Novelty and surprises in the theory of odd-order linear differential operators

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Introduction

* My main interest: boundary value problems for integrable PDEs such as NLS, KdV,.., but also any *linear* (constant coefficients) ones.

*Linear BVP on bounded domain involve linear differential operators that may not be self-adjoint, such as

$$L = \partial_x^3 \text{ on } \{f : I \to \mathbb{R},\ f \in \mathcal{S}(I), f \text{ satisfies given BC}\}$$

with

$$I = [0, \infty) \quad or \quad I = [0, 1].$$

*This talk is mainly about what the PDE approach can contribute to the understanding of the spectral structure of the operators

*Main example: the *Stokes*, or *Airy, equation*

$$u_t = Lu, \quad u = u(x, t), \ \ x \in I, \ \ t > 0,$$

Boundary value problems on $[0,1]$ - a question

on $[0,1] \times (0,\infty)$:

$$u_t = u_{xxx}, \qquad u(x,0) = u_0(x), \qquad \text{3 homogeneous BC (?)}$$

Separate variables, and use eigenfunctions of

$$L = \frac{d^3}{dx^3} \text{ on } \mathcal{D} = \{f \in C^\infty([0,1]) : f \text{ satisfies 3 } bc's\} \subset L^2[0,1]$$

L is not generally selfadjoint (because of BC), but may have infinitely many real eigenvalues λ_n, with eigenfunctions $\{\phi_n(x)\}$

Question: does it hold $\quad u(x,t) = \sum_n (u_0, \phi_n) \mathrm{e}^{i\lambda_n^3 t} \phi_n(x)$?

Eigenvalues as singularities of a complex-valued function

Classical idea in the context of elliptic linear PDE - Watson's transformation:

convert series to an integral, via a residue calculation, using a complex valued function with simple poles at the eigenvalues

$$\sum_{n=-\infty}^{\infty} f(n) = \int_C \frac{f(\lambda)}{1-\mathrm{e}^{2\pi i\lambda}} d\lambda,$$

where C is any contour enclosing the real λ-axis but none of the singularities (including possible singularities at ∞) of f .

The **Unified Transform** approach (*Fokas, P,...*) goes the other way,: derive, in general, a complex integral representation for the solution of linear BVP - which *may be* equivalent to a series

Integral representation of the solution of linear BVP

$$u_t(x,t) + iP(-i\partial_x)u(x,t) = 0, \quad x \in [0,1],\ t > 0, \qquad P \textit{ polynomial}$$

with given IC at $t = 0$ and BCs at $x = 0$, and $x = 1$

$$\{\overset{(IC)}{u_0(x)},\ \overset{(BC)}{f_j(t)}\} \xrightarrow{direct} \{\overset{(\lambda\in\mathbb{C})}{\zeta^{\pm}(\lambda),\ \Delta(\lambda)}\} \xrightarrow{inverse}$$

$$u(x,t) = \frac{1}{2\pi}\left\{\int_{\Gamma^+} \mathrm{e}^{i\lambda x - iP(\lambda)t}\frac{\zeta^+(\lambda)}{\Delta(\lambda)}d\lambda \int_{\Gamma^-} \mathrm{e}^{i\lambda(x-1) - iP(\lambda)t}\frac{\zeta^-(\lambda)}{\Delta(\lambda)}d\lambda\right\}$$

$$\Gamma^{\pm} = \{\lambda \in \mathbb{C} : Im\, P(\lambda) = 0\} \cap \mathbb{C}^{\pm}$$

(on this contour, $\mathrm{e}^{-iP(\lambda)t}$ *is purely oscillatory)*

Singularities in the representation (P, Smith)

$$u_t + Lu = 0,\ x \in I \qquad\qquad L = iP(-i\partial_x)(+\text{ b.c.})$$

$$u(x,t) = \frac{1}{2\pi}\left\{\int_{\Gamma^+} \mathrm{e}^{i\lambda x - iP(\lambda)t}\frac{\zeta^+(\lambda)}{\Delta(\lambda)}d\lambda \int_{\Gamma^-} \mathrm{e}^{i\lambda(x-1) - iP(\lambda)t}\frac{\zeta^-(\lambda)}{\Delta(\lambda)}d\lambda\right\}$$

• $\zeta^{\pm}(\lambda)$, are *transforms* of the given initial and boundary conditions
• $\Delta(\lambda)$ is a determinant (arising in the solution of the so-called *global relation*) whose zeros are (*essentially*) the discrete eigenvalues of L.

If the associated eigenfuctions form a basis (say the operator+bc is self-adjoint...), this representation is equivalent to the series one

* Uniformly convergent representation, in contrast to non-uniformly (slow) converging real integral/series representation

* Fast exponential decay can be harnessed for accurate numerical evaluations

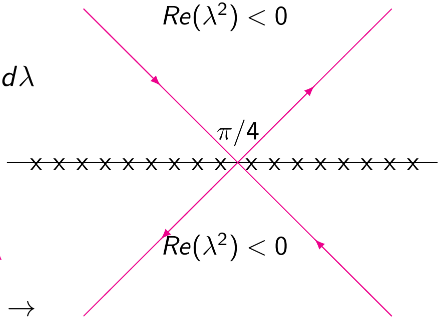
Example: homogeneous Dirichlet problem for the heat equation on $(0,1)$

UT solution representation:

$$2\pi u(x,t) =$$

$$\int_{\Gamma^+} \mathrm{e}^{i\lambda x-\lambda^2 t}\frac{(2\mathrm{e}^{-i\lambda}-\mathrm{e}^{i\lambda})\hat{u}_0(\lambda)-\mathrm{e}^{i\lambda}\hat{u}_0(-\lambda)}{\mathrm{e}^{-i\lambda}-\mathrm{e}^{i\lambda}}d\lambda$$

$$+\int_{\Gamma^-} \mathrm{e}^{i\lambda(x-1)-\lambda^2 t}\frac{\hat{u}_0(-\lambda)-\hat{u}_0(\lambda)}{\mathrm{e}^{-i\lambda}-\mathrm{e}^{i\lambda}}d\lambda.$$



$\lambda_n = \pi n$ zeros of $\Delta(\lambda) = \mathrm{e}^{-i\lambda} - \mathrm{e}^{i\lambda}$

Using Cauchy+residue calculation $\rightarrow$

$$u(x,t) = \frac{2}{\pi}\sum_n \mathrm{e}^{-\lambda_n^2 t}\sin(\lambda_n x)\hat{u}_0^s(\lambda_n) \qquad \textit{sine series}$$

$u_t = u_{xxx}$ on $[0,1]$, $u(x,0) = u_0(x)$, 3 BCs

Th zeros of $\Delta(\lambda)$ are an infinite set accumulating only at infinity; (asymptotic) location is given by general results in complex analysis , and **depends crucially on the boundary conditions**

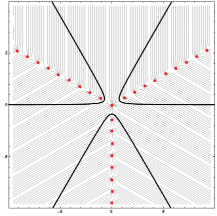
boundary conditions : $u(0,t) = u(1,t) = 0, \quad u_x(0,t) = \beta u_x(1,t),$

$$\Delta(\lambda) = \mathrm{e}^{-i\lambda} + \alpha \mathrm{e}^{-i\alpha\lambda} + \alpha^2 \mathrm{e}^{-i\alpha^2\lambda} + \beta(\mathrm{e}^{i\lambda} + \alpha \mathrm{e}^{i\alpha\lambda} + \alpha^2 \mathrm{e}^{i\alpha^2\lambda}), \ \alpha = \mathrm{e}^{\frac{2i\pi}{3}}.$$

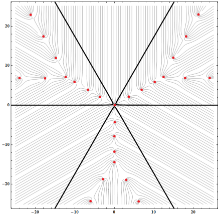
- $\beta = 1$: the zeros are on the integration contour $\rightarrow$ residue computation (with no contour deformation)
- $0 < \beta < 1$: the zeros are asymptotic to the integration contour $\rightarrow$ residue computation
- $\beta = 0$: the contour of integration cannot be deformed as far the asymptotic directions of the zeros
  $\Longrightarrow$ the underlying differential operator does not admit a Riesz basis of eigenfunctions

Zeros of $\Delta(\lambda)$ as a function of β solid lines = integration contour

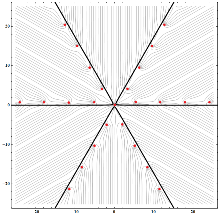
BC: $u(0,t) = u(1,t) = 0,\ u_x(0,t) = \beta u_x(1,t)$



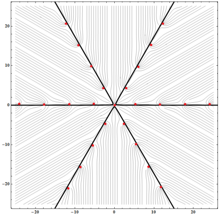
(a) $\beta = 0$



(b) $\beta = 0.001$



(c) $\beta = 0.5$



(d) $\beta = 0.8$

A transform pair - examples tailored to a specific BVP

$$u_t = u_{xxx}, \qquad IC: \ u(x,0) = u_0(x), \qquad 3BCs$$

$$u(x,t) = \frac{1}{2\pi}\left\{\int_{\Gamma^+} \mathrm{e}^{i\lambda x - i\lambda^3 t}\frac{\zeta^+(\lambda)}{\Delta(\lambda)}d\lambda + \int_{\Gamma^-} \mathrm{e}^{i\lambda(x-1) - i\lambda^3 t}\frac{\zeta^-(\lambda)}{\Delta(\lambda)}d\lambda\right\}$$

Problem 1:

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ u_x(0,t) = \frac{1}{2}u_x(1,t). \end{cases}$$

The integral representation is equivalent to a series, by calculating the residues around the poles on the contour.

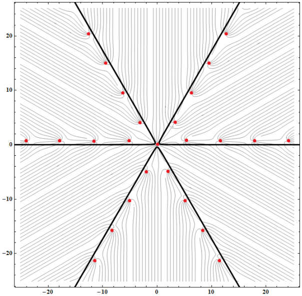
Problem 2:

$$u(0,t) = u(1,t) = u_x(0,t) = 0.$$

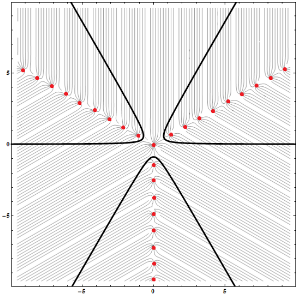
The integral representation cannot be deformed to a series representation.

Zeros of $\Delta(\lambda)$ as a function of β

BC: $u(0,t) = u(1,t) = 0,\ u_x(0,t) = \beta u_x(1,t)$



(a) Problem 1 - $\beta = \frac{1}{2}$



(b) Problem 2 - $\beta = 0$

On the solid lines, Re$(-i\lambda^3)$=0 - separating the regions where the t exponential decays or grows as $\lambda \to \infty$

Examples of transform pair tailored to a specific BVP

Problem 1:

$u(x,0) = f(x),\ u(0,t) = u(1,t) = 0,\ u_x(0,t) = \frac{1}{2} u_x(1,t).$

$$f(x) \to F[f](\lambda) = \begin{cases} \int_0^1 E^+(x,\lambda) f(x) dx & \lambda \in \Gamma^+ \\ \int_0^1 E^-(x,\lambda) f(x) dx & \lambda \in \Gamma^- \end{cases}$$

$$E^+(x,\lambda) = \frac{1}{2\pi\Delta(\lambda)} \left[\mathrm{e}^{-i\lambda x} (\mathrm{e}^{i\lambda} + 2\alpha \mathrm{e}^{-i\alpha\lambda} + 2\alpha^2 \mathrm{e}^{-i\alpha^2\lambda}) + \ldots \right]$$

$$E^-(x,\lambda) = \frac{-\mathrm{e}^{-i\lambda}}{2\pi\Delta(\lambda)} \left[\mathrm{e}^{-i\lambda x} (2 + \alpha^2 \mathrm{e}^{-i\alpha\lambda} + \alpha \mathrm{e}^{-i\alpha^2\lambda}) + \ldots \right]$$

with $\Delta(\lambda) = \mathrm{e}^{i\lambda} + \alpha \mathrm{e}^{i\alpha\lambda} + \alpha^2 \mathrm{e}^{i\alpha^2\lambda} + 2(\mathrm{e}^{-i\lambda} + \alpha \mathrm{e}^{-i\alpha\lambda} + \alpha^2 \mathrm{e}^{-i\alpha^2\lambda})$.

$$F(\lambda) \to f[F](x) = \left(\int_{\Gamma^+} + \int_{\Gamma^-} \right) \mathrm{e}^{i\lambda x} F(\lambda) d\lambda = \sum_{\sigma:\Delta(\sigma)=0} \int_{C_\sigma} \mathrm{e}^{i\lambda x} F(\lambda) d\lambda.$$

Examples of transform pair tailored to a specific BVP

Problem 2:

$u(x,0) = f(x),\ u(0,t) = u(1,t) = u_x(0,t) = 0.$

$$f(x) \to F[f](\lambda) = \begin{cases} \int_0^1 E^+(x,\lambda) f(x) dx & \lambda \in \Gamma^+ \\ \int_0^1 E^-(x,\lambda) f(x) dx & \lambda \in \Gamma^- \end{cases}$$

$$E^+(x,\lambda) = \frac{1}{2\pi\Delta(\lambda)} \left[\mathrm{e}^{-i\lambda x}(\alpha \mathrm{e}^{-i\alpha\lambda} + \alpha^2 \mathrm{e}^{-i\alpha^2\lambda}) - \alpha \mathrm{e}^{-i\alpha\lambda x} \ldots \right]$$

$$E^-(x,\lambda) = \frac{-\mathrm{e}^{-i\lambda}}{2\pi\Delta(\lambda)} \left[\mathrm{e}^{-i\lambda x} + \mathrm{e}^{-i\alpha\lambda x} + \alpha^2 \mathrm{e}^{-i\alpha^2\lambda x} \right]$$

with $\Delta(\lambda) = \mathrm{e}^{-i\lambda} + \alpha \mathrm{e}^{-i\alpha\lambda} + \alpha^2 \mathrm{e}^{-i\alpha^2\lambda}$.

$$F(\lambda) \to f[F](x) = \left(\int_{\Gamma^+} + \int_{\Gamma^-} \right) \mathrm{e}^{i\lambda x} F(\lambda) d\lambda, \quad \text{no series.}$$

Spectral decomposition of differential operators: Gel'fand generalised eigenfunctions

There are no nonzero eigenfunctions of

$$(Sf)(x) = -f''(x), \qquad \forall f \in \mathcal{S}[0,\infty) \text{ such that } f(0) = 0.$$

Define instead a functional $F[\cdot](\lambda) \in (\mathcal{S}[0,\infty))'$:

$$F[Sf](\lambda) = \lambda^2 F[f](\lambda), \quad \forall \lambda \in \mathbb{R}$$

For this example,

$$F[f](\lambda) = \frac{2}{\pi}\int_0^\infty \sin(\lambda x) f(x) dx, \quad \text{(sine transform on } [0,\infty)).$$

Gel'fand called this *eigenfunctional*, or *generalised eigenfunctions* (and $\lambda \in \mathbb{R}$ eigenvalues)
This notion depends on self-adjointness to prove any completeness result and to define the spectral representation of the operator.

More general spectral decomposition of differential operators: augmented eigenfunctions

Example: $u_t + \partial_x^n u = 0$, $x \in (0,1)$ + initial and homogeneous BC

Augmented eigenfunctions of $L = \partial_x^n$ on

$$\mathcal{D} = \{f \in C^\infty : f \text{ satisfies the boundary conditions } \} \subset L^2:$$

are (eigen)functionals

$$F_\lambda[f],\ \lambda \in \Gamma, \quad \Gamma = \{\lambda : Im\lambda^n = 0\}$$

such that there exist reminder functionals $R[\cdot](\lambda)$ with

$$F_\lambda[Lf] = \lambda^n F_\lambda[f] + R_\lambda[f],\ \ \lambda \in \Gamma,\ and \begin{cases} \int_\Gamma \mathrm{e}^{i\lambda x} R_\lambda[f] d\lambda = 0 \\ or \\ \int_\Gamma \mathrm{e}^{i\lambda x} \frac{R_\lambda[f]}{\lambda^n} d\lambda = 0. \end{cases}$$

Diagonalisation of the operator

If the eigenfunctionals form a complete family ($F_\lambda[f] = 0$ iff $f = 0$), then integration over Γ gives rise to a non-self-adjoint analogue of the spectral representation of L:

$$\int_\Gamma \mathrm{e}^{i\lambda x} F_\lambda[Lf] d\lambda = \int_\Gamma \lambda^n \mathrm{e}^{i\lambda x} F_\lambda[f] d\lambda,$$

or

$$\int_\Gamma \frac{1}{\lambda^n} \mathrm{e}^{i\lambda x} F_\lambda[Lf] d\lambda = \int_\Gamma \mathrm{e}^{i\lambda x} F_\lambda[f] d\lambda.$$

Hence they provides an effective diagonalisation modulo functions analytic in a certain sector of the complex spectral plane

Important: *Completeness follows from the PDE theory, rather than from self-adjointness*

Diagonalisation of such operators in very general situations - talk of Dave Smith later in the meeting

Another application of complex analytical ideas: Time-periodic boundary conditions

The problem:

$$\partial_t u(x,t)+P(-i\partial_x)u(x,t)=0, \qquad u(x,0)=u_0(x), \qquad x\in[0,1],$$

given appropriate time-periodic boundary conditions at $x=0$ and $x=1$.

Is $u(x,t)$ time-periodic (exactly or asymptotically)?
With the same period as the BC?

Examples

$$\text{(free Schrödinger)} \quad u_t - iu_{xx} = 0,$$
$$\text{(Stokes)} \quad u_t + u_{xxx} = 0.$$

Necessary conditions for periodicity = analyticity constraint

Step 1 *Assuming* time periodicity, one can derive necessary conditions (based on analyticity constraints) for the solvability of the D-to-N map.

Step 2 To prove that the solution/unknown boundary values is (asymptotically) periodic, one needs to analyse the integral or series representation of the solution.

Assuming that the necessary conditions for periodicity hold,:

- For **free Schrödinger** with time-periodic Dirichlet boundary conditions of period τ, the solution is time periodic only if τ and $2/\pi$ are linearly dependent over $\mathbb{Q}$.
- For the **Stokes** equation, with time-periodic *Dirichlet-type* conditions, the solution is always asymptotically time periodic, with the same period as the boundary data.

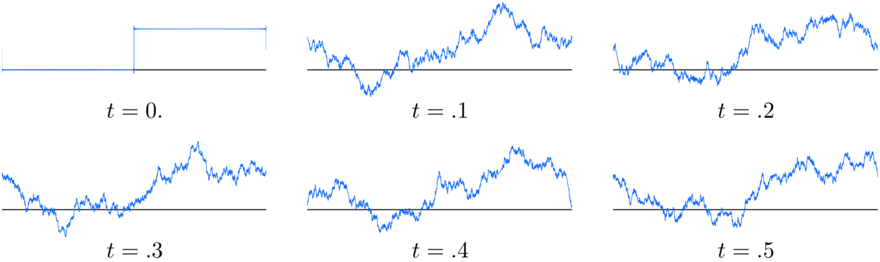
Finally: periodic revival for third order dispersion

Talbot effect, or the revival property: in linear periodic problem, it refers to the propagation, at rational values of the time, of any initial discontinuities - at other times, the solution is continuous but nowhere differentiable.

Studied for linear Schrödinger, then also for Stokes equation

$u_t = u_{xxx}$, $u(x, 0)$=step function, periodic boundary conditions

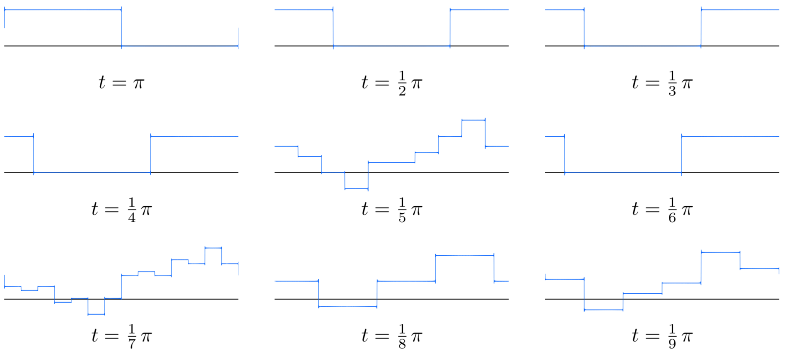
(Peter Olver)



Periodic Airy - solution at "rational" times

$u_t = u_{xxx}$, $u(x,0)$=step function, periodic boundary conditions

Revival of the initial discontinuities:

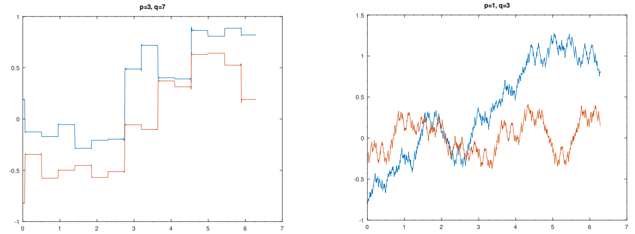


Revival property for quasi-periodic Stokes

Quasi-periodic conditions:

$$e^{i2\pi\theta}\partial_x^j u(t,0) = \partial_x^j u(t,2\pi), \ (j=0,1), \quad 0<\theta<1.$$

Revival property still hold for 2nd order problems (free space Schrödinger), for any value of θ - but it holds for Stokes *only* for $\theta \in \mathbb{Q}$.



(a) $t = 2\pi\frac{3}{7}$, $\theta = 1/4$ (b) $t = 2\pi\frac{1}{3}$, $\theta = \sqrt{2}/3$

More on this in the talk talk of George Farmakis later in the meeting

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UT: complex (RH) formulation of integral transforms

Example: The ODE

$$\mu_x(x,\lambda) - i\lambda\mu(x,\lambda) = u(x), \quad \lambda \in \mathbb{C}$$

encodes the Fourier transform

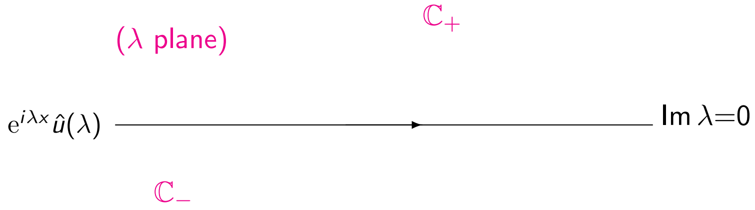
direct transform: via solving the ODE for $\mu(x,\lambda)$ *bounded in $\lambda \in \mathbb{C}$*
inverse transform: via solving a RH problem

Given $u(x)$ (smooth and decaying), solutions μ^+ and μ^- bounded (wrt λ) in $\mathbb{C}^+$ and $\mathbb{C}^-$ are

$$\mu^+ = \int_{-\infty}^{x} \mathrm{e}^{i\lambda(x-y)} u(y)dy,\ \lambda \in \mathbb{C}^+; \quad \mu^- = \int_{\infty}^{x} \mathrm{e}^{i\lambda(x-y)} u(y)dy,\ \lambda \in \mathbb{C}^-$$

$$\Longrightarrow \textit{for } \lambda \in \mathbb{R} \qquad (\mu^+ - \mu^-)(\lambda) = \mathrm{e}^{i\lambda x}\hat{u}(\lambda) \qquad \textbf{DIRECT}$$

Fourier inversion theorem



Given $\hat{u}(\lambda)$, $\lambda \in \mathbb{R}$, a function μ analytic everywhere in $\mathbb{C}$ except the real axis is the solution of a RH problem (via *Plemelj formula*):

$$\mu(\lambda, x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\zeta x}\hat{u}(\zeta)}{\zeta - \lambda} d\zeta$$

$$\Rightarrow u(x) = \mu_x - i\lambda\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\zeta x}\hat{u}(\zeta) d\zeta, \; x \in \mathbb{R}$$

INVERSE

Trasforms for BVP for (linear) PDEs

PDE as the *compatibility condition of a pair of linear ODEs*

Example: linear evolution problem

$$u_t + u_{xxx} = 0 \Longleftrightarrow \mu_{xt} = \mu_{tx} \text{ with } \mu : \begin{cases} \mu_x - i\lambda\mu = u \\ \mu_t - i\lambda^3\mu = u_{xx} + i\lambda u_x - \lambda^2 u \end{cases}$$

and $\lambda \in \mathbb{C}$.

BVP main idea: derive a transform pair (via RH) from this **system** of ODEs (with both x and t as parameters)

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equivalently, divergence form (classical for elliptic case)

$$u_t + u_{xxx} = 0 \Longleftrightarrow [\mathrm{e}^{-i\lambda x - i\lambda^3 t} u]_t - [\mathrm{e}^{-i\lambda x - i\lambda^3 t}(u_{xx} + i\lambda u_x - \lambda^2 u)]_x = 0$$