Using the Quasi-basis Structure to Understand the Mechanisms of Fluid Mechanics Phenomena

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Structure

- What are the Fluid Mechanics Phenomena in question
- Test Equations – RnsaGL and CnsaGL
- Non-normality with Nonlinearity
- Quasi-basis structure and Implications
- Further Research

Fluid Mechanics Phenomena

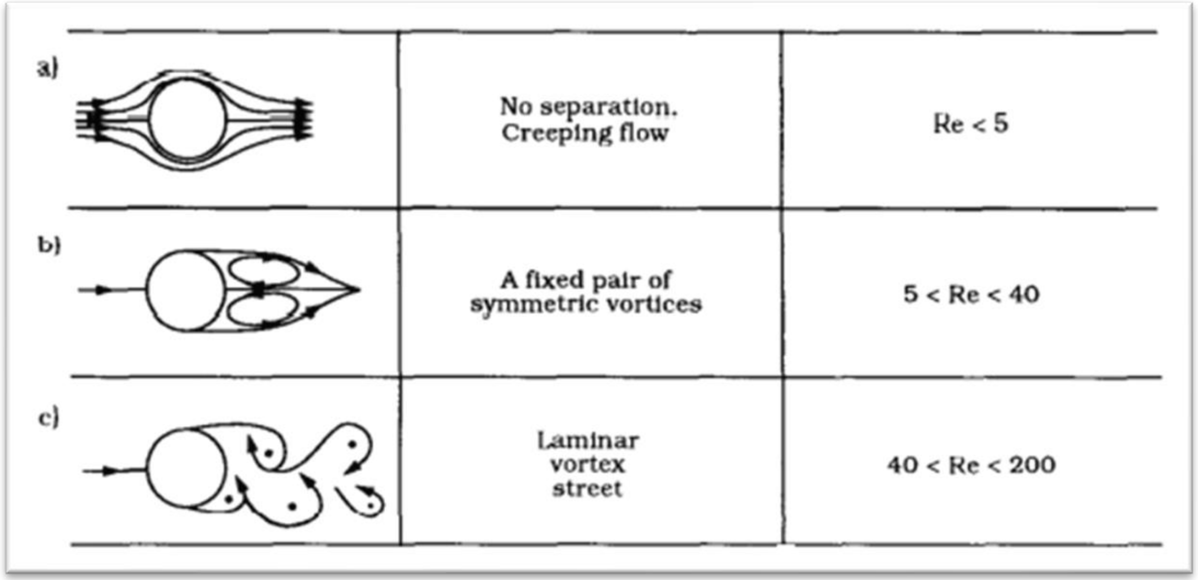


Figure 1a (top) Example of a Hopf Bifrucation. Regimes of flow around a smooth circular cylinder in steady current [1].

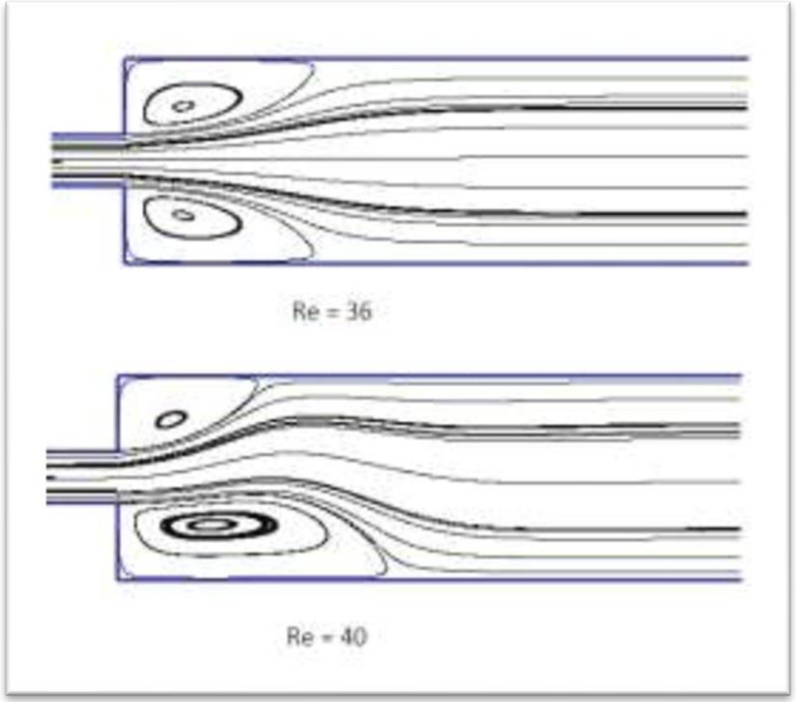


Figure 1b (below) Example of a Pitchfork bifurcation. Sudden expansions flow geometry including schematics of expected flow patters for Reynolds number shown in plot.

Test Equations - RnsaGL

The Real-non-self-adjoint Ginzburg-Landau (RnsaGL) equation is given by;

$$\frac{du}{dt} = \mathcal{L}^{RGL} u + \delta u - u^3$$

where

$$\mathcal{L}^{RGL} = \frac{\partial^2}{\partial x^2} - U \frac{\partial}{\partial x} + \left(\frac{U^2}{4} + \sqrt{c_2} - c_2 x^2 \right)$$

and smooth initial data.

As U increases, the operator becomes *more* « non-self-adjoint »… When $U = 0$, this equation is self-adjoint. The leading eigenvalue is always zero, i.e. $\lambda_0 = 0$.

The eigenvectors of this operator do not form a basis in L^2 .

Test Equations - CnsaGL

The Complex-non-self-adjoint Ginzburg-Landau (CnsaGL) equation is given by;

$$\frac{du}{dt} = \mathcal{L}^{CGL}u + \delta u - |u|^2 u$$

where

$$\mathcal{L}^{CGL} = (1-i)\frac{\partial^2}{\partial x^2} - (U + 0.2i)\frac{\partial}{\partial x} + \left[C_1 + \frac{1}{8}(U^2 - 0.4U \; - 0.04)\right] - c_2 x^2$$

with $C_1 = Re\left\{\sqrt{c_2(1-i)}\right\}$ with smooth initial data.

When U increases, the operator becomes more « non-normal »… The leading eigenvalue is always imaginary, i.e. $\lambda_0 = i\,\omega_0$.

The eigenvectors of this operator do not form a basis in L^2 .

Non-normality versus nonlinearity

The importance of non-normality in relation to nonlinearity has been in interest for several decades in Fluid Mechanics. The discussions have been driven by the success of various reduced order models.

It is instructive to derive a reduced order model before discussing the literature. We derive the reduced order model via WNLE.

The method of weakly nonlinear expansions can be described roughly into three steps:

1. Introduce a **diffusive scaling** centered around a **small parameter** that allows the linear operator to remain dominant.
2. Introduce an expansion in terms of the small parameter in order to get a hierarchy of equations.
3. Use the **Fredholm Alternative** repeatedly to obtain solvability conditions at each order to elaborate the temporal development of each term of the expansion.

1. We choose a diffusive timescaling $u(x,t) = \epsilon^{\frac{1}{2}} v(x,\tau), \tau = \epsilon t$ where ϵ is a small parameter.

   We also let $\delta = \epsilon\tilde{\delta}$ where $\tilde{\delta} = O(1)$. This results in the following equation

$$-\mathcal{L}^{RGL} v = \epsilon(-\frac{\partial v}{\partial \tau} + \tilde{\delta} v + v^3)$$

2. Expand the rescaled solution in powers of the small parameter.

$$v = v_0 + \epsilon\, v_1 + \ \dots$$

   This results in the following hierarchy of equations;

$$-\mathcal{L}^{RGL} v_0 = 0, \qquad\qquad -\mathcal{L}^{RGL} v_1 = -\frac{\partial v_0}{\partial \tau} + \delta v_0 - v_0^3$$

3. At first order, we obtain

$$v_0 = A(\tau)\ \hat{e}_0$$

where $\hat{e}_0$ is the zeroth eigenvector.

Substituting this in the next order, we get

$$-\mathcal{L}^{RGL} v_1 = -\frac{\partial A(\tau)}{\partial \tau} + \tilde{\delta} A(\tau)\ \hat{e}_0 - \big(A(\tau)\big)^3\ (\hat{e}_0)^3$$

We apply the Fredholm alternative in order to obtain

$$\frac{dA}{d\tau} = \tilde{\delta} A\ - \left\langle \hat{e}_0^{\dagger}, \hat{e}_0^{3} \right\rangle A^3.$$

We put the amplitude equation back on the original timescale to get the first order amplitude equation

$$\frac{dB}{dt} = \delta B - \lambda^1 B^3$$

where we have let $B = \epsilon^{\frac{1}{2}} A$ and $\lambda^1 = \left\langle \hat{e}_0^{\dagger}, \hat{e}_0^3 \right\rangle$.

The resulting first-order approximation is given by

$$u_{app} \approx B\ \hat{e}_0.$$

Stuart-Landau Equation (Real-Case)

How well does this approximate the solution to the RnsaGL?

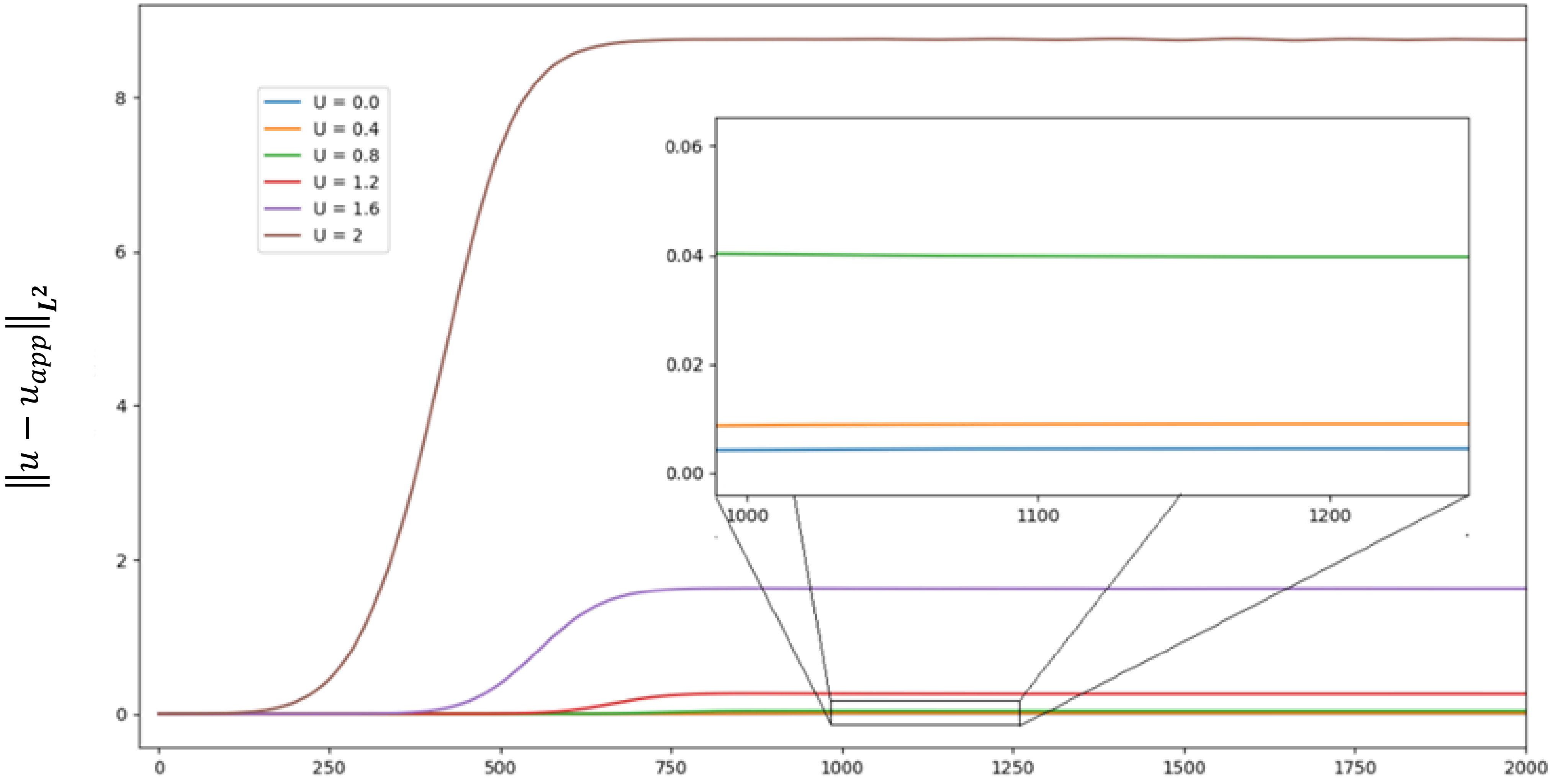


Figure. A plot of $\|u - u_{app}\|_{L^2}$ against t for various values of U, $\delta = 0.01$.

Timeline

(1940s) Landau [3] derived the Stuart-Landau equation, but did not determine the coefficients.

(1960) Stuart [4] derived the coefficients for the amplitude equation. Watson [5] created a method to find higher orders of the amplitude equation (Amplitude Expansion Method).

(1967) Numerical Experiments (Pekeris and Skholler [5], Reynolds and Potter [6] that show the approximation is valid for **only early times** and small values of δ.

This motivated the building of higher order amplitude equations and higher order approximations to capture better results.

Important Papers

- Coullet and Spiegel (1983) [13] created a new methodology for creating **higher order amplitude equations** for bifurcation problems in Fluid Mechanics based on the Krylov-Bogoliubov method.

- Hebert (1983) [7] established a way of making **higher order approximations**. He provided an extra condition in terms of a point $u(x_0) = u_0$ that normalised higher order terms.

- Fujimura (1989, 1991) [8, 9] showed the **equivalence** between making amplitude equations with normal form theory, WNLE and the Amplitude Expansion method [12].

- Chomaz (2005) [10] in the review paper entitled "Non-normality and Nonlinearity" firstly connected the failure of WNLE to non-normality and argued to for a nonlinear framework.

- Sipp and Lebedev (2007) [11] and Carini et al. (2015) [12] derived first order amplitude equations via two different methods and established a **well-approximated saturation frequency and not amplitude**.

Illustration of Sipp and Lebedev on CnsaGL

Recall

$$\frac{du}{dt} = \mathcal{L}^{CGL}u + \delta u - |u|^2 u \quad \text{with}$$

$$\mathcal{L}^{CGL} = (1-i)\frac{\partial^2}{\partial x^2} - (U + 0.2i)\frac{\partial}{\partial x} + \left[C_1 + \tfrac{1}{8}(U^2 - 0.4U \; - 0.04)\right] - c_2 x^2.$$

We consider the solution $u = Re^{\mathrm{i}\Phi}$ to get the following operators

$$\mathcal{L}_R = \frac{\partial^2}{\partial x^2} - U\frac{\partial}{\partial x} + \left[C_1 + \frac{1}{8}(U^2 - 0.4U \; - 0.04)\right] - c_2 x^2 \qquad \mathcal{L}_\Phi = \frac{\partial^2}{\partial x^2} - U\frac{\partial}{\partial x}.$$

We plot the matrix exponential for these two operators. For the intent and purpose here, it describes the maximum response to the most dangerous initial condition.

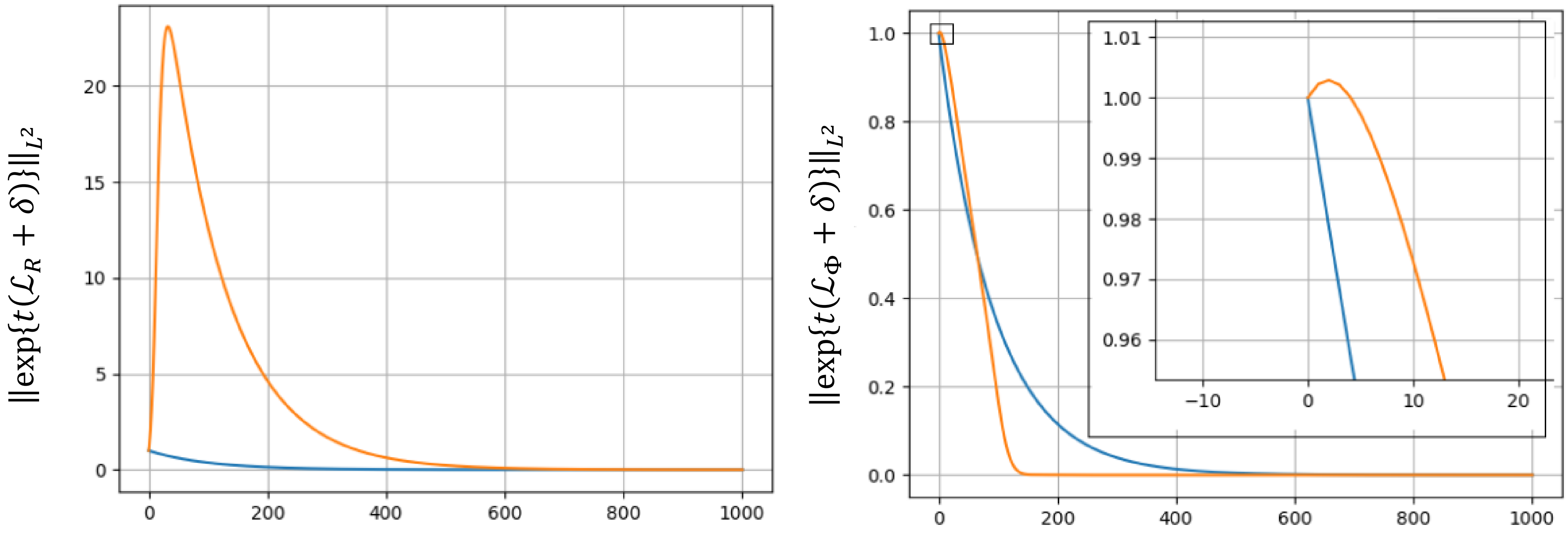


Figure. Plots of $\|\exp\{t(\mathcal{L}_R + \delta)\}\|_{L^2}$ (left) and $\|\exp\{t(\mathcal{L}_\Phi + \delta)\}\|_{L^2}$ (right) for $U = 0$ (blue) and $U = 1$ (orange) with $\delta = -0.01$.

Quasi-Basis Structure and Implications

We firstly give the following definitions;

- Definition (Metric Operator) A metric operator in a Hilbert space $\mathcal{H}$ is a strictly-positive self-adjoint operator G, that is $G > 0$.

- Definition (Quasi-Hermitian) A closed operator $\mathcal{L}$, with dense domain $D(\mathcal{L})$ is called quasi-Hermitian if there exists a metric operator G, with a dense domain $D(\mathcal{L}) \subset D(G)$ and

$$\langle \mathcal{L}\, \xi | G\, \eta \rangle = \langle G\, \xi | \, \mathcal{L}\, \eta \rangle, \quad \xi, \eta \in D(\mathcal{L}) \quad (*)$$

In the case of the RnsaGL, we have that $(*)$ holds with $\mathcal{L} = \mathcal{L}^{RGL}$ and $G = e^{-Ux}$.

Let $\mathcal{H}$ be a Hilbert space. Let us consider the following space of functions

$$D\left(G^{\frac{1}{2}}\right) = \left\{ u \in \mathcal{H} : \left\|G^{\frac{1}{2}} u\right\|_{\mathcal{H}} \leq \infty \right\}.$$

We let the $\|\cdot\|_G$ be the norm be induced by the inner product,

$$\|\cdot\|_G = \left\|G^{1/2} \cdot\right\|_{\mathcal{H}} = \sqrt{\left\langle G^{\frac{1}{2}} \cdot, G^{\frac{1}{2}} \right\rangle} .$$

We denote the Hilbert-Space $\mathcal{H}(\mathrm{G}) := \left(D\left(G^{\frac{1}{2}}\right), \|\cdot\|_G\right)$.

Furthermore, we have the extreme spaces $\mathcal{H}(G) \cap \mathcal{H}(G^{-1})$

$$\|f\|_{\mathcal{H}(G) \cap \mathcal{H}\left(G^{-1}\right)} = \|f\|_G + \|f\|_{G^{-1}}.$$

$$
\begin{tikzcd}
 & & \mathcal{H}(G^{-1}) \\
 & \mathcal{H}(1+G^{-1}) \arrow[ur] \arrow[dr] & \\
\mathcal{H}(G) \cap \mathcal{H}(G^{-1}) \arrow[ur] \arrow[dr] & & \mathcal{H} \\
 & \mathcal{H}(1+G) \arrow[ur] \arrow[dr] & \\
 & & \mathcal{H}(G)
\end{tikzcd}
$$

<u>Figure.</u> The resulting lattice of Hilbert Spaces from the unbounded metric operator G. All of the embeddings represent continuous embeddings with dense range.

Definition (Quasi-Basis) Given a dense subspace X_1 of a Hilbert Space $\mathcal{H}$. Let $E = \{ e_n\}_{n=0}^{\infty}$ and $E^{\dagger} = \{ e_n^{\dagger}\}_{n=0}^{\infty}$ be two bi-orthogonal sets such that all $e_n \in X_1$ and all $e_n^{\dagger} \in X_1$. Then E and $E^{\dagger}$ are X_1 quasi-bases, if for all $f, g \in X_1$

$$\langle f, g \rangle = \sum_{n=0} \langle f, e_n \rangle \langle e_n^{\dagger}, g \rangle = \sum_{n=0} \langle f, e_n^{\dagger} \rangle \langle e_n, g \rangle$$

Theorem (Quasi-Basis for RnsaGL). The direct and adjoint eigenvectors of $\mathcal{L}^{RGL}$ form a quasi basis with $\mathcal{H} = L^2(\mathbb{R})$ and $G = e^{-Ux}$ in the space $\mathcal{H}(G^{-1}) \cap \mathcal{H}(G)$. Proof. Follows Proposition 9 for Bagarello 2013.

Theorem. Let $\hat{E} = \{ \hat{e}_n\}_{n=0}^{\infty}$ of a Hilbert Space $\mathcal{H}$. Let $E = \{ e_n\}_{n=0}^{\infty}$ and $E^{\dagger} = \{ e_n^{\dagger}\}_{n=0}^{\infty}$ be quasi-bases of $\mathcal{H}(G) \cap \mathcal{H}(G^{-1})$ where the metric operator $G^{\frac{1}{2}}$ where $G^{\frac{1}{2}} e_n = \hat{e}_n$.

Furthermore let $\left(G^{\frac{1}{2}}\right)^{-1} e_n^{\dagger} = \hat{e}_n$ and $G^{-1}\ e_n^{\dagger} = e_n$ and $G^{-1}\ \hat{e}_n$. Then each element $u \in \mathcal{H}(G) \cap \mathcal{H}(G^{-1})$, can be expressed as $u = \sum_{n=0}^{\infty} \langle e_n, u \rangle \mathrm{e}_{\mathrm{n}}^{\dagger}$ and $u = \sum_{n=0}^{\infty} \langle e_n^{\dagger}, u \rangle e_n$.

Implications – Visualisation of Mechanism

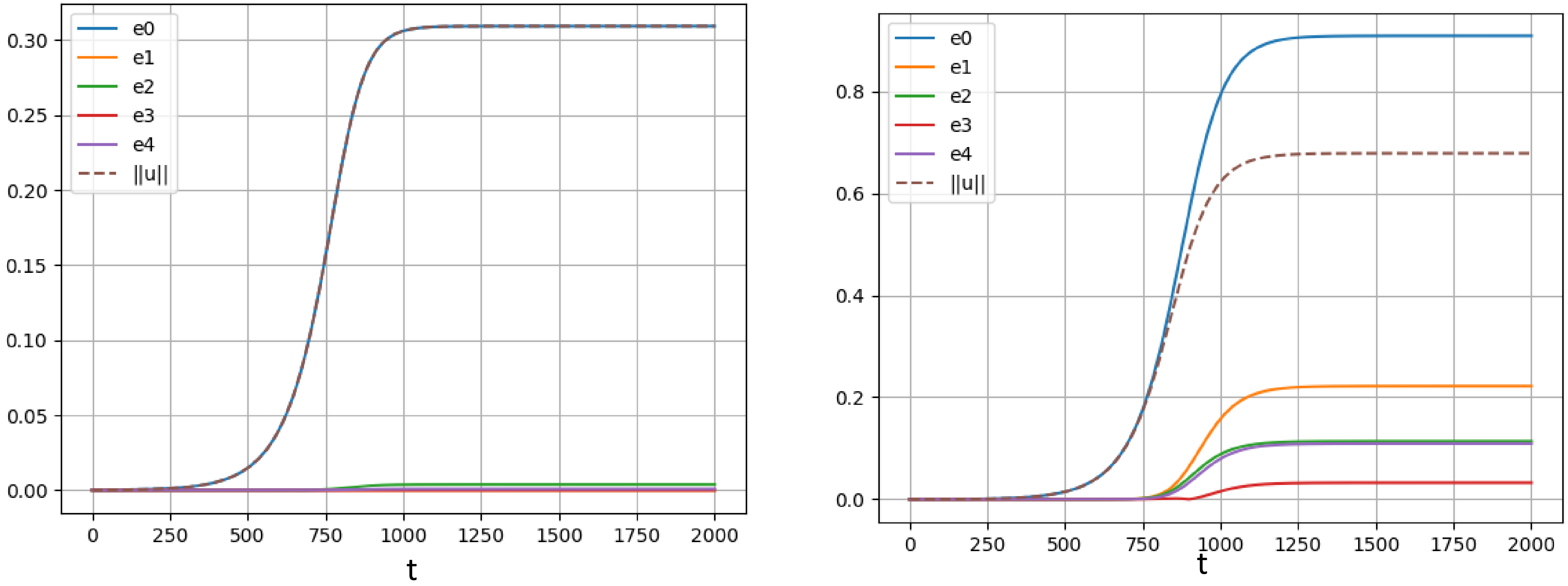


Figure. Plots of projections onto the direct modes $\hat{e}_n$, (i.e. $\langle \hat{e}_n^{\dagger}, u \rangle$).

The nonlinearity recharges the stable modes, but more so in the more non-self-adjoint cases!

Implications – Normalisation of higher order terms

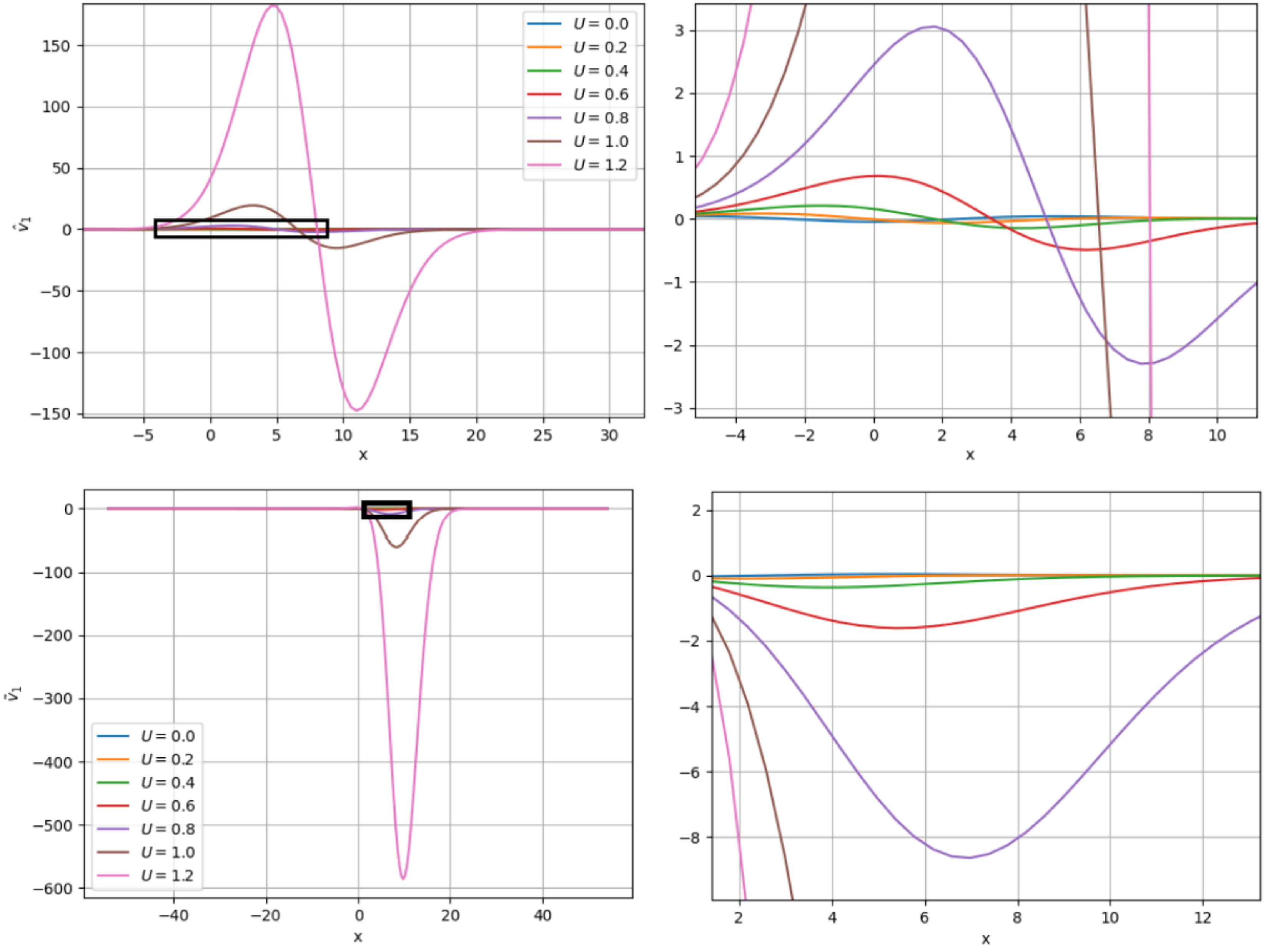
We can expand to get higher order terms, but owing to the Fredholm Alternative we need external conditions to work out the higher order terms.

For instance, we expand in the following way

$$v = \underbrace{C\hat{e}_0}_{v_0'} + \epsilon\,\underbrace{[v_1(C,x) + \gamma_1(C)\ \ \hat{e}_0]}_{v_1'} + \epsilon^2\,\underbrace{[v_2(C,x) + \gamma_2(C)\ \ \hat{e}_0]}_{v_2'} + \epsilon^3\,\underbrace{[v_3(C,x) + \gamma_3(C)\hat{e}_0]}_{v_3{}'} + \cdots$$

With this style of expansion, we want to have no overlap of the functions $\hat{e}_0$ with the terms $v_n(C,x)$ in the expansion. Owing to the quasi-basis, we can choose either

$$\langle \hat{e}_0, v_1\rangle = 0\ \ (v_1 = \hat{v}_1) \quad \text{and} \quad \langle \hat{e}_0^{\dagger}, v_1\rangle = 0\ \ (v_1 = \tilde{v}_1)$$

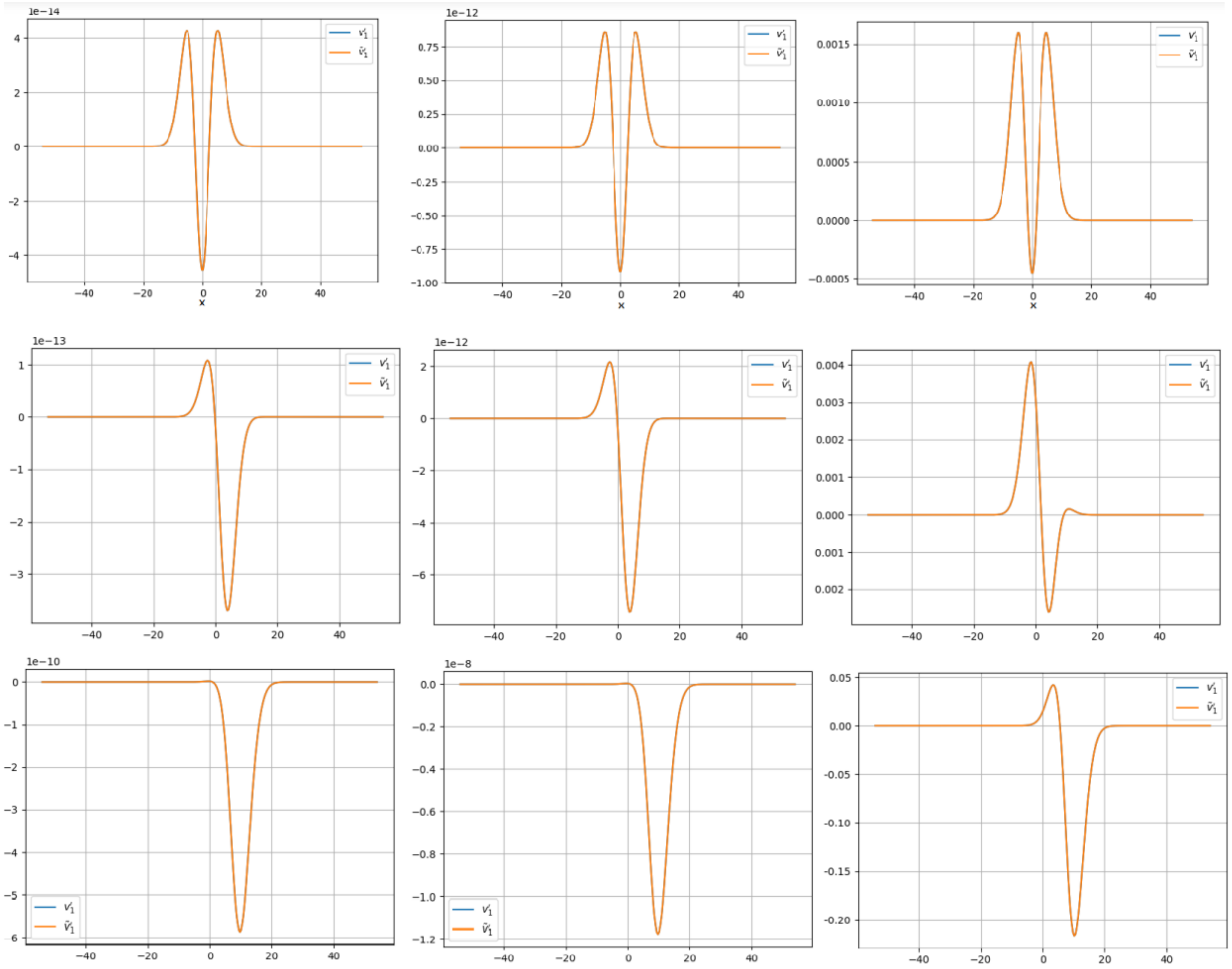


$\hat{\boldsymbol{v}}_1$

$\tilde{\boldsymbol{v}}_1$

Figure. (Left) Plots of $\hat{v}_1$ (top) and $\tilde{v}_1$ (bottom) against x for the values of the figures shown in plot. The pictures on the right correspond to close ups of the boxes on the left. $\delta = 0.01$

$t = 100s$ $t = 1000s$ $t = 2000s$



$U = 0$

$U = 0.6$

$U = 1.2$

Figure 10. Plots of $v_1'(x,t)$ and $\tilde{v}_1(x,t)$ against x for various values of U and T shown in the figure for $\delta = 0.01$

The plots overlap perfectly!

Implications – Stochastic Averaging

<u>Theorem (Expansion of noise in the eigenvectors of $\mathcal{L}^{RGL}$).</u> Let $\mathcal{W}(t)$ be a $\mathcal{H}$-valued Wiener process where $\mathcal{H} = \mathcal{H}(G) \cap \mathcal{H}(G^{-1})$. Let $\hat{e}_n^{\perp}$ be the eigenvectors of $\mathcal{L}^{RGL}$ with $U = 0$. Let Q be the corresponding covariance operator, such that there is a bounded sequence of sequence of non-negative real numbers λ_n, such that $Q\ \hat{e}_n^{\perp} = \lambda_n e_n^{\perp}$ where $\hat{e}_n^{\perp}$ be the eigenvectors of $\mathcal{L}^{RGL}$ with $U = 0$.

The noise can be expanded as

$$W(t) = \sum_{j=0}^{\infty} \sqrt{\lambda_j} \beta_j(t) \hat{e}_j^{\perp} = \sum_{j=0}^{\infty} \sqrt{\lambda_j} \hat{\beta}_j(t) \hat{e}_j$$

where $\beta_j = \frac{1}{\sqrt{\lambda_j}} \langle \hat{e}_j^{\perp}, W(t) \rangle$ and $\hat{\beta}_j = \frac{1}{\sqrt{\lambda_j}} \langle \hat{e}_j^{\dagger}, W(t) \rangle$.

Following the ability to expand noise in terms of the eigenfunctions of $\mathcal{L}^{RGL}$, we can attempt stochastic averaging in the non-self-adjoint case.

We choose a noise strength that is shown to give rise to a deterministic first order amplitude equation. We consider the following SPDE.

$$\frac{\partial u}{\partial t} = \mathcal{L}^{RGL}\, u + \epsilon\delta u \; - u^3 + \epsilon^{\frac{3}{4}} \frac{\partial W}{\partial t}$$

where we have that $W = \sum_{n=0}^{N} \alpha_n \hat{\beta}_n \, \hat{e}_n$ where α_n allows us to put the noise strength on each mode.

Stochastic averaging with this noise (Mohammed et al. 2014) results in the same amplitude equation to first order, i.e.

$$u_{app} \approx B \; \hat{e}_0 \;\text{ with}\; \frac{dB}{dt} = \delta B - \lambda^1 B^3$$

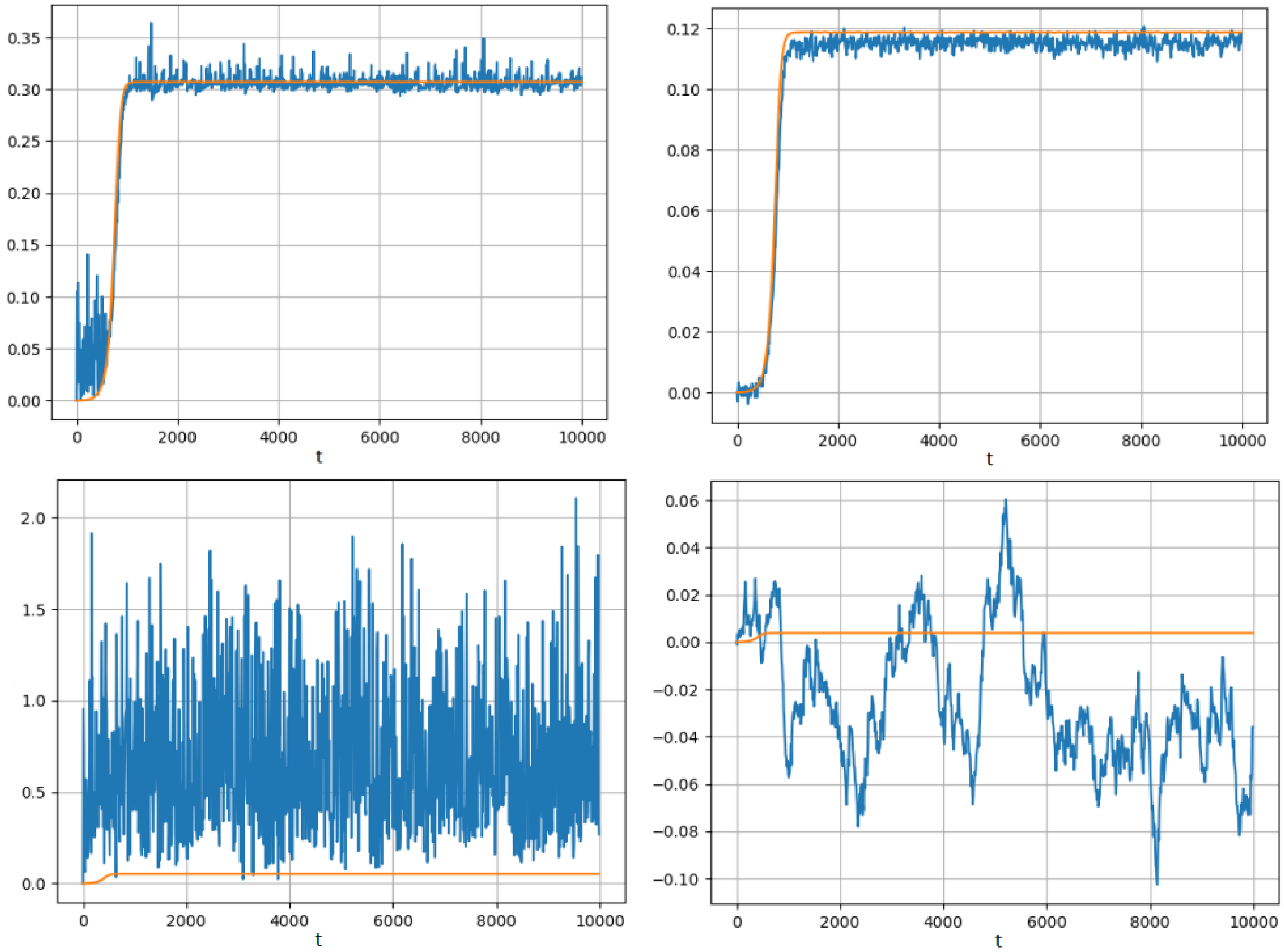


Figure. (Left) A plot of $\|u\|_{L^2}$ (blue) and $\|u_{app}\|_{L^2}$ (orange) against t for $U = 0$ (top) and (bottom) $U = 1$, and $\delta = 0.01$

(Right) A plot of $u(x = 0.22, t)$ (blue) and $u_{app}(x = 0.22, t)$ (orange) against t for $U = 0$ (top) and (bottom) $U = 1$, and $\delta = 0.01$

$\delta = -0.01$, pattern forming below criticality in the non-self-adjoint case

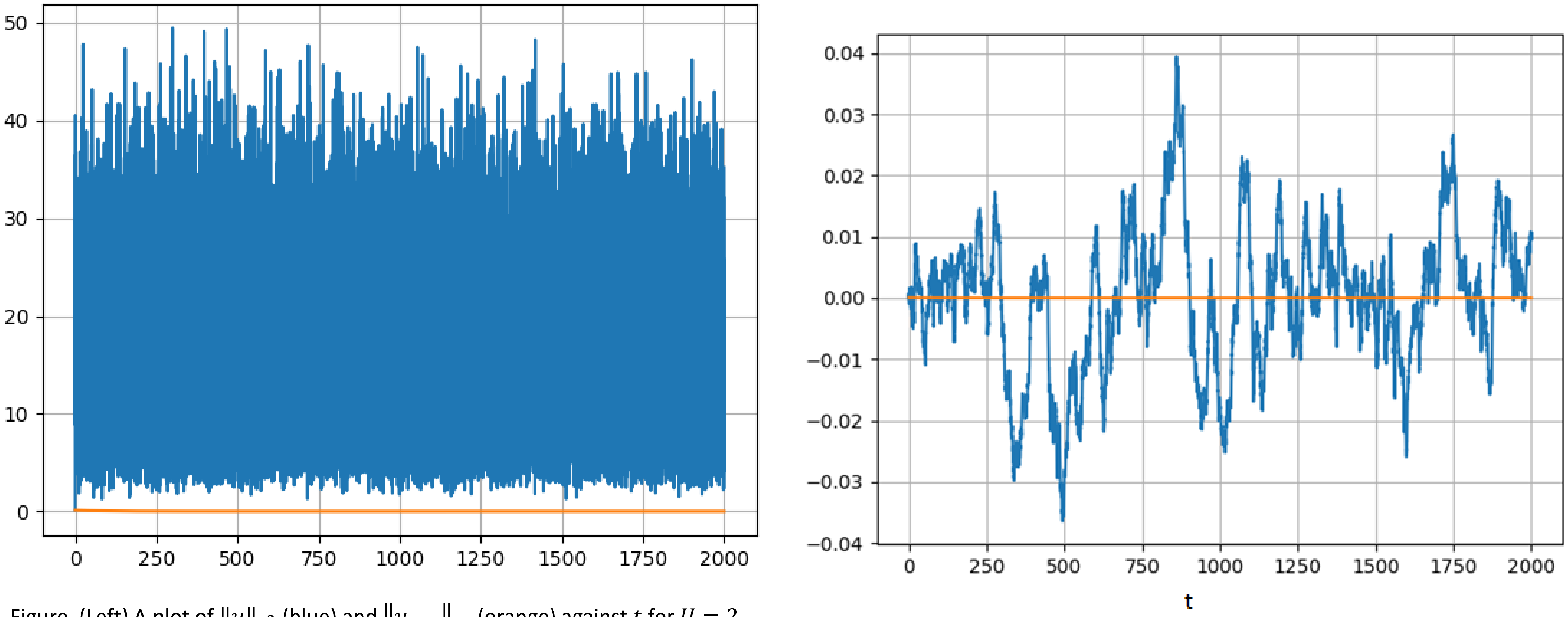


Figure. (Left) A plot of $\|u\|_{L^2}$ (blue) and $\|u_{app}\|_{L^2}$ (orange) against t for $U = 2$.
(Right) A plot of $u(x = 0.22, t)$ (blue) and $u_{app}(x = 0.22, t)$ (orange) against t for $U = 2$.

Further Research

In progress articles…

(Drysdale and Sipp) Second Order Approximations via WNLE for A Non-self-adjoint Problem in Fluid Dynamics

(Drysdale and Sipp) Using the Quasi-Basis to illustrate Non-normality and Nonlinearity Fluid Dynamics

(Drysdale and Needham) Exploring the “Weakly Nonlinear Limit for the Non-self-adjoint Ginzburg Landau Problem”

Other Ideas

- To use the quasi-basis for numerical purposes such as capturing boundary conditions for asymmetric problems.
- To investigate how to have higher order approximations in the stochastic case, and also the below-critical phenomena.
- To explore the semigroup properties if we define the domain of the operator to include the space, i.e. for sufficiently regular initial data in the space how do we prove that the solution stays in this space $\mathcal{H}(G) \cap \mathcal{H}(G^{-1})$.

Thank you for listening!

Bibliography

1. Sumer, B. & Fredsoe, Jorgen. (2006). Hydrodynamics Around Cylindrical Structures.

2. Praveen, T. & Eswaran, Vinayak. (2017). Transition to asymmetric flow in a symmetric sudden expansion: Hydrodynamics and MHD cases. Computers & Fluids. 148. 10.1016/j.compfluid.2017.02.011.

3. L. LANDAU and E. LIFSHITZ. Chapter iii - turbulence. In L. LANDAU and E. LIFSHITZ, editors, Fluid Mechanics (Second Edition), pages 95 – 156. Pergamon, second edition edition, 1987. ISBN 978-0-08-033933-7. doi: https://doi.org/10.1016/B978-0-08-033933-7.50011-8. URL http://www.sciencedirect.com/science/ article/pii/B9780080339337500118

4. J. Stuart. On the non-linear mechanics of wave disturbances in stable and unstable parallel flows part 1. the basic behaviour in plane poiseuille flow. Journal of Fluid Mechanics, 9(3):353–370, 1960.

5. W. C. Reynolds and M. C. Potter. Finite-amplitude instability of parallel shear flows. Journal of Fluid Mechanics, 27(3):465–492, 1967. doi: 10.1017/S0022112067000485

6. C. L. Pekeris and B. Shkoller. Stability of plane poiseuille flow to periodic disturbances of finite amplitude in the vicinity of the neutral curve. Journal of Fluid Mechanics, 29(1):31–38, 1967. doi: 10.1017/S0022112067000618

7. T. Herbert. On perturbation methods in nonlinear stability theory. Journal of Fluid Mechanics, 126:167–186, 1983.

8. K. Fujimura. The equivalence between two perturbation methods in weakly nonlinear stability theory for parallel shear flows. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 424(1867): 373–392, 1989.

9. K. Fujimura. Methods of centre manifold and multiple scales in the theory of weakly nonlinear stability for fluid motions. Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 434 (1892):719–733, 1991.

10. J.-M. Chomaz. Global instabilities in spatially developing flows: non-normality and nonlinearity. Annu. Rev. Fluid Mech., 37:357–392, 2005

11. D. Sipp and A. Lebedev. Global stability of base and mean flows: a general approach and its applications to cylinder and open cavity flows. Journal of Fluid Mechanics, 593:333–358, 2007.

12. M. Carini, F. Auteri, and F. Giannetti. Centre-manifold reduction of bifurcating flows. J. Fluid Mech, 767:109–145, 2015.

13. F. Bagarello. More mathematics for pseudo-bosons. Journal of Mathematical Physics, 54(6):063512, 2013.

Semigroup Properties and Ramifications

We have that both operators $(\mathcal{L}^{RGL}, D(\mathcal{L}^{RGL}))$ and $(\mathcal{L}^{CGL}, D(\mathcal{L}^{CGL}))$ generate C_0-semigroups.

We prove this by showing that

$$\hat{\mathcal{L}}_n^{RGL} = \frac{\partial^2}{\partial x^2} - U\frac{\partial}{\partial x} - (1 + c_2 x^2)$$

$$\hat{\mathcal{L}}_n^{CGL} = \frac{\partial^2}{\partial x^2} - (U + 0.2i)\frac{\partial}{\partial x} - (1 + c_2 x^2)$$

are both sectorial operators.

These sectorial operators are useful to us as it allows us to write our equations in integral form:

Semigroup Properties and Ramifications

For the RnsaGL, we have

$$u = e^{t\hat{\mathcal{L}}_n^{RGL}} u_0 + \int_0^t e^{(t-s)\hat{\mathcal{L}}_n^{RGL}} (\delta + C_2)u - u^3 \, dx$$

where $C_2 = \frac{U^2}{4} + \sqrt{c_2} + 1.$

For the CnsaGL, we have

$$u = e^{t\hat{\mathcal{L}}_n^{CGL}} u_0 + \int_0^t e^{(t-s)\hat{\mathcal{L}}_n^{CGL}} (\delta + C_2^c)u - |u|^2 u \, dx$$

where $C_2 = C_1 + \frac{1}{8}(U^2 - 0.4U \; - 0.04) + 1.$

This is useful when we derive the error bounds for our equations!

Compact Resolvent and Basis Ramifications

We also have both the operators generate a compact resolvent. This means that our operators have purely discrete spectrum.

This means that we can use theorems from Krejčiřík et al. [14] in order to determine whether the eigenvectors form a Riesz Basis or not.

We give the following definitions:

- Definition (Schauder Basis). Let X be a Banach space. A set of vectors $\{e_n\}_{n=0}^{\infty}$ is a basis if every $\phi \in X$ has a unique expansion in the vectors $\{e_n\}_{n=0}^{\infty}$.

- Definition (Riesz Basis). Let X be a Banach space. A set of vectors $\{e_n\}_{n=0}^{\infty}$ normalised to 1 in is a Riesz Basis or unconditional basis if it forms a basis and the inequality

$$\forall\, \phi \in X, \qquad C^{-1}\|\phi\| \leq \sum_{n=1}^{\infty} |\langle e_n, \phi\rangle|^2 \leq C\|\phi\|$$

holds with a positive constant C independent of ϕ.

Compact Resolvent and Basis Ramifications

We also present the following definitions of pseudospectra and trivial pseudospectra.

Definition. Let L be an operator and let us define its spectrum $\sigma(L)$. Given a positive number ϵ, we define the ϵ-pseudospectrum of an operator

$$\sigma_\epsilon(L) \coloneqq \sigma(L) \cup \{ z \in \mathbb{C} \mid \|(H - z)^{-1}\| > \epsilon^{-1}\}$$

Definition. We say that the pseudospectrum of H is trivial if there exists a fixed constant C such that for all $\epsilon > 0$,

$$\sigma_\epsilon(L) \subseteq \{ z \in \mathbb{C} \mid dist\left(z, \sigma(H)\right) \leq C\epsilon\}$$

Compact Resolvent and Basis Ramifications

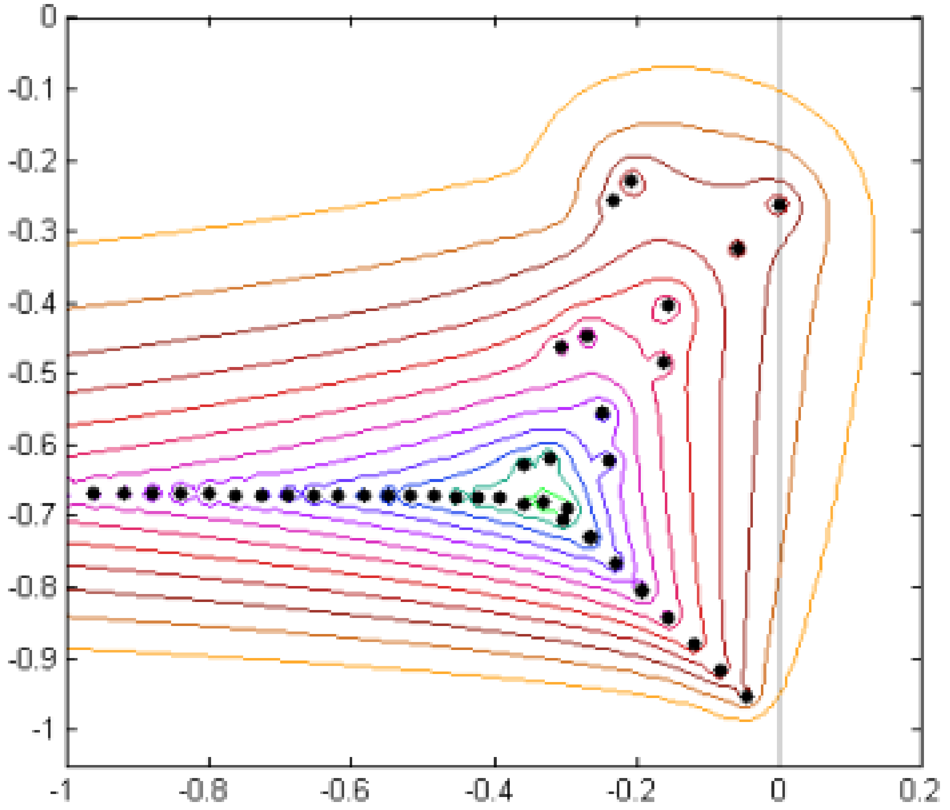


Figure 6a. (Trivial Pseudospectra) Pseudospectra of the Orr-Sommerfeld operator $i\alpha T^{-1}\left(\frac{1}{i\alpha R}T^2-(1-x^2)T^2\;-2\right)$ [16]

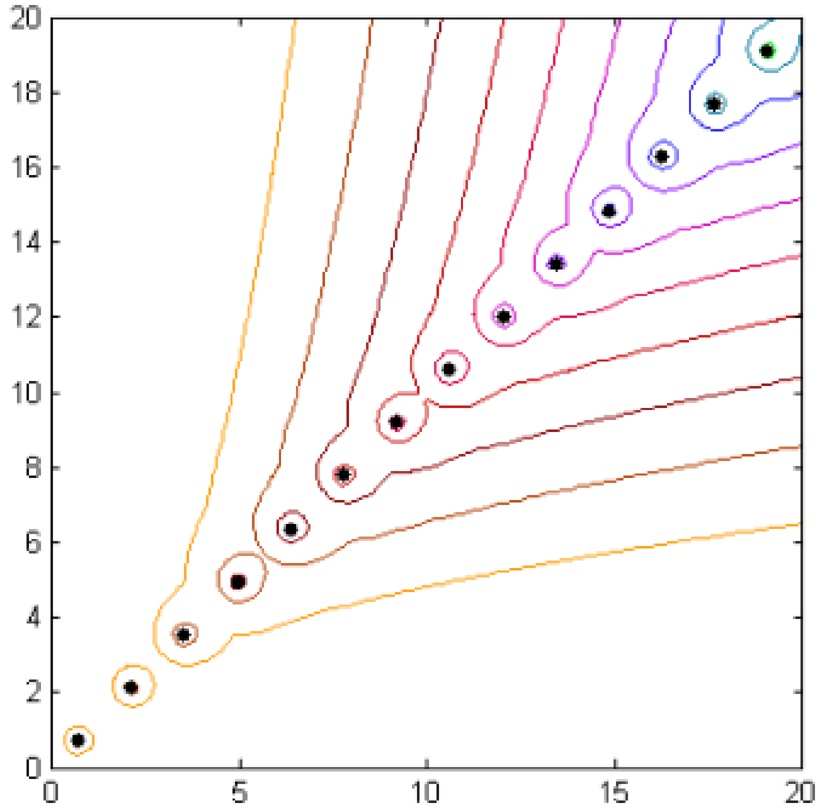


Figure 6b. (Non-trivial Pseudospectra) of the Davies Oscillator $-\frac{d^2}{dx^2}+ix^2$ operator [16]

Theoretical Framework– Domains of Test cases

Next consider the following two spaces. The first is a space of real valued functions, whereas the second is a space that maps from real valued functions to complex valued functions.

$$H^2_{Vr}(\mathbb{R}) = \{\, u \in H^2(\mathbb{R},\mathbb{R}):\quad Vu \in L^2(\mathbb{R},\mathbb{R})\},$$

and

$$H^2_{Vc}(\mathbb{R}) = \{\, u \in H^2(\mathbb{R},\mathbb{C}):\quad Vu \in L^2(\mathbb{R},\mathbb{C})\}.$$

We equip the spaces $H^2_{Vr}(\mathbb{R})$ and $H^2_{Vc}(\mathbb{R})$, with the respective norms

$$\|u\|_{H_{Vr}} = \|u\|_{H^2(\mathbb{R},\mathbb{R})} + \|Vu\|_{L^2(\mathbb{R},\mathbb{R})}$$

and

$$\|u\|_{H_{Vc}} = \|u\|_{H^2(\mathbb{R},\mathbb{C})} + \|Vu\|_{L^2(\mathbb{R},\mathbb{C})}.$$

Under these conditions, $H^2_{Vr}(\mathbb{R})$ and $H^2_{Vc}(\mathbb{R})$ are Banach spaces.

- Theorem (Biorthogonality of Basis vectors from Davies (2007) [3]). Let $\mathcal{B}$ be a Banach space and f be a function in $\mathcal{B}$. If $\{\psi_n\}_{n=1}^{\infty}$ is a basis in a Banach space $\mathcal{B}$, then there exists a $\phi_n \in \mathcal{B}^*$ such that the Fourier coefficients α_n are given by $\alpha_n := \langle f, \phi_n \rangle$. The pair of sequences $\{\psi_n\}_{n=1}^{\infty}$, $\{\phi_n\}_{n=1}^{\infty}$ is biorthogonal in the sense that $\langle f_n, \phi_m \rangle = \tilde{\delta}_{n,m}$ for all m, n,
  Proof. See Lemma 3.3.1 from Davies (2007) [3].
- Theorem (Basis Condition of the Uniform Boundedness of Projections). Let H be an operator with a compact resolvent and let us denote that has a basis of eigenvectors by ϕ_n. Let us denote the associated one-dimensional projections as
$$P_k := \psi_k \langle \phi_k, \cdot \rangle. \tag{85}$$
  If $\{\phi_k\rangle_{k=0}^{\infty}$ is a basis, then both P_k and $\sum_{k=0}^{N}$ are uniformly bounded in $\mathcal{B}$.
  Proof. See Krejcirik et al 2015 [6].

Theorem (Divergence of Projections for eigenvectors of $\mathcal{L}^{RGL}$). Let $\hat{e}_n$ and $\hat{e}_n^\dagger$ be defined as in (81) and (82) respectively, which form a biorthogonal set. We define the projections

$$P_n = \langle \hat{e}_n^\dagger, \cdot \rangle e_n. \tag{86}$$

We have that

$$||P_n|| = \frac{e^{2(nC)^{\frac{1}{2}}}}{2\sqrt{\pi}(C)^{\frac{1}{4}} n^{\frac{1}{2}}}(1 + O(n^{-\frac{1}{2}})) \tag{87}$$

and hence diverge as $n \longrightarrow \infty$.

Theorem (Divergence of Projections for eigenvectors of $\mathcal{L}^{CGL}$). Let $\hat{e}_n^c$ and $(\hat{e}_n^c)^\dagger$ be defined as in (81) and (82) respectively, which form a biorthogonal set. We define the projections

$$P_n = \langle (\hat{e}_n^c)^\dagger, \cdot \rangle e_n^c. \tag{96}$$

As $n \longrightarrow \infty$

$$||P_n|| \geq |\frac{e^{2(nC)^{\frac{1}{2}}}}{2\sqrt{\pi}(C)^{\frac{1}{4}} n^{\frac{1}{2}}}| \tag{97}$$

thus $||P_n||_{L^2}$ diverges as $n \longrightarrow \infty$.

- Theorem (The Existence of Quasi-Bases under certain conditions). Let $\hat{E} = \{\hat{e}_n\}_{n=0}^{\infty}$ be an orthonormal basis in a Hilbert Space $\mathcal{H}$. Consider the two sets $E = \{e_n\}_{n=0}^{\infty}{}^{\infty}$ and $E^{\dagger} = \{e_n^{\dagger}\}_{n=0}^{\infty}$, such that for the metric operator $G^{\frac{1}{2}}$ where $G^{\frac{1}{2}} e_n = \hat{e}_n$ and $(G^{\frac{1}{2}})^{-1} e_n^{\dagger} = \hat{e}_n$. Let the domain, $D(G^{\frac{1}{2}})$ of $G^{\frac{1}{2}}$ and $D((G^{\frac{1}{2}})^{-1})$ be defined as in (4.33) and (4.34). Given the following assumptions:

  - E_n and $\hat{E}_n$ are bi-orthogonal.
  - If $f \in \mathcal{H}(G)$ is orthogonal to all the e_n then $f = 0$.
  - If $f \in \mathcal{H}(G^{-1})$ is orthogonal to all the $e_n^{\dagger}$ then $f = 0$.

  The E_n and $\hat{E}_n$ are quasi-basis for the space $H(G^{-1}) \cap H(G)$.

Proof. See Bagarello (2013) [10].