On the stability of symmetric flows in a two-dimensional tunnel

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Joint work with: Bernard Helffer

Braude, Nantes

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Problem formulation

Navier-Stokes Equations

$$\begin{cases} \partial_t \mathbf{v} - \epsilon \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{b} & \text{in } \mathbb{R}_+ \times D \\ \mathbf{v} = 0 & \text{on } \mathbb{R}_+ \times \partial D \end{cases}$$

$\mathbf{v}$: velocity p : pressure $\mathbf{b} = b(x_2)\hat{i}_1$: body force

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$$(\mathfrak{T} - \Lambda)\mathbf{u} - \nabla q = \mathbf{F}.$$

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Restriction to $[0, 1]$ – even modes

$D(\mathcal{B}_{\lambda,\alpha,\beta}) = \{u \in H^4(0, 1)\,,\ u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\}.$

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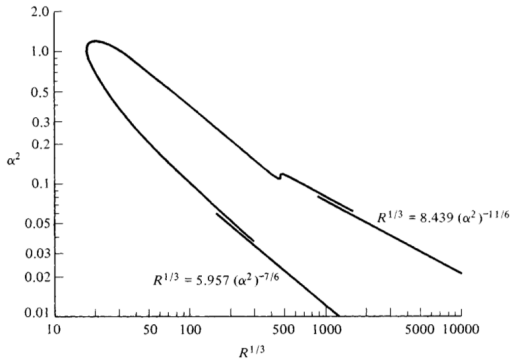Grenier, Guo, &Nguyen (2016) Rigorous analysis

Fig. 4.11. The curve of marginal stability for plane Poiseuille flow based on equation (28.33). (From Reid 1965.)

$R^{1/3} = 8.439 \, (\alpha^2)^{-11/6}$

$R^{1/3} = 5.957 \, (\alpha^2)^{-7/6}$

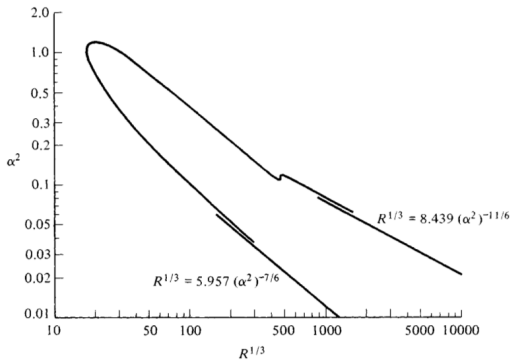with the vertical axis labeled α^2 and the horizontal axis labeled $R^{1/3}$.

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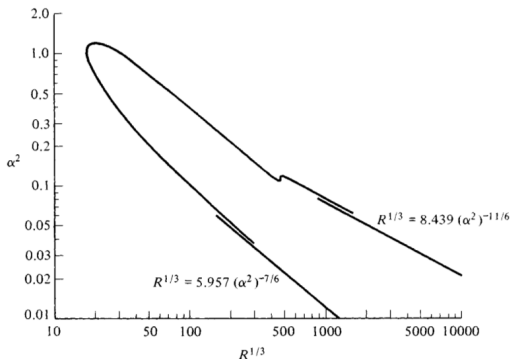
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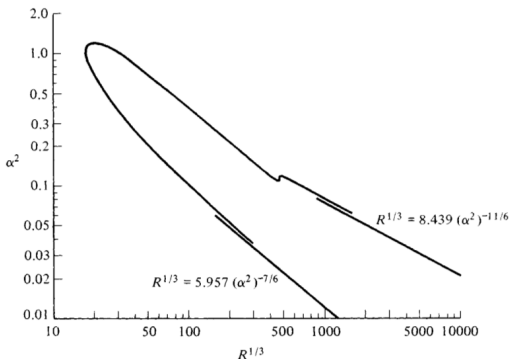
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$$\alpha\Re\lambda_0 \sim \mathcal{O}(\beta^{-1/2})$$

Main results

Almog & Helffer

Theorem

Let $\delta > 0$ and $U \in C^4([0,1])$ satisfy

$$U(1) = 0 \quad ; \quad \max_{x \in [-1,1]} U''(x) < 0 \quad ; \quad U'(0) = U^{(3)}(0) = 0$$

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Then $\exists (\alpha_L, C, \Upsilon) \in \mathbb{R}^3_+$, and $\beta_0 > 1$: $\forall \beta > \beta_0$ it holds that

$$\sup_{\substack{0 \leq \alpha \leq \alpha_L \beta^{-1/6} \\ \Re\lambda < \Upsilon \beta^{-1/2}}} \left\| (\mathcal{B}_{\lambda,\alpha,\beta})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda,\alpha,\beta})^{-1} \right\| \leq C\beta^{-1/2} \,.$$

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Proof outlines

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Alternative formulation of $\mathcal{B}_{\lambda,\alpha,\beta}\phi = f$

$$\begin{cases} (\mathcal{L} - \beta\lambda)v_{\mathfrak{D}} = g \\ v_{\mathfrak{D}}(1) = v_{\mathfrak{D}}'(0) = 0 \end{cases} \qquad \mathcal{L} = -\frac{d^2}{dx^2} + i\beta U$$

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$$\|\phi^{(3)}\|_2 \leq C\left(\beta^{-1/3}\|f\|_2 + \beta^{1/2}\|\phi\|_{1,2}\right)$$

$$\|\phi\|_{1,2} \leq C\|v\|_2 \leq C\Big(\beta^{-1/6}\|\phi\|_{1,2} + \beta^{-1}\|f\|_2\Big)$$

Proposition

Let $U \in C^4([0,1])$: $U(1) = U'(0) = U^3(0) = 0$; $U'' < 0$. Then, $\exists C > 0$, $\beta_0 > 0$: $\forall \beta \geq \beta_0$

$$\big\|(\mathcal{B}_{0,0,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}\,(\mathcal{B}_{0,0,\beta})^{-1}\Big\| + \beta^{-1/3}\Big\|\frac{d^2}{dx^2}\,(\mathcal{B}_{0,0,\beta})^{-1}\Big\| \leq C\beta^{-1}\,.$$

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$$

$$\Updownarrow$$

$$\mathcal{B}^{\mathfrak{D}}_{0,0,\beta}\phi = f + \beta\lambda(\phi'' - \alpha^2\phi) - \alpha^2\phi'' + i\alpha^2\beta U\phi$$

$$\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}}\phi = f$$

$$\Updownarrow$$

$$\mathcal{B}_{0,0,\beta}^{\mathfrak{D}}\phi = f + \beta\lambda(\phi'' - \alpha^2\phi) - \alpha^2\phi'' + i\alpha^2\beta U\phi$$

Corollary

Let $U \in C^4([0,1])$: $U(1) = U'(0) = U^3(0) = 0$; $U'' < 0$. Then,

$$\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}}\phi = f$$

$$\Updownarrow$$

$$\mathcal{B}_{0,0,\beta}^{\mathfrak{D}}\phi = f + \beta\lambda(\phi'' - \alpha^2\phi) - \alpha^2\phi'' + i\alpha^2\beta U\phi$$

Corollary

Let $U \in C^4([0,1])$: $U(1) = U'(0) = U^3(0) = 0$; $U'' < 0$. Then,

$$\exists C > 0, \ \beta_0 > 0, \ \alpha_0 > 0, \ \lambda_0 > 0 : \forall \beta \geq \beta_0$$

$$\sup_{\substack{0 \leq \alpha \leq \alpha_0 \\ |\lambda| < \lambda_0\beta^{-1/3}}} \left\| (\mathcal{B}_{\lambda,\alpha,\beta})^{-1} \right\| + \left\| \frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta})^{-1} \right\| \leq C\beta^{-1}.$$