Large pseudospectra for biharmonic operators with complex potentials

Duc Tho Nguyen

Czech Technical University in Prague

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Overview

1. Introduction and motivation
2. Biharmonic operator with a discontinuous potential
3. Biharmonic operator with a general potential

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- Not easy to calculate the spectrum $\sigma(\mathscr{L})$,
- Not easy to calculate the resolvent $(\mathscr{L} - z)^{-1}$,
- Even if you have an exact formula for $(\mathscr{L} - z)^{-1}$, not easy to calculate its norm.

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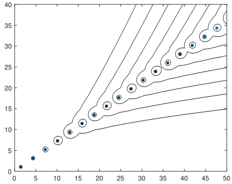

Figure: Pseudospectrum of $\mathscr{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^2$.

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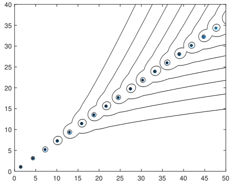

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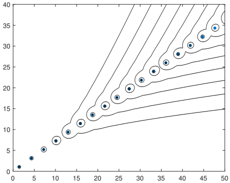

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$\rightsquigarrow$ Small perturbation of the operator does not mean small perturbation of the spectrum.

What is this talk about?

This talk is about the perturbed biharmonic operator by a complex potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$,

$$\mathscr{L}_V := \frac{\mathrm{d}^4}{\mathrm{d}x^4} + V(x),$$
$$\mathrm{Dom}(\mathscr{L}_V) := \left\{ u \in L^2(\mathbb{R}) : \mathscr{L}_V u \in L^2(\mathbb{R}) \right\}.$$

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- its numerical range $\mathrm{Num}(\mathscr{L}) := \left\{\langle \mathscr{L}u, u\rangle : u \in H^4(\mathbb{R}), \|u\|_{L^2} = 1\right\}$ is

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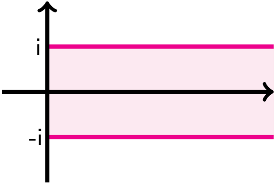

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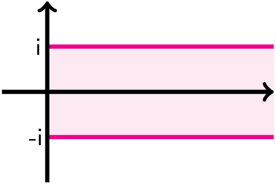

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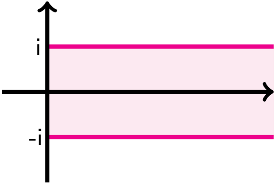

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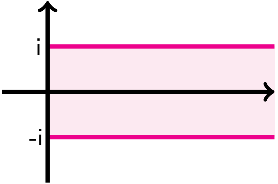

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$\implies \mathscr{L}$ is neither self-adjoint nor normal.

The Spectrum

Schrödinger (**Henry-Krejčiřík**-17)

The spectrum of $\mathscr{L}_{\text{Sch}} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + i\,\mathrm{sign}(x)$ is given by

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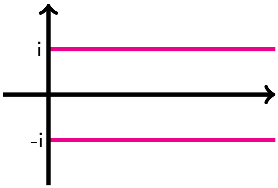

Resolvent estimate

Schrodinger and Biharmonic (**Henry-Krejčiřík**-17 and **N.**-22)

For all $\varepsilon > 0$, there exists a constant $C_0 > 0$ such that

$$(1-\varepsilon)\frac{\operatorname{Re} z}{\sqrt{1-(\operatorname{Im} z)^2}} \leq \|(\mathscr{L}_{\text{Sch}} - z)^{-1}\| \leq 4(1+\varepsilon)\frac{\operatorname{Re} z}{1-(\operatorname{Im} z)^2}$$

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$$\|\,(\mathscr{L} - z)^{-1}\,\|^2 \leq \left(\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{R}_z(x,y)|\, \mathrm{d}y\right)\left(\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |\mathcal{R}_z(x,y)|\, \mathrm{d}x\right)$$

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- The lower bound is obtained by choosing a "nice" function $f_0 \in L^2(\mathbb{R})$

$$\|\,(\mathscr{L} - z)^{-1}\,\| \geq \frac{\|(\mathscr{L} - z)^{-1} f_0\|_{L^2(\mathbb{R})}}{\|f_0\|_{L^2(\mathbb{R})}}$$

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What about the operator that we can not calculate the resolvent?

From the definition

$\lambda \in \sigma_\varepsilon(\mathscr{L})$ if and only if,

- $\lambda \in \sigma(\mathscr{L})$
  or
- λ is a pseudoeigenvalue, i.e. there exists $\Psi \in \mathrm{Dom}(\mathscr{L})$

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Question

Which region $\Omega \subset \mathbb{C}$ such that there exists Ψ_λ satisfying

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4. In 2022, **Krejčiřík** and **N.** (*J. Funct. Anal.*) extended the method of **Krejčiřík** and **Siegl** to to *relativistic quantum mechanics* by considering *Dirac operators*.

WKB analysis

Find Ψ_λ such that $\frac{\left\|\left(\frac{d^4}{dx^4}+V(x)-\lambda\right)\Psi_\lambda\right\|}{\|\Psi_\lambda\|} = o(1)$ when λ very large.

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  - ξ_λ is a cut-off function,
  - ψ_k satisfying the system of solvable *ODEs*

$$\textbf{The Eq. 1}: \quad (\psi_{-1}^{(1)})^4 = \frac{\lambda - V(x)}{\lambda^4},$$

$$\textbf{The Eq. 2}: \quad \psi_0^{(1)} = \frac{3}{2}\frac{\psi_{-1}^{(2)}}{\psi_{-1}^{(1)}},$$

$\dots$

What's new with this non-semiclassical WKB?

Semiclassical WKB

1) The parameter h is a small positive number.
2) The eikonal and all the transport solutions do not depend on h.
3) The pseudomodes always localize.
4) The potentials are always assumed very smooth.

Non-semiclassical WKB

1) The parameter λ is a large complex number.
2) The eikonal and all the transport solutions depend on λ. Therefore, it's a challenge to perform all estimations uniformly.
3) The pseudomodes do not localize, their supports can be extended in some cases.
4) It can cover the potentials with low regularity, even discontinuous ones.

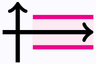
Let $\mathcal{V} : \mathbb{R} \to \mathbb{R}$ and $\limsup_{x \to -\infty} \mathcal{V}(x) < 0 < \liminf_{x \to +\infty} \mathcal{V}(x)$

For Schrödinger operators (**Krejčiřík**- **Siegl**-19)

Assume $\mathcal{V} \in W^{N+1,\infty}_{\mathrm{loc}}(\mathbb{R})$ for some $N \geq 0$ and let $\mathscr{L}_{\mathrm{Sch}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + i\mathcal{V}(x)$. Then

$$\frac{\|(\mathscr{L}_{\mathrm{Sch}} - \lambda)\Psi_{\lambda,N}\|_{L^2(\mathbb{R})}}{\|\Psi_{\lambda,N}\|_{L^2(\mathbb{R})}} = \mathcal{O}\left((\mathrm{Re}\lambda)^{-\frac{N+1}{2}}\right)$$

as $\lambda \to \infty \subset \mathbb{C}$ in the region parallel to the positive semi-axis



For Biharmonic operators (**N.**-22)

Assume $\mathcal{V} \in W^{N+3,\infty}_{\mathrm{loc}}(\mathbb{R})$ for some $N \geq 0$ and let $\mathscr{L}_{\mathrm{Bi}} = \frac{\mathrm{d}^4}{\mathrm{d}x^4} + i\mathcal{V}(x)$. Then

$$\frac{\|(\mathscr{L}_{\mathrm{Bi}} - \lambda)\Psi_{\lambda,N}\|_{L^2(\mathbb{R})}}{\|\Psi_{\lambda,N}\|_{L^2(\mathbb{R})}} = \mathcal{O}\left((\mathrm{Re}\lambda)^{-\frac{N+1}{4}}\right)$$

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Example (Bounded and smooth potentials)

$$\mathcal{V}(x) = \frac{2}{\pi}\arctan(x)$$

We have

$$\lim_{x\to-\infty} \mathcal{V}(x) = -1 \qquad \text{and} \qquad \lim_{x\to+\infty} \mathcal{V}(x) = 1.$$

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Then, for all $N \geq 0$,

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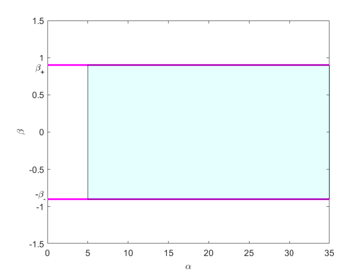
as $\lambda \to \infty$ in $\Omega = \{\alpha + i\beta : \alpha \gtrsim 1,\ \beta \in [-\beta_-, \beta_+] \subset (-1, 1)\}$.



Example (Bounded at $-\infty$ and unbounded at $+\infty$)

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The method also works for a wide classes of potentials:

- Polynomials $\mathcal{V}(x) = \text{sign(x)}|x|^\gamma$ for $\gamma \geq 0$,
- Logarithmic functions $\mathcal{V}(x) = \ln\left(x + \sqrt{x^2+1}\right), \ldots$
- Super-exponential functions $\mathcal{V}(x) = \sinh(x), \sinh(\sinh(x)), \ldots$

Example (Decay at $\pm\infty$)

$$\mathcal{V}(x) = \frac{\text{sgn}(x)}{|x|^\gamma}, \qquad |x| \gtrsim 1,\ 0 < \gamma < 1.$$

It does not satisfy the assumption of Theorem:

$$\lim_{x \to -\infty} \mathcal{V}(x) = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} \mathcal{V}(x) = 0.$$

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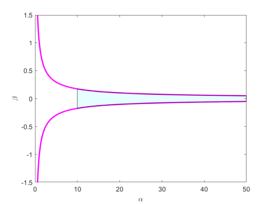
$$\Omega = \left\{\alpha + i\beta \in \mathbb{C} : \alpha \gtrsim 1,\ |\beta| \lesssim \alpha^{-\frac{3}{4}\frac{\gamma}{1-\gamma}-\varepsilon}\right\}.$$



WKB around turning point

$$\mathscr{L} := \frac{d^4}{dx^4} + i\,\mathcal{V}(x) \text{ on } L^2(\mathbb{R}_+)$$

- Assume that $\mathcal{V}$ is smooth enough and strictly increasing near $+\infty$ such that

$$\lim_{x\to+\infty} \mathcal{V}(x) = +\infty,$$

- We write $\lambda = \alpha + i\beta$, for sufficiently large $\beta > 0$, the turning point x_β of $\mathcal{V}$ by

$$\mathcal{V}(x_\beta) = \beta,$$

- By doing WKB analysis around the turning point, we can construct a region bounded by two curves in which we get the pseudomode.

Example (Polynomial potential)

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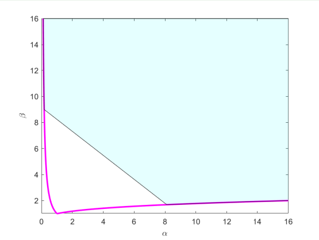
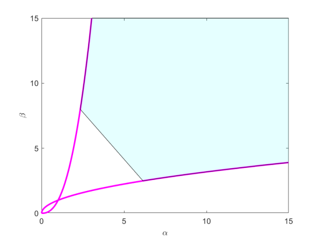
$$\Omega := \begin{cases} \left\{\alpha + i\beta \in \mathbb{C} : \beta \gtrsim 1 \text{ and } \beta^{\frac{4}{5}\left(1-\frac{1}{\gamma}\right)} \lesssim \alpha \lesssim \beta^{\frac{4}{3}\left(1+\frac{1}{\gamma}\right)-\varepsilon}\right\}, & 0 < \gamma < 1, \\ \left\{\alpha + i\beta \in \mathbb{C} : \beta \gtrsim 1 \text{ and } \beta^{\frac{4}{5}\left(1-\frac{1}{\gamma}\right)+\varepsilon} \lesssim \alpha \lesssim \beta^{\frac{4}{3}\left(1+\frac{1}{\gamma}\right)-\varepsilon}\right\}, & \gamma \geq 1. \end{cases}$$

for arbitrary small $\varepsilon > 0$.



(a) $V(x) = ix^{\frac{1}{2}}$.



(b) $V(x) = ix^2$.

Example (Super-exponential potential)

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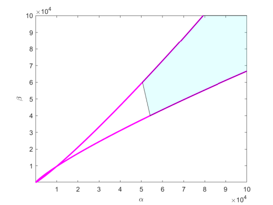

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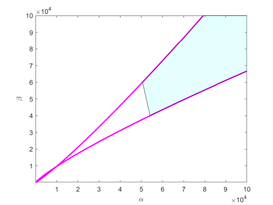

Thank you for your attention!