Controllability of infinite dimensional quantum systems based on Quantum Graphs.

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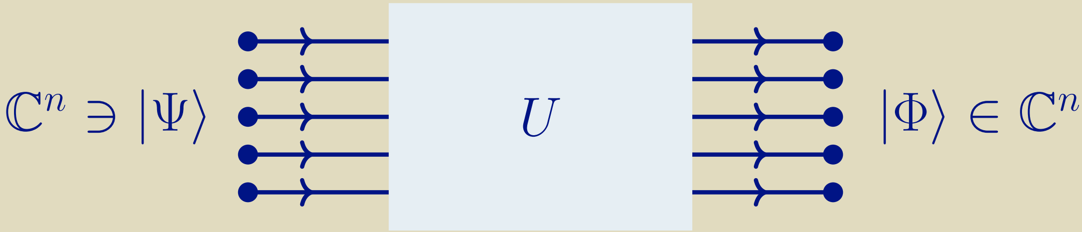
Mathematical Aspects of the Physics with non-self-Adjoint Operators

Joint work with A. Balmaseda and D. Lonigro

Outline

- The Control Problem in Quantum Mechanics
- Time dependent Boundary Conditions and the Schrödinger Equation
- Controllability on Quantum Graphs

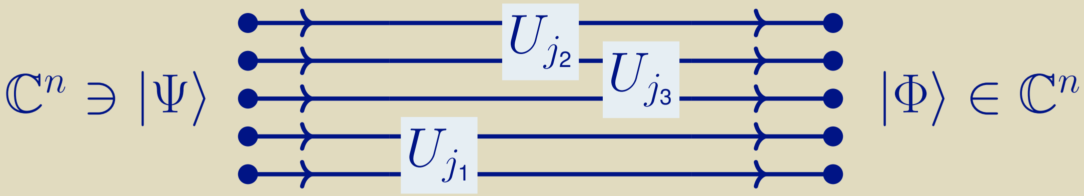
- A simplified scheme of a quantum computer



- $|\Psi\rangle$ input state; $|\Phi\rangle$ output state; U is a unitary operator in $\mathcal{U}(\mathbb{C}^n)$.
- Building a Quantum Computer $\Rightarrow$ Design a system capable of implementing any possible unitary operator
- This problem is equivalent to simultaneously control the evolution of n linearly independent states. Fixing n orthonormal states as input and other n as output.

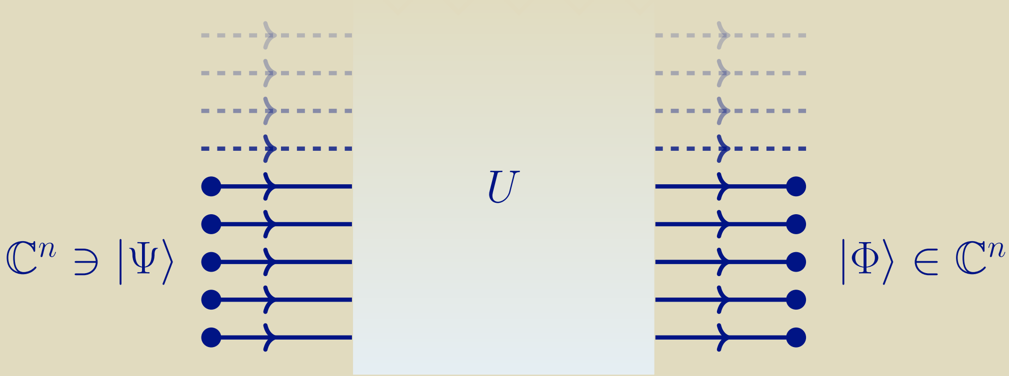
Universal set of gates

- $\mathcal{G} = \{U_1, U_2, U_3, \ldots, U_n\}$



- Any unitary might be obtained by combining elements of this finite set.
- A drawback of this approach is that the number of elemental operations grows exponentially with n.
- In practice one works with more energy levels (auxiliary levels). They are used to mediate the interactions and control errors.

Quantum Control in Infinite Dimensions



- One considers dynamics in the complete Hilbert space.
- This can be done even if one wants to codify information in a finite dimensional subspace.
- This is a systematic way of handling with the auxiliary levels.
- Opens the possibility of a new type of control. We will use the space of self-adjoint extensions as space of controls.

Consider the Schrödinger equation:

$$i\frac{\partial}{\partial t}\Phi(t) = (H_0 + u(t)H_1)\,\Phi(t)$$

We want to steer the state of the system:

- From an initial state Ψ_0 at $t = t_0$
- To a target state Ψ_T at a later time $t = T$

Does it exist a function $u\colon [t_0, T] \to \mathbb{R}$ such that:

- $\Phi(t) = U(t, t_0)\Psi_0$ solves the Schrödinger equation
- $\Phi(0) = \Psi_0$ and $\Phi(T) = U(T, t_0)\Psi_0 = \Psi_T$

- The classic theory of control has been applied successfully to finite dimensional quantum systems.
- The success in the development of recent quantum technologies is a proof of this.
- This can be used even for infinite dimensional systems. What is the main idea?
  - Pick a suitable basis $\{\Phi_n\} \subset \mathcal{H}$
  - $\langle \Phi_n, H(c(t))\Phi_m \rangle = H_{nm}(c(t))$
  - Consider the truncated Schrödinger eq.:

$$i\dot{x}_n = \sum_{m=1}^{N} H_{nm}\left(c(t)\right) x_m$$

- Results of control on finite dimensions cannot be applied directly to infinite dimension.
- The notion of controllability introduced in the previous slides is not appropriate for infinite dimensions
  - One can find examples where all the finite dimensional truncations are controllable but the infinite dimensional system is not, for instance the Harmonic oscillator.
  - This is reasonable. Suppose that the target state Ψ_T expressed in the basis $\{\Phi_n\}$ has countably many non-zero coefficients. Then $\|\Psi_T - \Psi_T^N\| > 0$ for any N.
  - *Approximate Controllability:* $\|\Psi_T - \Phi(T)\| < \epsilon$

From finite dimensional to infinite dimensional

Finite dimensional	Infinite dimensional
$i\dot{x} = (H_0^N + u(t)H_1^N)x$	$i\dot{\Psi} = (H_0 + u(t)H_1)\Psi$
$x(T_N) = U_u^N(T_N, t_0)x_0$	$\Psi(T_N) = U_u(T_N, t_0)\Psi_0$

- From the initial and target states Ψ_0, Ψ_t one can choose approximations x_0 and x_t such that

$$\|x_0 - \Psi_0\| < \epsilon \qquad \|x_t - \Psi_t\| < \epsilon$$

- The finite dimensional control will be a solution of the infinite dimensional control problem one only if

$$\|U_u^N(T_N, t_0) - U_u(T_N, t_0)\| \xrightarrow[N \to \infty]{} 0$$

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Operators and Quadratic Forms

Semibounded self-adjoint operator

$$T : \mathcal{D}(T) \to \mathcal{H}$$

$\Leftrightarrow$

Semibounded Hermitean Quadratic Form

$$Q : \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{C}$$

$$Q(\Phi, \Psi) = \langle \Phi, T\Psi \rangle$$

- We say that $\mathfrak{Q}$ is the form domain of the operator T.
- For Hamiltonians with constant form domain one can define the weak Schrödinger equation

$$\frac{d}{dt}\langle \Psi, \Phi(t) \rangle = Q_t(\Psi, \Phi(t)) \qquad \Phi(t), \Psi \in \mathfrak{Q}$$

- The domain of the operator might depend on time while the form domain doesn't.

Some particular boundary conditions on $\mathcal{L}^2([0,2\pi])$

- Quasi-periodic Boundary Conditions:

$$\mathcal{D}_\alpha = \left\{ \phi \in \mathcal{H}^2 \middle| \begin{array}{c} \phi(0) = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) = e^{i2\pi\alpha}\phi'(2\pi) \end{array} \right\}$$

$$\mathfrak{Q}_\alpha = \{ \phi \in \mathcal{H}^1 | \quad \phi(0) = e^{i2\pi\alpha}\phi(2\pi) \quad \}$$

- Delta like Boundary Conditions:

$$\mathcal{D}_\delta = \left\{ \phi \in \mathcal{H}^2 \middle| \begin{array}{c} \phi(0) = \phi(2\pi) \\ \phi'(0) - \phi'(2\pi) = \delta\phi(0) \end{array} \right\}$$

$$\mathfrak{Q}_\delta = \{ \phi \in \mathcal{H}^1 | \quad \phi(0) = \phi(2\pi) \quad \}$$

- Quasi-Delta Boundary Conditions:

$$\mathcal{D}_{\alpha,\delta} = \left\{ \phi \in \mathcal{H}^2 \middle| \begin{array}{c} \phi(0) = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) - e^{i2\pi\alpha}\phi'(2\pi) = \delta\phi(0) \end{array} \right\}$$

$$\mathfrak{Q}_{\alpha,\delta} = \{ \phi \in \mathcal{H}^1 | \quad \phi(0) = e^{i2\pi\alpha}\phi(2\pi) \quad \}$$

Stability

An important property that we were able to prove is the stability of the dynamics under perturbations/deformations of the Hamiltonian:

Theorem [Balmaseda, Lonigro, PP]:

Let $\{H_n(t)\}_{n=1,2}$ be two time-dependent Hamiltonians with constant form domain $\mathcal{H}_+$ that satisfy the conditions of Kisyński's Theorem and [a certain uniform bound on their derivatives]. Then the following inequality holds:

$$\|U_1(t,s) - U_2(t,s)\|_{+,-} \leq L \int_s^t \|H_1(\tau) - H_2(\tau)\|_{+,-} \mathsf{d}\tau,$$

where the constant L is independent of t and s.

The norm $\|\cdot\|_{+-}$ is the norm of linear operators $L : \mathcal{H}_+ \to \mathcal{H}_-$, where $\mathcal{H}_-$ is the canonical dual space of $\mathcal{H}_+$.

Improves previous results of B. Simon (1971) and A. Sloan (1981).

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Varying Quasiperiodic Boundary Conditions

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

$$\mathcal{D}_\alpha = \left\{ \phi \in \mathcal{H}^2 \middle| \begin{array}{rl} \phi(0) & = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) & = e^{i2\pi\alpha}\phi'(2\pi) \end{array} \right\}$$

- This is a family of self-adjoint operators depending on α
- We want to consider $\alpha(t)$ the control parameter. These Hamiltonians do not have constant form domain.
- One can tackle with these systems by the unitary transformation $T(t) : \Phi(x) \mapsto \exp(-ix\alpha(t))\Phi(x)$
- Assuming that the parameter α depends smoothly with time this is equivalent to:

$$H(t) = \left[i\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(t) \right]^2 + \dot{\alpha}(t)x$$

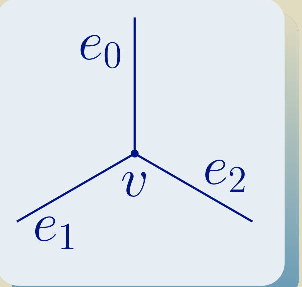
$\mathcal{D}_0 =$ "Periodic Boundary Conditions"

Laplacians on Quantum Graphs I

- Consider a planar Graph (V, E) and associate to each edge e a Hilbert space $\mathcal{H}_e = \mathcal{L}^2([0, l_e])$
- Take $\mathcal{H} = \bigoplus_{e\in E} \mathcal{H}_e$ and $\Delta = \bigoplus \Delta_e$ densely defined in it.
- The structure of the graph arises when one selects the boundary conditions.
- At each vertex we choose quasi-δ-boundary conditions:

$$\exp(-i\chi_{e_i,v})\Phi_e(v) = \Phi_{e_0}(v) \qquad i = 1, \ldots, n-1$$

$$\sum_e \exp(i\chi_{e_i,v})\dot{\varphi}_e = \delta_v \Phi_{e_0}(v)$$



Laplacians on Quantum Graphs II

- Laplacian with quasi-delta boundary conditions does not have constant form domain.
- There exist also time-dependent unitary maps that transform these Laplacians:

$$\Delta_e \rightsquigarrow \left[i\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(t,x)\right]^2 + \frac{d}{dt}\Theta(t,x)$$

$$\alpha(t,x) = \mathrm{d}\Theta(t,x)$$

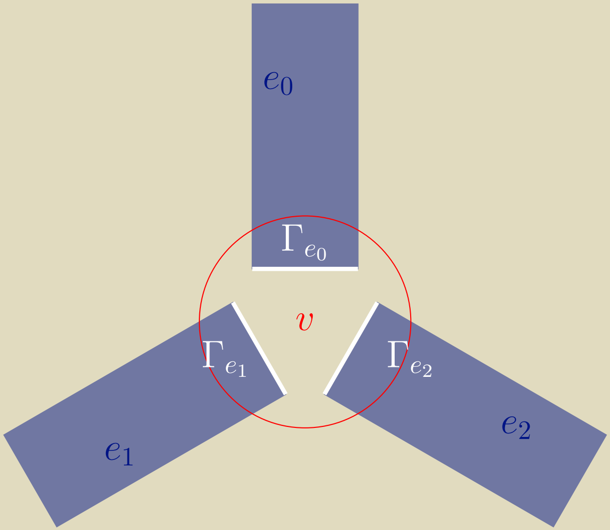
$$\Phi_e(v) = \Phi_{e_0}(v)$$
$$\sum_e \dot{\varphi}_{\alpha,e} = \delta_v \Phi_{e_0}(v)$$

$$\dot{\varphi}_{\alpha,e} = \frac{d}{d\vec{n}}\Phi(v) + \alpha_{\vec{n}(t,v)}$$

- The magnetic potential depends on time $\Rightarrow$ Domains are time dependent.
- The form domain is constant.

Laplacians on Thick Quantum Graphs

- Instead of associating an interval to each edge e one can associate a Riemannian manifold Ω_e.
- Magnetic Laplacians can be defined in an analogous way.
- There is a generalisation of the quasi-δ-type boundary conditions to Thick Quantum Graphs [Balmaseda, Lonigro, PP].



Γ_{e_i}: Boundary of the manifold Ω_{e_i}
Diffeomorphic to each other

Theorem [Balmaseda, Lonigro, PP]:
Let $u \in C^1_p(\mathbb{R})$, piecewise differentiable, and consider $\chi_{v,e}(t) := u(t)\chi_{v,e}$. Let $H(t)$ be the time-dependent Hamiltonian defined by the Laplacian on a thick Quantum Graph (V, E) with quasi-delta boundary conditions. Then, the linear system defined by $H(t)$ is approximately controllable w.r.t $\|\cdot\|_-$.

Sketch of the proof:

$$[id_x - \alpha(t,x)]^2 + \frac{d}{dt}\Theta(t,x)$$

$$\mathrm{d}\Theta(t,x) = \alpha(t,x)$$

- Auxiliary System

$$[id_x - \alpha_0]^2 + u(t)\Theta(x)$$

- Construct a differentiable magnetic potential such that $\alpha(t,x) \simeq \alpha_0$ and $\alpha'(t,x) = u'(t)\mathrm{d}\Theta(x)$
- Use stability to prove convergence

Controllability of a particle by moving walls

- Free particle in a cavity $[d(t) - l(t)/2, d(t) + l(t)/2] = \Omega(t)$

$$\Delta,\ \mathcal{D}(\Delta) = \mathcal{H}^1\big(\Omega(t)\big) \cap \mathcal{H}^2\big(\Omega(t)\big)$$

- Using a time dependent unitary operator one can transform this problem into the following [Martino, Anza, et. al. J. of Phys. A 46(36), 2013]

$$H(t) = -\frac{1}{l(t)^2}\Delta_{\text{Dir}} - \frac{\dot{l}(t)}{l(t)} x \circ p - \frac{\dot{d}(t)}{l(t)} p$$

$$\Omega = [0, 1]$$

Theorem [Balmaseda, Lonigro, PP]:

$$d(t) = d_0 + \delta f(t) \qquad l(t) = l_0 + \lambda f(t) \qquad f \in C^1_p(\mathbb{R})$$

- If $\delta \neq 0$ then the system is approximately controllable w.r.t $\|\cdot\|_-$.
- If $\delta = 0$ then the reduced subsystems with well defined parity are approximately controllable w.r.t $\|\cdot\|_-$.

THANKS!

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