$, where $f : G \to \{0,1\}$ and G is any *finite* group.

CONJECTURE III: OPERATOR THEORY

Conjecture III [**H**HH21]: If $\|M\|_\mu \leq c \implies M$ can be written as a linear combination of k_c blocky matrices.

CONJECTURE III: OPERATOR THEORY

Conjecture III [HHH21]: If $\|M\|_\mu \leq c \implies M$ can be written as a linear combination of k_c blocky matrices.

Conjecture ★: The idempotent Schur multipliers are exactly those Boolean matrices that can be written as a linear combination of *finitely* many contractive idempotents.

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Matrix **Algebra of Schur multipliers**

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Matrix	**Algebra of Schur multipliers**
Boolean matrix	

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Matrix		**Algebra of Schur multipliers**
Boolean matrix	=	**Idempotent**

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Matrix		Algebra of Schur multipliers
Boolean matrix	=	Idempotent
Blocky matrix		

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Matrix		Algebra of Schur multipliers
Boolean matrix	=	Idempotent
Blocky matrix		Contractive Idempotent

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Matrix		**Algebra of Schur multipliers**
Boolean matrix	=	**Idempotent**
Blocky matrix		**Contractive Idempotent**

Theorem [HHH21]: Conjecture III is equivalent to Conjecture★.

ALICE AND BOB

Thank you!