Open problem: Stability of solitons in the 1D Dirac model

Atanas Stefanov, University of Alabama - Birmingham

Mathematical aspects of the Physics with Non-self-Adjoint Operators
Banff Research Station
July 12^{th}, 2022

Dirac equation

The stability of solitons all Dirac models is wide open.

Dirac equation

The stability of solitons all Dirac models is wide open. In its simplest, 1D Soler version is

$$\begin{cases} iu_t = \partial_x v + (1 - (|u|^2 - |v|^2)^k)u \\ iv_t = -\partial_x u - (1 - (|u|^2 - |v|^2)^k)v \end{cases} \tag{1}$$

Dirac equation

The stability of solitons all Dirac models is wide open.
In its simplest, 1D Soler version is

$$\begin{cases} iu_t = \partial_x v + (1 - (|u|^2 - |v|^2)^k)u \\ iv_t = -\partial_x u - (1 - (|u|^2 - |v|^2)^k)v \end{cases} \tag{1}$$

Solitons are special solutions in the form
$\begin{pmatrix} u \\ v \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$, $|\omega| < 1$, which satisfy

$$\begin{cases} \omega\phi = \partial_x \psi + (1 - (\phi^2 - \psi^2)^k)\phi \\ \omega\psi = -\partial_x \phi - (1 - (\phi^2 - \psi^2)^k)\psi \end{cases} \tag{2}$$

Explicit solutions

With $\sigma = \sqrt{1-\omega^2}$, the exact solutions are localized and (Lee et. al. in 1975; Chugunova-Pelinovsky'06, Cooper et. al'10)

$$\begin{aligned}
\phi(x) &= \cosh(k\sigma x)\sqrt{\frac{1+\omega}{1+\omega\cosh(2k\sigma x)}}\left[\frac{(k+1)\sigma^2}{1+\omega\cosh(2k\sigma x)}\right]^{\frac{1}{2k}} \\
\psi(x) &= \sinh(k\sigma x)\sqrt{\frac{1-\omega}{1+\omega\cosh(2k\sigma x)}}\left[\frac{(k+1)\sigma^2}{1+\omega\cosh(2k\sigma x)}\right]^{\frac{1}{2k}}
\end{aligned}$$

Explicit solutions

With $\sigma = \sqrt{1-\omega^2}$, the exact solutions are localized and (Lee et. al. in 1975; Chugunova-Pelinovsky'06, Cooper et. al'10)

$$
\begin{aligned}
\phi(x) &= \cosh(k\sigma x)\sqrt{\frac{1+\omega}{1+\omega\cosh(2k\sigma x)}}\left[\frac{(k+1)\sigma^2}{1+\omega\cosh(2k\sigma x)}\right]^{\frac{1}{2k}} \\
\psi(x) &= \sinh(k\sigma x)\sqrt{\frac{1-\omega}{1+\omega\cosh(2k\sigma x)}}\left[\frac{(k+1)\sigma^2}{1+\omega\cosh(2k\sigma x)}\right]^{\frac{1}{2k}}
\end{aligned}
$$

Open problems: Uniqueness?

Explicit solutions

With $\sigma = \sqrt{1-\omega^2}$, the exact solutions are localized and (Lee et. al. in 1975; Chugunova-Pelinovsky'06, Cooper et. al'10)

$$\begin{aligned} \phi(x) &= \cosh(k\sigma x)\sqrt{\frac{1+\omega}{1+\omega\cosh(2k\sigma x)}}\left[\frac{(k+1)\sigma^2}{1+\omega\cosh(2k\sigma x)}\right]^{\frac{1}{2k}} \\ \psi(x) &= \sinh(k\sigma x)\sqrt{\frac{1-\omega}{1+\omega\cosh(2k\sigma x)}}\left[\frac{(k+1)\sigma^2}{1+\omega\cosh(2k\sigma x)}\right]^{\frac{1}{2k}} \end{aligned}$$

Open problems: Uniqueness? Non-existence of the waves for $|\omega| > 1$?

Linearized problem

For the simplest case $k = 1$, linearization leads to the following **Hamiltonian** eigenvalue problem

$$\begin{cases} \mathcal{L}_+ z_1 = -\lambda z_2 \\ \mathcal{L}_- z_2 = \lambda z_1 \end{cases} \tag{3}$$

where

Linearized problem

For the simplest case $k = 1$, linearization leads to the following **Hamiltonian** eigenvalue problem

$$\begin{cases} \mathcal{L}_+ z_1 = -\lambda z_2 \\ \mathcal{L}_- z_2 = \lambda z_1 \end{cases} \tag{3}$$

where

$$\begin{cases} \mathcal{L}_- = \begin{pmatrix} 1 - \omega - (\phi^2 - \psi^2) & \partial_x \\ -\partial_x & -1 + (\phi^2 - \psi^2) - \omega \end{pmatrix} \\ \mathcal{L}_+ = \mathcal{L}_- - 2 \begin{pmatrix} \phi^2 & -\phi\psi \\ -\phi\psi & \psi^2 \end{pmatrix} \end{cases}$$

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \tag{4}$$

has no solutions.

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \tag{4}$$

has no solutions.
Open problem: Prove spectral stability for ALL waves.

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \tag{4}$$

has no solutions.
Open problem: Prove spectral stability for ALL waves.
Main issue: $\mathcal{L}_\pm$ have spectrum at both $\pm\infty$.

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \tag{4}$$

has no solutions.
Open problem: Prove spectral stability for ALL waves.
Main issue: $\mathcal{L}_\pm$ have spectrum at both $\pm\infty$.

Why NLS is better?

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \tag{4}$$

has no solutions.
Open problem: Prove spectral stability for ALL waves.
Main issue: $\mathcal{L}_\pm$ have spectrum at both $\pm\infty$.

Why NLS is better?

Since $\mathcal{L}_- \geq 0, z \perp Ker[\mathcal{L}_-]$, (4) is transformed into

$$\sqrt{\mathcal{L}_-}\mathcal{L}_+\sqrt{\mathcal{L}_-} w = -\lambda^2 w, w = \mathcal{L}_-^{-\frac{1}{2}} z.$$

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \qquad (4)$$

has no solutions.
Open problem: Prove spectral stability for ALL waves.
Main issue: $\mathcal{L}_\pm$ have spectrum at both $\pm\infty$.

Why NLS is better?

Since $\mathcal{L}_- \geq 0, z \perp Ker[\mathcal{L}_-]$, (4) is transformed into

$$\sqrt{\mathcal{L}_-}\mathcal{L}_+\sqrt{\mathcal{L}_-} w = -\lambda^2 w, w = \mathcal{L}_-^{-\frac{1}{2}} z.$$

But this is self-adjoint problem, so $-\lambda^2$ is real.

Spectral stability?

Spectral stability means $\mathcal{L}_+ z_1 = -\lambda z_2, \mathcal{L}_- z_2 = \lambda z_1$ has no solutions for $\lambda : \Re\lambda > 0$.
Equivalently,

$$\mathcal{L}_- \mathcal{L}_+ z = -\lambda^2 z \tag{4}$$

has no solutions.
Open problem: Prove spectral stability for ALL waves.
Main issue: $\mathcal{L}_\pm$ have spectrum at both $\pm\infty$.

Why NLS is better?

Since $\mathcal{L}_- \geq 0, z \perp Ker[\mathcal{L}_-]$, (4) is transformed into

$$\sqrt{\mathcal{L}_-} \mathcal{L}_+ \sqrt{\mathcal{L}_-} w = -\lambda^2 w, w = \mathcal{L}_-^{-\frac{1}{2}} z.$$

But this is self-adjoint problem, so $-\lambda^2$ is real. A huge step.
Open problem: Can one at least prove that λ^2 in (4) is real (and not $\lambda = a + ib : a, b \neq 0$).

Thank you for your attention!