Model-free portfolio theory: a rough path approach

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Model uncertainty in portfolio theory

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Portfolio theory:

Find "optimal" investment strategies on financial markets.

(initiated by H. Markowitz '59, see also de Finetti '40)

Optimal portfolios often require unobservable quantities:

$\rightsquigarrow$ error of statistical estimation

Optimal portfolios are highly sensitives to model misspecifications:

$\rightsquigarrow$ model risk (e.g. Chopra & Ziemba '93, DeMiguel, Garlappi & Uppal '07, ...).

Today: model-free portfolio theory in continuous time

$\rightsquigarrow$ no underlying probabilistic models (goes back to Hobson '98)

Model-free portfolio theory

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Two major approaches determining “model-free” optimal portfolios:

- Universal Portfolio Theory (initiated by T. Cover ’91)
  aims to find preference-free well preforming investment strategies.
- Stochastic Portfolio Theory (initiated by R. Fernholz ’99)
  aims to construct portfolios using only observable market quantities.

Traditionally, the constructed portfolios are analyzed in prob. models.

Model-free treatments in continuous time:

- Schied, Speiser & Voloshchenko ’18
- Cuchiero, Schachermayer & Wong ’19

$\rightsquigarrow$ requires pathwise integration

Portfolio theory using Föllmer integration

Setting (roughly speaking)

of Schied, Speiser & Voloshchenko '18 and Cuchiero, Schachermayer & Wong '19

Fix a seq. of partitions $(\mathcal{P}^n) \subset [0,\infty)$ s.t. $|\mathcal{P}^n| \to 0$ as $n \to \infty$.

A price path is a pair $(S, [S])$ with

$$S \in C([0,\infty); \mathbb{R}^d) \quad \text{and} \quad [S] \in C([0,\infty); \mathbb{R}^{d\times d})$$

such that the quadratic variation

$$[S]_t = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} (S_{t^n_{k+1}\wedge t} - S_{t^n_k \wedge t}) \otimes (S_{t^n_{k+1}\wedge t} - S_{t^n_k \wedge t})$$

exists along $(\mathcal{P}^n)$.

Note: All classical models in math finance possess quadratic variation.

Portfolio theory using Föllmer integration

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Föllmer '81: Assume $f \in C^2$, then

$$\int_0^T \mathrm{D}f(S_s)\,\mathrm{d}(S,[S])_s := \lim_{n\to\infty} \sum_{k=1}^{N_n} \mathrm{D}f(S_{t_k^n})(S_{t_{k+1}^n} - S_{t_k^n}).$$

This allows (Schied, Speiser & Voloshchenko '18 and Cuchiero, Schachermayer & Wong '19)

- to develop pathwise Universal and Stochastic Portfolio Theory
- for portfolios "generated" by gradients of functions.

Remark on Föllmer integration

- extensions of Föllmer integration:
  Würlmi '80,..., Cont & Fournié '10,...,
- applications to pathwise hedging:
  Bick & Willinger '94, Lyons '95,...,

Example: log-optimal portfolio

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Probabilistic model for the market portfolio

$$\mu_t = \mu_0 + \int_0^t c(\mu_s)\lambda(\mu_s)\,\mathrm{d}s + \int_0^t \sqrt{c(\mu_s)}\,\mathrm{d}W_s, \qquad t \in [0,\infty),$$

where W is a d-dimensional Brownian motion and $\mu_0 \in \Delta_+^d$.

Fix c, λ s.t. $\mu_t \in \Delta_+^d$ and "no unbounded profit with bounded risk" holds.

For a given $T > 0$, the log-optimal portfolio $\widehat{\pi}$ is the maximizer of

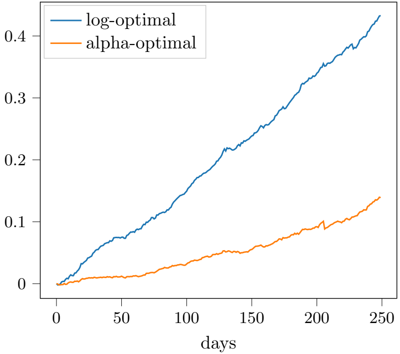
$$\sup_{\pi} \mathbb{E}[\log V_T^{\pi}] \quad \text{where } V_T^{\pi} \text{ is the relative wealth of } \pi.$$

The log-optimal portfolio $\widehat{\pi} = (\widehat{\pi}^1, \dots, \widehat{\pi}^d)$ is

$$\widehat{\pi}_t^i = \mu_t^i \Big(\lambda^i(\mu_t) + 1 - \sum_{j=1}^d \mu_t^j \lambda^j(\mu_t) \Big).$$

Example: log-optimal portfolio

Note: log-optimal portfolios are not "generated" by gradients of functions!



Expected utility of the log-optimal vs. the alpha-optimal portfolio

There are pathwise portfolios outperforming all gradient-type portfolios.

Financial market – Property (RIE)

Aim: Model-free portfolio theory allowing for more general portfolios

The price paths are all pairs $\mathbf{S} = (S, \mathbb{S})$ with $S \in C([0,\infty); \mathbb{R}^d)$ satisfying

Property (RIE)

Let $p \in (2,3)$ and $(\mathcal{P}^n)$ be a seq. of partitions s.t. $|\mathcal{P}^n| \to 0$ as $n \to \infty$.
S^n denotes the piecewise constant approximation of S along $\mathcal{P}^n$.
We assume that:

- $\int_0^t S_u^n \otimes \mathrm{d}S_u := \sum_{k=0}^{N_n-1} S_{t_k^n} \otimes (S_{t_k^n \wedge t} - S_{t_{k+1}^n \wedge t})$ *converge uniformly as $n \to \infty$ to a limit $\mathbb{S}$,*
- *there exists a control function c s.t.*

$$\sup_{s \neq t} \frac{|S_{s,t}|^p}{c(s,t)} + \sup_{n \in \mathbb{N}} \sup_{0 \leq k < \ell \leq N_n} \frac{\left| \int_{t_k^n}^{t_\ell^n} S_u^n \otimes \mathrm{d}S_u - S_{t_k^n} \otimes (S_{t_k^n} - S_{t_\ell^n}) \right|^{\frac{p}{2}}}{c(t_k^n, t_\ell^n)} \leq 1.$$

Financial market – Property (RIE)

Property (RIE) is satisfied by sample paths of

- semimartingales,
- Young semimartingales (e.g. fractional Black–Scholes models),
- typical price paths (in the sense of Vovk),

along partitions $\mathcal{P}^n := \{\tau_k^n \,:\, k \geq 0\}$ given by the stopping times

$$\tau_k^n := \inf\{t \geq \tau_{k-1}^n : |S_t - S_{\tau_{k-1}^n}| \geq 2^{-n}\} \quad \text{with} \quad \tau_0^n := 0.$$

Financial market – Property (RIE)

Note: (RIE) leads to a canonical rough path lift $\mathbf{S} = (S, \mathbb{S})$ of S.

Theorem (Allan, Cuchiero, Liu & P. '21)

Suppose that S satisfies Property (RIE) *with respect to $(\mathcal{P}^n)_{n\in\mathbb{N}}$.*

Let (Y, Y') be a controlled path with respect to S, and let $f \in C^{p+\varepsilon}$ for some $\varepsilon > 0$. Then the rough integral of (Y, Y') against $f(S)$ is given by

$$\int_0^t Y_u \,\mathrm{d}f(S)_u = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} Y_{t_k^n}\big(f(S_{t_{k+1}^n\wedge t}) - f(S_{t_k^n\wedge t})\big),$$

where the convergence is uniform in $t \in [0, T]$.

- E.g. $(Y, Y') = (g(S), \mathrm{D}g(S))$ is a controlled path, for $g \in C^2$.

Remarks

- Rough integral is defined for non-gradient-type integrands.
- Financial interpretation of rough integrals under (RIE).
  (The rough integral is defined as compensated Riemann sums.)
- Rough integration extends Föllmer integration.
  (Rough integral depend on the choice of partitions $(\mathcal{P}^n)$.)
- Full access to stability results and pathwise Itô formulae.

The wealth process

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Assume: Price paths are all $S \in C([0,\infty);\mathbb{R}^d)$ satisfying (RIE).

Let π be a portfolio – that is, a controlled path (π,π') such that $\pi_t \in \Delta^d$ for all $t \in [0,\infty)$.

The corresponding wealth process W^π satisfies

$$\frac{\mathrm{d}W_t^\pi}{W_t^\pi} = \sum_{i=1}^d \pi_t^i \frac{\mathrm{d}\mathbf{S}_t^i}{S_t^i}, \qquad W_0^\pi = 1,$$

and is given by

$$W_t^\pi = \exp\left(\int_0^t \frac{\pi_s}{S_s}\,\mathrm{d}\mathbf{S}_s - \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\pi_s^i \pi_s^j}{S_s^i S_s^j}\,\mathrm{d}[\mathbf{S}]_s^{ij}\right).$$

Examples of admissible portfolios

Functionally generated portfolio:

$$\pi_t^i := \mu_t^i \Big(\frac{\partial}{\partial x_i} \log G(\mu_t) + 1 - \sum_{j=1}^d \mu_t^j \frac{\partial}{\partial x_j} \log G(\mu_t) \Big).$$

"Functionally controlled portfolio":

$$\pi_t^i := \mu_t^i \Big(F^i(\mu_t) + 1 - \sum_{j=1}^d \mu_t^j F^j(\mu_t) \Big).$$

Portfolios generated by controlled equations:

$$\mathrm{d}\pi_t = f(\pi_t)\,\mathrm{d}\mu_t$$

admits a unique solution $(\pi, \pi') = (\pi_0 + \int_0^\cdot f(\pi_t)\,\mathrm{d}\mu_t, f(\pi))$.

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Universal Portfolio Theory (initiated by T. Cover '91)
aims to construct preference-free asymptotically "optimal" portfolios.

Cover's universal portfolio (Cover '91, Jamshidian '92, Cover & Ordentlich '96) requires trading according to a mixture of all admissible portfolios:

$$\pi_t^{\nu} := \frac{\int_{\mathcal{A}} \pi_t V_t^{\pi} \,\mathrm{d}\nu(\pi)}{\int_{\mathcal{A}} V_t^{\pi} \,\mathrm{d}\nu(\pi)}, \qquad t \in [0,\infty),$$

where

- $\mathcal{A}$ stands for all admissible portfolios,
- ν is a given probability measure on $\mathcal{A}$,

following the pathwise version of Cuchiero, Schachermayer & Wong '19.

Functionally controlled universal portfolio

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How well does Cover's universal portfolio do asymptotically?

Let $\mathcal{A}^{K,\alpha}$ be a suitable set of functionally controlled portfolios.

The pathwise version of Cover's universal portfolio

$$\pi_t^\nu := \frac{\int_{\mathcal{A}^{K,\alpha}} \pi_t V_t^\pi \,\mathrm{d}\nu(\pi)}{\int_{\mathcal{A}^{K,\alpha}} V_t^\pi \,\mathrm{d}\nu(\pi)}, \qquad t \in [0,\infty),$$

is well-defined and a controlled path w.r.t. the market portfolio μ.

By compactness, there exists a best retrospectively chosen portfolio, i.e. an $\pi_T^* \in \mathcal{A}^{K,\alpha}$ such that

$$V_T^{\pi_T^*} = \sup_{\pi_T^* \in \mathcal{A}^{K,\alpha}} V_T^{\pi_T}.$$

Functionally controlled universal portfolio

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Theorem (Allan, Cuchiero, Liu & P. '21)

Suppose that $\lim_{T\to\infty}(1+\|\mu\|^2_{p,[0,T]})\xi_T = \infty$, *where* $\xi_T = \xi_T(S,\mathbb{S})$.

Then

$$\lim_{T\to\infty} \frac{1}{(1+\|\mu\|^2_{p,[0,T]})\xi_T}\Big(\log V_T^{\pi_T^*} - \log V_T^{\pi^\nu}\Big) = 0.$$

Remark:

- It is a generalization of Cuchiero, Schachermayer & Wong '19.
- The above pathwise rate is non-trivial.
- We can take different sets of admissible portfolios.

Summary

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- A rough path based framework for financial modelling was provided, allowing more general portfolios than previous model-free approaches.
- Model-free Cover's universal portfolios were introduced and their asymptotic optimality were shown.
- Model-free master formulae in the spirit of Fernholz's stochastic portfolio theory were established.

Thank you very much for your attention!

References:

Allan, A. L., Cuchiero, C., Liu, C., and Prömel, D. J. (2021).
Model-free Portfolio Theory: A Rough Path Approach.
Preprint arXiv:2109.01843.